A Property for Certain Derivatives

CLIFFORD E. WEIL

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Introduction. The problem of determining when a real valued function of a real variable has a primitive is still unsolved. The first step towards resolving a question of this nature is to find properties possessed by every derivative. The best known of these is the intermediate value property or Darboux property. Less well-known but easy to prove is that every derivative is a function of Baire class one. In 1916, Denjoy enlarged the list of properties of derivatives, and in 1950, Zahorski proved a property for derivatives which includes Denjoy's.

In this work there is introduced a property of derivatives stronger than Zahorski's. It is shown that this property is also a long way from classifying derivatives by showing that this property is also possessed by every approximate derivative and by every k^{th} Peano derivative. It follows by a technique given in [1], that this property is also possessed by every k^{th} L_p derivative for $p \geq 1$.

Definitions and conventions. All functions will be real valued functions of a real variable and measurable.

Definition. A function f has an approximate derivative $f_{ap}'(x)$ at a point x if there is a measurable set E whose density at x is one such that

$$E - \lim_{y \rightarrow x} (f(y) - f(x))/(y - x) = f_{ap}'(x)$$

where $E - \lim_{y \rightarrow x}$ means y must be in E .

For a concise development of the properties of approximate derivatives see [2].

Definition. Let k be a positive integer. A function f has a k^{th} Peano derivative at x if there are numbers $f_1(x), \dots, f_k(x)$ such that

$$f(x + h) = f(x) + hf_1(x) + \dots + (h^k/k!)f_k(x) + o(h^k).$$

The number $f_k(x)$ is called the k^{th} Peano derivative of f at x . The elementary properties of k^{th} Peano derivatives can be found in [3].

It is assumed that the reader knows that if either of these generalized derivatives is bounded above or below on an interval, it is an ordinary derivative on that interval. See [2] and [3].

Definition. A function f has property Z if for each x , each $\epsilon > 0$ and each sequence $\{I_n\}$ of closed intervals converging to x (every neighborhood of x contains all but finitely many intervals) such that for each n $f(y) \geq f(x)$ on I_n or $f(y) \leq f(x)$ on I_n ,

$$\lim_{n \rightarrow \infty} \frac{m\{y \in I_n : |f(y) - f(x)| \leq \epsilon\}}{mI_n + d(x, I_n)} = 0$$

where mA is the Lebesgue measure of A and $d(x, I_n)$ is the distance from x to I_n .

Property Z and Zahorski's property. To indicate most transparently how property Z is stronger than Zahorski's, define Zahorski's property as follows:

Definition. A function f has Zahorski's property if for each open interval (a, b) and each x in $f^{-1}(a, b)$ and each sequence $\{I_n\}$ of closed intervals converging to x (in the same sense as used in the definition of property Z) such that for each n

$$f^{-1}(a, b) \cap I_n = \emptyset,$$

$$\lim_{n \rightarrow \infty} mI_n/d(x, I_n) = 0.$$

Remark. Zahorski's property is usually defined with the condition $f^{-1}(a, b) \cap I_n = \emptyset$ replaced by the weaker assumption $m(f^{-1}(a, b) \cap I_n) = 0$. All of the types of functions dealt with here belong to the class of functions for which the two conditions are equivalent; that is, the functions having Denjoy's property.

Theorem. If a Darboux function f has property Z , then it has Zahorski's property.

Proof. Let (a, b) be an open interval, x in $f^{-1}(a, b)$ and $\{I_n\}$ a sequence of closed intervals such that $f^{-1}(a, b) \cap I_n = \emptyset$ for each n . Then for each interval I_n , $f(y) \geq f(x)$ for all $y \in I_n$ or $f(y) \leq f(x)$ for all $y \in I_n$ for otherwise the Darboux property would imply that there would be a y in I_n with $f(y) = f(x)$ contrary to $f^{-1}(a, b) \cap I_n = \emptyset$. So since f has Property Z , for each $\epsilon > 0$

$$\lim_{n \rightarrow \infty} \frac{m\{y \in I_n : |f(y) - f(x)| \geq \epsilon\}}{mI_n + d(x, I_n)} = 0.$$

Let

$$\epsilon = \min [b - f(x), f(x) - a].$$

Then since

$$f^{-1}(a, b) \cap I_n = \emptyset,$$

$$\{y \in I_n : |f(y) - f(x)| \geq \epsilon\} = I_n.$$

Consequently

$$\lim_{n \rightarrow \infty} \frac{mI_n}{mI_n + d(x, I_n)} = 0$$

or

$$\lim_{n \rightarrow \infty} \frac{1}{1 + \frac{d(x, I_n)}{mI_n}} = 0,$$

from which it easily follows that

$$\lim_{n \rightarrow \infty} \frac{mI_n}{d(x, I_n)} = 0.$$

That property Z is strictly stronger than Zahorski's property can be seen from an example such as the following. Let $\{I_n\}$ be any sequence of disjoint closed intervals converging to 0 such that

$$\lim_{n \rightarrow \infty} mI_n / (mI_n + d(0, I_n)) = \alpha$$

where $0 < \alpha < 1$. Define a function f to be 0 off of the intervals I_n , in particular $f(0) = 0$. To define f on one of the intervals I_n , divide I_n into n subintervals $I_{n,1}, \dots, I_{n,n}$ of equal length. Let f be 0 at both endpoints of $I_{n,i}$, 1 on the middle half of $I_{n,i}$ and linear on the remainder. Consequently, $m\{y \in I_n : f(y) = 1\} = \frac{1}{2}mI_n$. Hence,

$$\lim_{n \rightarrow \infty} m\{y \in I_n : |f(y) - f(0)| \geq 1\} / (mI_n + d(0, I_n)) = \frac{1}{2} \alpha \neq 0.$$

Thus f does not have property Z , but, as will now be indicated, it does have Zahorski's property. Since f is continuous except at 0, only those open intervals (a, b) containing $f(0) = 0$ need be investigated and 0 is the only element of $f^{-1}(a, b)$ that needs to be considered. Any interval J with $J \cap f^{-1}(a, b) = \emptyset$ is a subinterval of $I_{n,j}$ for some pair of integers n, j with $j \leq n$ of course. Thus the length of J is no more than $(1/n)mI_n$ whereas its distance from I is at least $d(0, I_n)$. Thus

$$mJ/d(0, J) \leq \frac{1}{n} mI_n/d(0, I_n).$$

It is an easy exercise to show that

$$\lim_{n \rightarrow \infty} mI_n/d(0, I_n) = \alpha/(1 - \alpha).$$

In addition, in order for J to be close to 0, the associated integer n must be large. From both of these remarks it follows that if $\{J_k\}$ is a sequence of intervals converging to 0 with $J_k \cap f^{-1}(a, b) = \emptyset$ for each k , then

$$\lim_{k \rightarrow \infty} mJ_k/d(0, J_k) = 0.$$

Therefore f has Zahorski's property but not property Z .

Property Z for ordinary derivatives. That ordinary derivatives have property Z could be concluded from Theorem 2 or Theorem 3, but the proof is so much easier than the proofs of Theorems 2 and 3 that it is presented here for the purpose of contrast.

Theorem 1. *If f has an ordinary derivative f' everywhere, then f' has property Z .*

Proof. Let x be any real number, $\epsilon > 0$, $[a, b]$ any interval, and let

$$A = \{y \in [a, b] : |f'(y) - f'(x)| \geq \epsilon\}.$$

First assume $f'(y) \geq f'(x)$ for all $y \in [a, b]$. Then f' is integrable on $[a, b]$ and

$$\begin{aligned} f(b) - f(a) &= \int_a^b f'(y) dy \geq (f'(x) + \epsilon)mA + f'(x)m([a, b] - A) \\ &= f'(x)(b - a) + \epsilon mA; \end{aligned}$$

or

$$\epsilon mA \leq f(b) - f(a) - f'(x)(b - a).$$

Next assume $f'(y) \leq f'(x)$ for all $y \in [a, b]$. Then again f' is integrable on $[a, b]$ and

$$\begin{aligned} f(b) - f(a) &= \int_a^b f'(y) dy \leq (f'(x) - \epsilon)mA + f'(x)m([a, b] - A) \\ &= f'(x)(b - a) - \epsilon mA; \end{aligned}$$

or

$$\epsilon mA \leq -(f(b) - f(a) - f'(x)(b - a)).$$

So if $f'(y) \geq f'(x)$ for all $y \in [a, b]$ or if $f'(y) \leq f'(x)$ for all $y \in [a, b]$, then

$$\epsilon mA \leq |f(b) - f(a) - f'(x)(b - a)|.$$

Since $f'(x)$ exists given $\eta > 0$ there is a $\delta > 0$ such that $|y - x| < \delta$ implies

$$|f(y) - f(x) - f'(x)(y - x)| \leq |y - x| \epsilon \eta / 2.$$

Hence if $[a, b] \subset (x - \delta, x + \delta)$

$$\begin{aligned} &|f(b) - f(a) - f'(x)(b - a)| \\ &= |(f(b) - f(x) - f'(x)(b - x)) - (f(a) - f(x) - f'(x)(a - x))| \\ &\leq |f(b) - f(x) - f'(x)(b - x)| + |f(a) - f(x) - f'(x)(a - x)| \\ &\leq |b - x| \epsilon \eta / 2 + |a - x| \epsilon \eta / 2 = (\epsilon \eta / 2)(|b - x| + |a - x|) \\ &\leq \epsilon \eta (d(x, [a, b]) + (b - a)) \end{aligned}$$

because both $|b - x|$ and $|a - x|$ are no larger than $d(x, [a, b]) + (b - a)$. Consequently,

$$mA \leq (d(x, [a, b]) + (b - a))\eta$$

and from this inequality it is easy to show that f' has property Z .

Property Z for approximate derivatives. The proof that every approximate derivative has property Z follows along the same lines as that for the ordinary derivative. Only a few technical points and additional remarks need be added.

Theorem 2. *If f has an approximate derivative everywhere, then f_{ap}' has property Z .*

Proof. Let x be any real number, $\epsilon > 0$, $[a, b]$ any interval such that $f_{ap}'(y) \geq f_{ap}'(x)$ on $[a, b]$ or $f_{ap}'(y) \leq f_{ap}'(x)$ on $[a, b]$. Let $A = \{y \in [a, b] : |f_{ap}'(y) - f_{ap}'(x)| \geq \epsilon\}$. In addition let E be a set whose density at x is 1 such that

$$E - \lim_{y \rightarrow x} (f(y) - f(x))/(y - x) = f_{ap}'(x).$$

First obtain an estimate on $m(A \cap E)$. For this purpose assume $[a, b] \cap E \neq \emptyset$ and let $a \leq c \leq d \leq b$ with $c, d \in E$. Then using an argument similar to that of the preceding theorem, (recalling that $f_{ap}' = f'$ on $[a, b]$) it can easily be shown that

$$\epsilon m(A \cap [c, d]) \leq |f(d) - f(c) - f_{ap}'(x)(d - c)|.$$

By the choice of E , given an $\eta > 0$ there is a $\delta > 0$ such that:

$$\text{i) } |f(y) - f(x) - f_{ap}'(x)(y - x)| \leq |y - x| \epsilon \eta / 4 \text{ for } |y - x| < \delta \text{ and } y \in E,$$

and

$$\text{ii) } m(I \cap E^c) \leq m(I)\eta/2 \text{ for } I \text{ an interval contained in } (x - \delta, x + \delta) \text{ and } x \in I \text{ where } E^c \text{ is the complement of } E.$$

Then again similarly to the previous proof, if $[a, b] \subset (x - \delta, x + \delta)$, then

$$|f(d) - f(c) - f_{ap}'(x)(d - c)| \leq (d(x, [a, b]) + (b - a))\epsilon\eta/2.$$

Consequently,

$$m(A \cap [c, d]) < (d(x, [a, b]) + (b - a))\eta/2.$$

If $a' = \inf \{c \in [a, b] \cap E\}$ and $b' = \sup \{d \in [a, b] \cap E\}$, then it follows from the above inequality that

$$m(A \cap E) \leq m(A \cap [a', b']) \leq (d(x, [a, b]) + (b - a))\eta/2.$$

Clearly the same estimate holds if $[a, b] \cap E = \emptyset$. Now,

$$mA = m(A \cap E) + m(A \cap E^c) \leq m(A \cap E) + m(I \cap E^c)$$

where I is the smallest closed interval containing both x and $[a, b]$. Then $mI = d(x, [a, b]) + (b - a)$. Therefore

$$\begin{aligned} mA &\leq (d(x, [a, b]) + (b - a))\eta/2 + mI\eta/2 \\ &= (d(x, [a, b]) + (b - a))\eta, \end{aligned}$$

and once more property Z for $f_{a''}$ follows easily.

Property Z for k^{th} peano derivatives. The proof that every k^{th} Peano derivative has property Z is not like either of the proceeding proofs. It begins with a lemma where all of the work is done. After the lemma the proof is relatively easy.

Lemma. Suppose f is a function whose k^{th} derivative exists and is nonnegative on a closed interval $[a, b]$, and let

$$A = \{x \in [a, b] : f^{(k)}(x) \geq \epsilon\}$$

where ϵ is a fixed positive number. Then there is a partition $\{a = t_0 < t_1 < \cdots < t_\ell = b\}$ of the interval $[a, b]$ with $\ell \leq 2^k$ and such that for each $i = 1, \dots, \ell$

$$|f(t_i) - f(t_{i-1})| \geq (\epsilon/k!)(m(A \cap [t_{i-1}, t_i]))^k.$$

Proof. It will be shown by induction that for each integer $j = 1, \dots, k$, there is a partition $\{a = t_{0,i} < t_{1,i} < \cdots < t_{\ell(j),i} = b\}$ of $[a, b]$ with $\ell(j) \leq 2^j$ and such that for each $i = 1, \dots, \ell(j)$ one of the following holds for all x in $[t_{i-1,i}, t_{i,i}]$:

- 1(j) : $f^{(k-i)}(x) - f^{(k-i)}(t_{i-1,i}) \geq (\epsilon/j!)(m(A \cap [t_{i-1,i}, x]))^j$
and $f^{(k-i)}(t_{i-1,i}) \geq 0$
- 2(j) : $f^{(k-i)}(x) - f^{(k-i)}(t_{i,i}) \leq -(\epsilon/j!)(m(A \cap [x, t_{i,i}]))^j$
and $f^{(k-i)}(t_{i,i}) \leq 0$
- 3(j) : $f^{(k-i)}(x) - f^{(k-i)}(t_{i-1,i}) \leq -(\epsilon/j!)(m(A \cap [t_{i-1,i}, x]))^j$
and $f^{(k-i)}(t_{i-1,i}) \leq 0$
- 4(j) : $f^{(k-i)}(x) - f^{(k-i)}(t_{i,i}) \geq (\epsilon/j!)(m(A \cap [x, t_{i,i}]))^j$
and $f^{(k-i)}(t_{i,i}) \geq 0$.

The desired partition is then the one corresponding to $j = k$ and the desired inequality is obtained by replacing in each of the four statements 1(k), 2(k), 3(k) and 4(k) x be the endpoint of the interval $[t_{i-1,k}, t_{i,k}]$ not already appearing in the statement and then taking absolute values, where, of course, $f^{(0)} = f$.

The induction assertion is first proved for $j = 1$. Since $f^{(k)}(x) \geq 0$ for all $x \in [a, b]$, $f^{(k-1)}$ is nondecreasing on $[a, b]$. Let $|f^{(k-1)}|$ attain its minimum value on $[a, b]$ at $t \in [a, b]$. There are three cases to consider: $t = a$, $t = b$, and $a < t < b$.

If $t = a$, then since $f^{(k-1)}$ is nondecreasing on $[a, b]$, $0 \leq f^{(k-1)}(a)$, and for all x in $[a, b]$.

$$f^{(k-1)}(x) - f^{(k-1)}(a) = \int_a^x f^{(k)}(s) ds \geq \epsilon m(A \cap [a, x]).$$

In this case 1(1) holds for all x in $[a, b]$; so take $t_{0,1} = a$ and $t_{1,1} = b$.

If $t = b$, then a similar argument shows that, 2(1) holds for all x in $[a, b]$; so take again $t_{0,1} = a$ and $t_{1,1} = b$.

If $a < t < b$, then since $f^{(k-1)}$ is nondecreasing on $[a, b]$, $f^{(k-1)}(t) = 0$. Arguments like the one just presented show that 2(1) holds on $[a, t]$ and 1(1) holds on $[t, b]$. In this case take $t_{0,1} = a$, $t_{1,1} = t$ and $t_{2,1} = b$. The induction assertion is established for $j = 1$. (Note that only cases 1 and 2 occurred. In the induction step it will be seen that 3 and 4 arise from 2.)

Assume the assertion true for j , and let i be an integer between 1 and $\ell(j)$. As a notational convenience denote the interval $[t_{i-1,i}, t_{i,i}]$ by $[c, d]$ and the function $f^{(k-(i+1))}$ by g so that $f^{(k-i)} = g'$. It must be shown that if 1(j), 2(j), 3(j) or 4(j) holds on $[c, d]$, then $[c, d]$ can be divided into no more than two subintervals so that on each subinterval 1($j+1$), 2($j+1$), 3($j+1$), or 4($j+1$) holds. It will suffice to present in detail the arguments for a few representative cases. From them, the reader will see the pattern and be able to supply the missing ones if desired. Before starting the proof of the first case, some facts are established that will be needed below.

Let $[\alpha, \beta]$ be a subinterval of $[c, d]$ and let

$$F(t) = m(A \cap [\alpha, t]) = \int_{\alpha}^t \chi(s) ds$$

where χ denotes the characteristic function of A . Then F is absolutely continuous and $F'(t) = \chi(t)$ a.e. Moreover, almost everywhere

$$\frac{d}{dt} F^{i+1} = (j+1)F^i \chi \leq (j+1)F^i.$$

Consequently,

$$\begin{aligned} \int_{\alpha}^{\beta} (\epsilon/j!)(m(A \cap [\alpha, s]))^i ds &\geq (\epsilon/(j+1)!)(m(A \cap [\alpha, s]))^{i+1} \Big|_{\alpha}^{\beta} \\ &= (\epsilon/(j+1)!)(m(A \cap [\alpha, \beta]))^{i+1}. \end{aligned}$$

Likewise if

$$G(t) = m(A \cap [t, \beta]) = \int_t^{\beta} \chi(s) ds,$$

then G is absolutely continuous and $G'(t) = -\chi(t)$ a.e. In addition almost everywhere

$$\frac{d}{dt} G^{i+1} = (j+1)G^i(-\chi) \geq -(j+1)G^i.$$

Therefore,

$$\begin{aligned} \int_{\alpha}^{\beta} -(\epsilon/j!)(m(A \cap [s, \beta]))^i ds &\leq (\epsilon/(j+1)!)(m(A \cap [s, \beta]))^{i+1} \Big|_{\alpha}^{\beta} \\ &= -(\epsilon/(j+1)!)(m(A \cap [\alpha, \beta]))^{i+1}. \end{aligned}$$

First assume that 1(j) holds for all $x \in [c, d]$. Then

$$\begin{aligned} g'(x) &\geq (\epsilon/j!)(m(A \cap [c, x]))^i + g'(c) \\ &\geq (\epsilon/j!)(m(A \cap [c, x]))^i \geq 0 \end{aligned}$$

since $g'(c) \geq 0$. Hence g is nondecreasing on $[c, d]$. Let $|g|$ attain its minimum value on $[c, d]$ at $t \in [c, d]$. There are as before three cases to consider: $t = c$, $t = d$, and $c < t < d$.

If $t = c$, then $g(c) \geq 0$ and for all $x \in [c, d]$

$$\begin{aligned} g(x) - g(c) &= \int_c^x g'(s) ds \geq \int_c^x (\epsilon/j!)(m(A \cap [c, s]))^i ds \\ &\geq (\epsilon/(j+1)!)(m(A \cap [c, x]))^{i+1}. \end{aligned}$$

Thus, 1($j+1$) holds for all x in $[c, d]$.

If $t = d$, then $g(d) \leq 0$ and for all $x \in [c, d]$

$$\begin{aligned} g(d) - g(x) &= \int_x^d g'(s) ds \geq \int_x^d (\epsilon/j!)(m(A \cap [c, s]))^i ds \\ &\geq \int_x^d (\epsilon/j!)(m(A \cap [x, s]))^i ds \\ &\geq (\epsilon/(j+1)!)(m(A \cap [x, d]))^{i+1}. \end{aligned}$$

Multiplying by -1 yields

$$g(x) - g(d) \leq -(\epsilon/(j+1)!)(m(A \cap [x, d]))^{i+1},$$

so 2($j+1$) holds for all x in $[c, d]$.

If $c < t < d$, then $g(t) = 0$ and it can be shown by arguments like the two just given that 2($j+1$) holds for all $x \in [c, t]$, and that 1($j+1$) holds for all $x \in [t, d]$.

Second, assume that 2(j) holds for all $x \in [c, d]$. Then

$$\begin{aligned} g'(x) &\leq -(\epsilon/j!)(m(A \cap [x, d]))^i + g'(d) \\ &\leq -(\epsilon/j!)(m(A \cap [x, d]))^i \leq 0. \end{aligned}$$

Consequently, g is nonincreasing on $[c, d]$. Let $|g|$ attain its minimum value on $[c, d]$ at t . There are three cases.

If $t = c$, then $g(c) \leq 0$, and for all $x \in [c, d]$,

$$\begin{aligned} g(x) - g(c) &= \int_c^x g'(s) ds \leq \int_c^x -(\epsilon/j!)(m(A \cap [s, d]))^j ds \\ &\leq \int_c^x -(\epsilon/j!)(m(A \cap [s, x]))^j ds \\ &\leq -(\epsilon/(j+1)!) (m(A \cap [c, x]))^{j+1}. \end{aligned}$$

Therefore, $3(j+1)$ holds for all $x \in [c, d]$.

If $t = d$, then $g(d) \geq 0$ and it is easily shown that $4(j+1)$ holds for all $x \in [c, d]$. If $c < t < d$, then $g(c) = 0$ and arguing as has already been indicated will verify that $4(j+1)$ holds for all $x \in [c, t]$ while $3(j+1)$ holds for all $x \in [t, d]$. (It should be observed that cases 3 and 4 arise from 2 and thus are essential.)

Third assume that $3(j)$ holds for all $x \in [c, d]$. Then

$$g'(x) \leq -(\epsilon/j!)(m(A \cap [c, x]))^j \leq 0$$

for all $x \in [c, d]$ and hence g is nonincreasing on $[c, d]$. Let $|g|$ attain its minimum value on $[c, d]$ at t . If $t = c$, then it can be shown by arguing similarly to above that $3(j+1)$ holds for each $x \in [c, d]$.

If $t = d$, it is again easily shown that $4(j+1)$ holds for all $x \in [c, d]$.

If $c < t < d$, then it can be established that $4(j+1)$ holds on $[c, t]$ and $3(j+1)$ on $[t, d]$.

Finally, assume $4(j)$ holds for all $x \in [c, d]$. Then it is easily seen that g is nondecreasing on $[c, d]$. Let $|g|$ attain its minimum value on $[c, d]$ at t . If $t = c$, then $1(j+1)$ holds for all $x \in [c, d]$. If $t = d$, then $g(d) \leq 0$ and it can be proved that $2(j+1)$ holds for all $x \in [c, d]$. If $c < t < d$, then $g(t) = 0$, and proceeding as has already been demonstrated it can be established that $2(j+1)$ holds on $[c, t]$ and $1(j+1)$ on $[t, d]$. This completes the proof of the lemma.

Theorem 3. If f has a k^{th} Peano derivative everywhere, then f_k has property Z.

Proof. Let x be any real number, $\epsilon > 0$, and $[a, b]$ any interval. Let $A = \{y \in [a, b] : |f_k(x) - f_k(y)| \geq \epsilon\}$. First assume that $f_k(y) \geq f_k(x)$ on $[a, b]$. (Recall that this implies that $f_k(y) = f^{(k)}(y)$ on $[a, b]$.) Apply the lemma to the function $g(y) = f(y) - f(x) - f_1(x)(y-x) - \cdots - f_k(x)(y-x)^k/k!$ which satisfies $g_k(y) = f_k(y) - f_k(x) \geq \epsilon$ on $[a, b]$. So there is a partition $\{a = t_0 < t_1 < \cdots < t_\ell = b\}$ with $\ell \leq 2^k$ such that for each $i = 1, \dots, \ell$

$$|g(t_i) - g(t_{i-1})| \geq (\epsilon/k!)(m(A \cap [t_{i-1}, t_i]))^k.$$

Second, if $f_k(y) \leq f_k(x)$ on $[a, b]$, then apply the lemma to $-g$ and obtain precisely the same inequality. Since $f_k(x)$ exists given $\eta > 0$ there is a $\delta > 0$ such that $|x - y| < \delta$ implies

$$\begin{aligned} |g(y)| &= |f(y) - f(x) - f_1(x)(y-x) - \cdots - f_k(x)(y-x)^k/k!| \\ &\leq (\epsilon\eta^k/(k! 2^{k(k+1)})) |y-x|^k. \end{aligned}$$

Thus if $[a, b] \subset (x - \delta, x + \delta)$, then

$$\begin{aligned}
 m(A \cap [a, b]) &= \sum_{i=1}^{\ell} m(A \cap [t_{i-1}, t_i]) \\
 &\leq (k!/\epsilon)^{1/k} \sum_{i=1}^{\ell} |g(t_i) - g(t_{i-1})|^{1/k} \\
 &\leq (k!/\epsilon)^{1/k} \sum_{i=1}^{\ell} (|g(t_i)|^{1/k} + |g(t_{i-1})|^{1/k}) \\
 &\leq (k!/\epsilon)^{1/k} (\epsilon \eta^k / (k! 2^{k(k+1)}))^{1/k} \sum_{i=1}^{\ell} |t_i - x| + |t_{i-1} - x| \\
 &\leq (\eta/2^{k+1}) 2\ell(d(x, [a, b]) + (b - a)) \\
 &\leq \eta(d(x, [a, b]) + (b - a)),
 \end{aligned}$$

and again property Z is easily deduced for f_k .

Concluding remarks.

1. Possessing a k^{th} Peano derivative can be interpreted as demanding that the L_{∞} -norm on the interval between 0 and h of $f(x+t) - f(x) - f_1(x)t - \cdots - f_k(x)t^k/k!$ be $o(h^k)$ as $h \rightarrow 0$. Then it is natural to replace the L_{∞} -norm on the interval between 0 and h by the L_p -norm on that interval multiplied by $|h|^{-1/p}$ so that the function 1 has norm 1. The resulting derivative is called the k^{th} L_p derivative. In [1] Evans shows that every k^{th} L_p derivative for $p \geq 1$ is a $(k+1)^{\text{th}}$ Peano derivative in L_{∞} . It follows that every k^{th} L_p derivative ($p \geq 1$) has property Z.

2. The derivatives discussed here actually have a stronger version of property Z obtained by replacing the assumption that $f(y) \geq f(x)$ for all $y \in [a, b]$ or $f(y) \leq f(x)$ for all $y \in [a, b]$ by the weaker condition $f(y) \geq f(x)$ for *a.e.* $y \in [a, b]$ or $f(y) \leq f(x)$ for *a.e.* $y \in [a, b]$. But since ordinary derivatives, approximate derivatives, and Peano derivatives all have the Denjoy property, the weaker condition implies the stronger one.

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Michigan State University

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