A Decomposition for Some Operators

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Let H be a complex Hilbert space and let $\mathfrak{B}(H)$ denote the algebra of all bounded linear operators on H . Then $T \in \mathfrak{B}(H)$ is abnormal (sometimes, completely non-normal) if there is no non-trivial subspace $M \subseteq H$ which reduces T and such that the restriction of T to M is normal. Every $T \in \mathfrak{B}(H)$ may be written uniquely as the direct sum of a normal operator T_0 with an abnormal operator T_1 . We shall refer to T_0 and T_1 as the normal and abnormal parts of T , respectively.

A theorem of von Neumann ([8], p. 96), rediscovered and sharpened by Halmos ([3]), asserts that every isometry V on a Hilbert space H is unitarily equivalent to the direct sum of a unitary operator and a pure isometry of multiplicity $d = \dim [(VH)^\perp]$ (cf. [4], problem 118). It develops that the scalar d is a complete set of unitary invariants for the abnormal part of the isometry V . An operator T is quasinormal if T commutes with T^*T . In particular, every isometry is quasinormal. In [1] Brown obtains both a canonical form and a complete set of unitary invariants for the abnormal part of a quasinormal operator. In the isometric case Brown's results specialize to those of von Neumann.

In section 1 of this paper we obtain a decomposition for operators, which, as is shown in section 3, is a generalization of Brown's work on quasinormal operators. We associate with each $T \in \mathfrak{B}(H)$ a (not necessarily proper) subspace $H_1(T)$ of H which is invariant under T^* and reduces $[T] = T^*T - TT^*$. If $V \in \mathfrak{B}(H)$ is isometric, for instance, one has $H_1(V) = (VH)^\perp$. We establish that the abnormal part of T is completely determined up to unitary equivalence by the restrictions of T^* and $[T]$ to $H_1(T)$. In case $d = \dim (H_1(T)) < \infty$, the structure of the abnormal part of T is determined by two d -by- d matrices.

The results of section 1 are of little interest if $H_1(T)$ is too large. In section 2 we study conditions under which $H_1(T) = H$. We show that if T is abnormal and nearly a finite-dimensional operator (in some appropriate sense), then $H_1(T) = H$. This suggests that the results of section 1 will be of most interest if the operator being studied is far from being finite-dimensional.

The main result of section 3 is that if T is subnormal, then $H_1(T)$ is the closure of the range of $[T]$. This means, for example, that the structure results given in section 1 may be easily applied to subnormal operators whose self-commutator is of finite rank. It also enables us to deduce the results of Brown and von Neumann mentioned above from our results in section 1.

In section 4 we give an application of our results to the study of quasitriangular operators.

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§1. We begin with several lemmas which may be of interest independent of their application here.

Lemma 1.1: *Let H be a Hilbert space and let $A, B \in \mathfrak{B}(H)$. Then the largest subspace M of H for which $BM \subseteq M$ and $ABv = BA v$ for every $v \in M$ is*

$$M = \bigcap_{s=1}^{\infty} \ker (AB^s - B^s A).$$

Proof: It is clear that M is a subspace of H . Pick $v \in M$ and let $w = Bv$. Then for all integers $s \geq 1$, we have $AB^s w = AB^{s+1}v = B^{s+1}Av = B^s(BAv) = B^s(ABv) = B^s Aw$, since $v \in M$. Hence $BM \subseteq M$. The relation $M \subseteq \ker (AB - BA)$ implies that $ABv = BA v$ for all $v \in M$.

Next, let Y be a subspace of H such that $BY \subseteq Y$ and $AB y = BA y$ for all $y \in Y$. Then $B^s Y \subseteq Y$ for all $s \geq 1$. If $y \in Y$, then $AB^2 y = AB(By) = BA(By) = B(AB y) = B^2 Ay$. By induction, $B^s Ay = AB^s y$ for every $y \in Y$ and all $s \geq 1$. Hence $Y \subseteq M$.

An easy modification of the proof of Lemma 1.1 yields a proof of the following result:

Lemma 1.2: *Let $A, B \in \mathfrak{B}(H)$. Then the largest subspace $M \subseteq H$ such that $AM \subseteq M$, $BM \subseteq M$ and $ABv = BA v$ for every $v \in M$ is*

$$M = \bigcap_{r=1}^{\infty} \bigcap_{s=1}^{\infty} \ker (A^r B^s - B^s A^r).$$

Although we shall not use the results in this generality, we note that Lemmas 1.1 and 1.2 both hold in case the underlying space is a Banach space.

An immediate consequence of Lemma 1.2 and the definition of reducing subspace is the following:

Corollary 1.3: *Let $T \in \mathfrak{B}(H)$. Then the largest subspace H_0 of H which reduces T and such that $T|_{H_0}$ is normal is*

$$H_0 = \bigcap_{r=1}^{\infty} \bigcap_{s=1}^{\infty} \ker ((T^*)^r T^s - T^s (T^*)^r).$$

Furthermore, $T|(H_0)^\perp$ is abnormal.

Our next lemma is the basis for the subsequent decomposition theorems.

Lemma 1.4: Let $T \in \mathfrak{B}(H)$. Put $M_0 = H$ and, for all $k \geq 1$, define

$$M_k = \bigcap_{r=1}^k \bigcap_{s=1}^{\infty} \ker ((T^*)^r T^s - T^s (T^*)^r).$$

Then

- i.) $M_k \supseteq M_{k+1}$ for all $k \geq 0$,
- ii.) $TM_k \subseteq M_k$ for all $k \geq 0$,
- iii.) $T^*M_k \subseteq M_{k-1}$ for all $k \geq 1$,
- iv.) $T^*M_k^\perp \subseteq M_k^\perp$ for all $k \geq 0$,
- v.) $TM_k^\perp \subseteq M_{k+1}^\perp$ for all $k \geq 0$.

Proof: Parts iv.) and v.) of the assertion follow immediately from parts ii.) and iii.) upon taking orthogonal complements, and part i.) follows directly from the definition of the subspaces M_k .

Part ii.) is trivially true if $k = 0$. For $r \geq 1$, Lemma 1.1 implies that the intersection (taken over $s \geq 1$) of the subspaces $\ker [(T^*)^r T^s - T^s (T^*)^r]$ is itself an invariant subspace for T , and hence, so also is M_k for every $k \geq 1$.

Part iii.) is trivial if $k = 1$, so suppose that $k \geq 2$. Let $x \in M_k$ and put $z = T^*x$. Then since $x \in M_k \subseteq M_1$, we have $(T^*)^r T^s z = (T^*)^r (T^s T^* x) = (T^*)^{r+1} T^s x$. But if $r + 1 \leq k$, then, since $x \in M_p$ for every $p \leq k$, we have $(T^*)^r T^s z = T^s (T^*)^{r+1} x = T^s (T^*)^r z$. Thus $z \in M_{k-1}$ and iii.) holds.

If we let $\{e_k | k \geq 0\}$ be the standard orthonormal basis for ℓ^2 and if we let T be the unilateral shift on ℓ^2 , then it is instructive to note that

$$M_k = \text{sp}\{e_j | j \geq k\} \quad \text{for } k \geq 0.$$

Theorem 1.5: Let $T \in \mathfrak{B}(H)$. Then there exists a (finite or infinite) sequence $\{H_j | j \geq 0\}$ of pairwise orthogonal subspaces of H such that

- i.) $H = H_0 \oplus H_1 \oplus \cdots \oplus H_k \oplus \cdots$,
- ii.) H_0 reduces T , $T|_{H_0}$ is normal, and $T|_{H_0^\perp}$ is abnormal,
- iii.) $T^*H_1 \subseteq H_1$,
- iv.) $T^*H_k \subseteq H_{k-1} \oplus H_k$ for all $k \geq 2$,
- v.) $TH_k \subseteq H_k \oplus H_{k+1}$ for all $k \geq 1$,
- vi.) $H_k \oplus H_{k+1} = \vee \{H_k, TH_k\}$ for all $k \geq 1$,
- vii.) $\dim H_k \geq \dim H_{k+1}$ for all $k \geq 1$.

Proof: We associate with T the subspaces M_k as was done in Lemma 1.4. Put

$$H_0 = \bigcap_{k=1}^{\infty} M_k = \bigcap_{r=1}^{\infty} \bigcap_{s=1}^{\infty} \ker [(T^*)^r T^s - T^s (T^*)^r].$$

From Corollary 1.3, H_0 reduces T , $T|_{H_0}$ is normal, and $T|(H_0)^\perp$ is abnormal.

Next, define $H_k = (M_k)^\perp \cap M_{k-1}$ for all $k \geq 1$. Since H_0 is the intersection of the subspaces M_k , $k \geq 1$, we have $H_0 \perp H_j$ for every $j > 0$. Noting that $H_j \subseteq M_{j-1}$ for every $j \geq 1$ and that $H_i \subseteq (M_i)^\perp \subseteq (M_{i-1})^\perp$ whenever $i \leq j-1$, we conclude that $H_i \perp H_j$ if $i < j$, or equivalently, $H_i \perp H_j$ if $i \neq j$. By induction,

$$M_k^\perp = H_1 \oplus H_2 \oplus \cdots \oplus H_k, \quad k \geq 1.$$

It follows immediately that

$$H = H_0 \oplus H_1 \oplus \cdots \oplus H_k \oplus \cdots.$$

Thus, both *i.*) and *ii.*) hold.

Part *iii.*) follows from Lemma 1.4, since

$$T^*H_1 = T^*(M_1)^\perp \subseteq (M_1)^\perp = H_1.$$

A glance at the matrix representation of T relative to the decomposition $H = H_0 \oplus H_1 \oplus \cdots \oplus H_k \oplus \cdots$ reveals that *iv.*) and *v.*) are equivalent and that they both follow immediately from parts *ii.*) and *v.*) of Lemma 1.4.

Since $TH_k \subseteq H_k \oplus H_{k+1}$ for every $k \geq 1$, we have $\vee \{H_k, TH_k\} \subseteq H_k \oplus H_{k+1}$ for every $k \geq 1$. Assume that $v \notin H_k \oplus H_{k+1}$ and that v is orthogonal to $\vee \{H_k, TH_k\}$, where $k \geq 1$ is fixed. Clearly, $v \notin H_{k+1}$. The fact that $\langle v, Tx \rangle = 0$ for every $x \in H_k$ together with *v.*) implies $\langle v, Tx \rangle = 0$ for every $x \in H_1 \oplus \cdots \oplus H_k$. Thus, T^*v is orthogonal to $H_1 \oplus \cdots \oplus H_k$; that is, $T^*v \in M_k$. But since $v, T^*v \in M_k$, we have

$$\begin{aligned} T^s(T^*)^{k+1}v &= T^s(T^*)^k(T^*v) \\ &= (T^*)^k T^s(T^*v) \\ &= (T^*)^k(T^*T^s v) \\ &= (T^*)^{k+1}T^s v \end{aligned}$$

for every $s \geq 1$. Hence $v \in M_{k+1}$. Then $v \in [H_{k+1} \cap M_{k+1}] = \{0\}$ and *vi.*) holds. Part *vii.*) follows immediately from *vi.*)

It is worth noting that if T is the unilateral shift, then $H_0 = \{0\}$, while for $k \geq 1$, H_k is precisely the one-dimensional subspace spanned by e_{k-1} .

If $T \in \mathfrak{B}(H)$, then we shall use the notation $H_k(T)$, $k = 0, 1, 2, \dots$, to denote the subspaces associated with T as in Theorem 1.5. Note that in case $H = H_0(T) \oplus H_1(T)$, Theorem 1.5 is nothing more than the decomposition of T into a normal and an abnormal part. In case $H = H_0(T) \oplus H_1(T)$, we shall say that T has a trivial decomposition.

If $T \in \mathfrak{B}(H)$ is abnormal (so that $H_0(T) = \{0\}$) and if we let $P_k : H \rightarrow H_k(T)$ denote the orthogonal projection of H onto $H_k(T)$ for $k \geq 1$, and if we define

$$T_{ij} = P_i T|_{H_j}$$

for all $i, j \geq 1$, then T is represented by the matrix of operators $\{T_{ij}\}$ acting on

the direct sum of the spaces H_k , $k \geq 1$. Parts *iii.*), *iv.*), and *v.*) of Theorem 1.5 assert that $T_{ij} = 0$ if either $j > i$ or $j < i - 1$. Thus T is represented by a matrix of operators whose non-zero (operator) entries lie on either the main diagonal or the first subdiagonal of the matrix. To simplify the notation, let $D_i = T_{i,i}$ and let $S_i = T_{i+1,i}$ for every $i \geq 1$. From part *vi.*) of Theorem 1.5, we have $\text{ran } S_i$ dense in H_{i+1} for $i \geq 1$, or, equivalently, that $\ker (S_i)^* = \{0\}$ for $i \geq 1$.

In the next two theorems we exhibit a canonical form for operators which have a non-trivial decomposition. The technique to be used is a modification of the proof of the fact that every weighted shift is unitarily equivalent to a weighted shift with non-negative weights (cf. [4], problem 75). Roughly speaking, we wish to show that the matrix of operators $\{T_{ij}\}$ described above is unitarily equivalent to a matrix of operators of the same form with non-negative operator weights along the first subdiagonal. Some technical difficulties arise from the fact that the spaces $\{H_k\}$ may be of different dimensions.

For simplicity, we break the reduction to canonical form into two parts.

Theorem 1.6: *Let $T \in \mathfrak{B}(H)$ be abnormal. Then there exists a (finite or infinite) sequence of Hilbert spaces $J_1 \supseteq J_2 \supseteq \cdots$ and corresponding sequences of operators $\tilde{D}_i : J_i \rightarrow J_i$ and $\tilde{S}_i : J_i \rightarrow J_{i+1}$ with $\ker (\tilde{S}_i)^* = \{0\}$ and $\ker \tilde{S}_i = J_i \ominus J_{i+1}$, such that T is unitarily equivalent to the operator $\tilde{T}$ defined on $J_1 \oplus J_2 \oplus \cdots$ by the matrix $\{\tilde{T}_{i,j}\}$ of operators given by $\tilde{T}_{i,i} = \tilde{D}_i$, $\tilde{T}_{i+1,i} = \tilde{S}_i$, and $\tilde{T}_{i,j} = 0$ if $i \neq j, j + 1$.*

Further, $H_k(\tilde{T}) = J_k$ for $k \geq 1$.

Proof: We shall assume that all of the subspaces $H_k(T)$, $k \geq 1$, are non-zero. Put $J_1 = H_1(T)$ and define $(W_1)^* : H_1(T) \rightarrow J_1$ by $(W_1)^* = I$ on $H_1(T)$. Put $\tilde{D}_1 = D_1$.

Recall that $S_1 : J_1 \rightarrow H_2(T)$ and that $\text{cl}(\text{ran } S_1) = H_2(T)$. Let $J_2 = (\ker S_1)^\perp \subseteq J_1$. Then $\dim J_2 = \dim [(\ker S_1)^\perp] = \dim [\text{cl}(\text{ran } S_1)] = \dim H_2(T)$. Pick a unitary operator $(W_2)^* : H_2(T) \rightarrow J_2$ (onto J_2) and define $\tilde{S}_1 = (W_2)^* S_1 W_1 = (W_2)^* S_1$. Then $\tilde{S}_1 : J_1 \rightarrow J_2$. Since $\ker (S_1)^* = \{0\}$, we have $\ker (\tilde{S}_1)^* = \ker [(S_1)^* W_1] = \{0\}$, and since $J_2 = (\ker S_1)^\perp$, we have $\ker (\tilde{S}_1) = \ker S_1 = J_1 \ominus J_2$. Setting $\tilde{D}_2 = (W_2)^* D_2 W_2$, we see that $\tilde{D}_2 : J_2 \rightarrow J_2$.

Suppose that we have defined Hilbert spaces $J_1 \supseteq J_2 \supseteq \cdots \supseteq J_m$, that we have picked unitary operators $(W_i)^*$ mapping $H_i(T)$ onto J_i , $i = 1, 2, \dots, m$, and that we have put $\tilde{D}_i = (W_i)^* D_i W_i$ for $i = 1, 2, \dots, m$, and $\tilde{S}_i = (W_{i+1})^* S_i W_i$ for $i = 1, 2, \dots, m - 1$. Then define $J_{m+1} = [\ker (S_m W_m)]^\perp$. Since $\text{cl}(\text{ran } S_m) = H_{m+1}(T)$, we have $\dim (J_{m+1}) = \dim (H_{m+1}(T))$. Pick a unitary operator $(W_{m+1})^*$ mapping $H_{m+1}(T)$ onto J_{m+1} and define $\tilde{S}_m = (W_{m+1})^* S_m W_m$. Then, as above, $\tilde{S}_m$ maps J_m into J_{m+1} , $\ker (\tilde{S}_m)^* = \{0\}$, and $\ker \tilde{S}_m = J_m \ominus J_{m+1}$. Next, put $\tilde{D}_{m+1} = (W_{m+1})^* D_{m+1} W_{m+1}$, noting that $\tilde{D}_{m+1}$ is an operator on J_{m+1} .

Continuing this process, we obtain a sequence $J_1 \supseteq J_2 \supseteq \cdots$ of Hilbert spaces and a sequence of unitary operators $\{W_i\}$ with W_i mapping J_i onto

$H_i(T)$ for all $i \geq 1$. The associated sequences of operators $\{\tilde{D}_i\}$ and $\{\tilde{S}_i\}$ are as in the statement of the theorem.

Next put $J = J_1 \oplus J_2 \oplus \cdots$ and define $W = W_1 \oplus W_2 \oplus \cdots$. Then W is a unitary operator mapping J onto H , and, of course, the operator $\tilde{T} = W^*TW \in \mathfrak{B}(J)$ is unitarily equivalent to T . A straightforward computation with the representations of T and W as matrices of operators shows that the matrix of $\tilde{T}$ relative to the decomposition $J = J_1 \oplus J_2 \oplus \cdots$ is as desired.

Since $\tilde{T} = W^*TW$, we find that for all $r, s \geq 1$, the equation

$$(\tilde{T}^*)^r(\tilde{T})^s - (\tilde{T})^s(\tilde{T}^*)^r = W^*[(T^*)^rT^s - T^s(T^*)^r]W$$

holds, so that W^* maps $M_k(T)$ onto $M_k(\tilde{T})$ in a one-to-one fashion for all $k \geq 1$. It follows immediately that $H_k(\tilde{T}) = J_k$ for all $k \geq 1$.

Definition: Let $J_1 \supseteq J_2 \supseteq \cdots$ be a finite or infinite sequence of Hilbert spaces and let $J = J_1 \oplus J_2 \oplus \cdots \oplus J_k \oplus \cdots$. Let E_i denote the partial isometry of J_i onto J_{i+1} defined for all $i > 0$ by $E_iv = F_iv$ for all $v \in J_i$, where $F_i \in \mathfrak{B}(J_i)$ is the orthogonal projection on $J_{i+1} \subseteq J_i$. Let π_i denote the partial isometry of J onto J_i defined analogously for all $i > 0$. Then we say that $T \in \mathfrak{B}(J)$ is in standard form if $M_k(T) = J_{k+1} \oplus J_{k+2} \oplus \cdots$ for $k \geq 1$ and if there exist operators $\hat{D}_i \in \mathfrak{B}(J_i)$ and non-negative operators $P_i \in \mathfrak{B}(J_i)$ with $\ker P_i = J_i \ominus J_{i+1}$ such that $\pi_i T|J_k = 0$ if $k > i$ or $k < i - 1$, $\pi_i T|J_i = \hat{D}_i$ and $\pi_{i+1} T|J_i = E_i P_i$ for all $i > 0$.

Theorem 1.7: Let $T \in \mathfrak{B}(H)$ be abnormal. Then T is unitarily equivalent to an operator in standard form.

Proof: We may as well replace T by the operator $\tilde{T}$ of Theorem 1.6. We also retain the notation of Theorem 1.6.

Define $V_1 = I$ on J_1 . Next, write $\tilde{S}_1 = U_1[(\tilde{S}_1)^* \tilde{S}_1]^{1/2} = U_1 P_1$, so that P_1 is a non-negative operator on J_1 and U_1 is a partial isometry with initial space $(\ker \tilde{S}_1)^\perp = J_2 \subseteq J_1$ and final space $\text{cl}(\text{ran } \tilde{S}_1) = J_2$. Since U_1 is onto J_2 , it follows that $(U_1)^* : J_2 \rightarrow J_1$ is an isometry with range J_2 . Thus, $(V_2)^* = E_1(U_1)^*$ is a unitary operator on J_2 and $(V_2)^* \tilde{S}_1 V_1 = (V_1)^* U_1 P_1 = E_1(U_1)^* U_1 P_1 = E_1 P_1$. Suppose that we have defined unitary operators $V_i \in \mathfrak{B}(J_i)$ such that $(V_i)^* (\tilde{S}_{i-1})^* V_{i-1} = E_{i-1} P_{i-1}$ for $1 \leq i \leq m$ and non-negative operators $P_i \in \mathfrak{B}(J_i)$ by $P_i = [(V_i)^* (\tilde{S}_i)^* \tilde{S}_i V_i]^{1/2}$ for $1 \leq i \leq m$. We consider the polar factorization $\tilde{S}_m V_m = U_m P_m$ of $\tilde{S}_m V_m$. Arguing as above, $(U_m)^* : J_{m+1} \rightarrow J_m$ is an isometry with final space J_{m+1} , so that $(V_{m+1})^* = E_m(U_m)^*$ is a unitary operator on J_{m+1} . Thus we obtain a (finite or infinite) sequence of unitary operators $V_i \in \mathfrak{B}(J_i)$ and a sequence $P_i \in \mathfrak{B}(J_i)$ of non-negative operators such that $\ker P_i = J_i \ominus J_{i+1}$ and $(V_{i+1})^* \tilde{S}_i V_i = E_i P_i$ for all $i > 0$. We define $\hat{D}_i = (V_i)^* \tilde{D}_i V_i$ for all $i > 0$ and put $V = V_1 \oplus V_2 \oplus \cdots \oplus V_k \oplus \cdots$. Then $V \in \mathfrak{B}(J)$ is unitary, so that $\hat{T} = V^* \tilde{T} V$ is unitarily equivalent to $\tilde{T}$, and hence, to T . A straightforward calculation shows that $\pi_i \hat{T}|J_k = 0$ if $k > i$ or $k < i - 1$, that $\pi_i \hat{T}|J_i = V^* \tilde{D}_i V_i = \hat{D}_i$ for $i > 0$, and that $\pi_{i+1} \hat{T}|J_i = (V_{i+1})^* \tilde{S}_i V_i = E_i P_i$ for all $i > 0$.

Finally, we note that an argument analogous to that used in Theorem 1.6 shows that $H_k(\hat{T}) = J_k$ for all $k > 0$. We note also that $H_k(\hat{T}) = VWH_k(T)$ and $H_k(T) = (VW)^*H_k(\hat{T})$ for all $k > 0$.

To simplify our notation, we shall assume in the future that if $T \in \mathfrak{B}(H)$ is abnormal and is in standard form, then $H = H_1 \oplus H_2 \oplus \cdots$, where $H_1 \supseteq H_2 \supseteq \cdots$. The diagonal (operator) entries of the matrix representation for T will be denoted by D_k and the subdiagonal entries by $S_k = E_k P_k$ for all $k \geq 1$.

Definition: Let $T^{(1)} \in \mathfrak{B}(H^{(1)})$ and $T^{(2)} \in \mathfrak{B}(H^{(2)})$ be abnormal and in standard form and suppose that there exists a unitary operator U from $H^{(1)}$ onto $H^{(2)}$ such that $T^{(1)} = U^*T^{(2)}U$ and such that the matrix representation of U relative to the decompositions of $H^{(1)}$ and $H^{(2)}$ is a diagonal matrix of operators with unitary entries along the main diagonal. We shall say that $T^{(1)}$ and $T^{(2)}$ are *equivalent* if the diagonal entries of U satisfy $U_{k+1} = E_k^{(2)}U_k(E_k^{(1)})^*$ for all $k \geq 1$.

In case $\dim H_k^{(i)} = \dim H_1^{(i)}$ for all $k \geq 1$, $i = 1, 2$, we note that the $E_k^{(i)}$'s are unnecessary and that the last condition in the preceding definition devolves to $U_k = U_1$ for all $k \geq 1$. In the general case, this condition amounts to $U_{k+1} = U_k|H_{k+1}^{(1)}$ for all $k \geq 2$.

Although it is clear that the equivalence of $T^{(1)}$ and $T^{(2)}$ implies their unitary equivalence, the special form of U which we require in the definition of equivalence suggests that equivalence is a much stronger relation than unitary equivalence. Surprisingly, perhaps, this is not the case.

Theorem 1.8: Let $T^{(i)} \in \mathfrak{B}(H^{(i)})$, $i = 1, 2$, be abnormal operators in standard form. Then $T^{(1)}$ and $T^{(2)}$ are equivalent if and only if they are unitarily equivalent.

Proof: As noted above, one half of the result is trivial. Let the unitary operator U satisfy $T^{(1)} = U^*T^{(2)}U$. Arguing as in Theorem 1.6, we see that U maps $M_k(T^{(1)})$ onto $M_k(T^{(2)})$ for all $k \geq 1$. Viewing $T^{(1)}$, $T^{(2)}$, and U as matrices of operators, this means that U is a diagonal matrix with diagonal (operator) entries U_k , where U_k is a unitary operator mapping $H_k(T^{(1)})$ onto $H_k(T^{(2)})$ for all $k \geq 1$.

An elementary matrix computation shows that $D_k^{(1)} = (U_k)^*D_k^{(2)}U_k$ and that $S_k^{(1)} = (U_{k+1})^*S_k^{(2)}U_k$ for all $k \geq 1$. The last equation yields

$$\begin{aligned} (S_k^{(1)})^*S_k^{(1)} &= U_k^*(S_k^{(2)})^*U_{k+1}U_{k+1}^*S_k^{(2)}U_k \\ &= U_k^*(S_k^{(2)})^*S_k^{(2)}U_k, \end{aligned}$$

or,

$$P_k^{(1)}(E_k^{(1)})^*E_k^{(1)}P_k^{(1)} = U_k^*P_k^{(2)}(E_k^{(2)})^*E_k^{(2)}P_k^{(2)}U_k$$

for every $k \geq 1$. Recalling that $\text{cl}(\text{ran } P_k^{(i)}) = H_{k+1}^{(i)}$ for $i = 1, 2$, $k \geq 1$, we obtain the identity $(E_k^{(i)})^*E_k^{(i)}P_k^{(i)} = P_k^{(i)}$ for $i = 1, 2$, $k \geq 1$. Substituting above, we obtain $(P_k^{(1)})^2 = (U_k)^*(P_k^{(2)})^2U_k$ for all $k \geq 1$. Hence, $P_k^{(1)} = \{(U_k)^*(P_k^{(2)})^2U_k\}^{1/2}$ for all $k \geq 1$. Finally, since $(U_k)^*(P_k^{(2)})^2U_k \geq 0$ and since $[(U_k)^*P_k^{(2)}U_k]^2 = (U_k)^*(P_k^{(2)})^2U_k$ we have $P_k^{(1)} = (U_k)^*P_k^{(2)}U_k$ for all $k \geq 1$.

Thus, $S_k^{(1)} = E_k^{(1)} P_k^{(1)} = E_k^{(1)} (U_k)^* P_k^{(2)} U_k$ for $k \geq 1$. From above, however, we have $S_k^{(1)} = (U_{k+1})^* S_k^{(2)} U_k = (U_{k+1})^* E_k^{(2)} P_k^{(2)} U_k$ for all $k \geq 1$. Hence, $E_k^{(1)} (U_k)^* = (U_{k+1})^* E_k^{(2)}$ on $\text{cl}(\text{ran } P_k^{(2)}) = H_{k+1}^{(2)} \subseteq H_k^{(2)}$, for all $k \geq 1$. Since both $E_k^{(1)} (U_k)^*$ and $(U_{k+1})^* E_k^{(2)}$ are zero on $\ker P_k^{(2)} = \ker E_k^{(2)}$, we have $E_k^{(1)} (U_k)^* = (U_{k+1})^* E_k^{(2)}$ for all $k \geq 1$. Thus, $E_k^{(2)} U_k (E_k^{(1)})^* = E_k^{(2)} \cdot (E_k^{(2)})^* U_{k+1} = U_{k+1}$ for all $k \geq 1$, and $T^{(1)}$ and $T^{(2)}$ are equivalent.

The next theorem deals with the relations that hold among the entries in the standard form for T . It shows that the structure of the abnormal part of an operator is determined by the action of the operator on the subspace $H_1(T)$.

Theorem 1.9: *Let $T \in \mathfrak{B}(H)$ be abnormal and in standard form. Put $C = [T] \upharpoonright H_1(T)$. Then*

- i.) $[D_1] = C - S_1^* S_1$,
- ii.) $[D_i] = S_{i-1} S_{i-1}^* - S_i^* S_i$ for $i \geq 2$.
- iii.) $S_i^* D_{i+1} = D_i S_i^*$ for $i \geq 1$.

Further, the operators D_i , $i \geq 2$ and the operators P_i , E_i , $i \geq 1$, may be determined explicitly in terms of C and D_1 .

Proof: Note first that

$$\ker [T] \supseteq M_1(T) = \bigcap_{r=1}^{\infty} \ker (T^* T^r - T^r T^*).$$

Hence, $\text{cl}(\text{ran } [T]) \subseteq (M_1(T))^{\perp} = H_1(T)$. It follows that $H_1(T)$ reduces $[T]$, so that $C = [T] \upharpoonright H_1(T)$ is well-defined. In fact, $[T] = C \oplus 0$. If we represent $[T]$ as a matrix of operators relative to the decomposition

$$H = H_1 \oplus H_2 \oplus \cdots \oplus H_k \oplus \cdots,$$

then we obtain a matrix $\{C_{i,j}\}$ of operators with $C_{1,1} = C$ and $C_{i,j} = 0$ if $i + j > 2$.

Using the matrix representation for T relative to this same decomposition for H , we obtain another expression for $[T]$. Direct comparison of the entries in these two representations for $[T]$ yields equations i.), ii.), and iii.).

To complete the proof, note first that $\ker S_1 = \ker ((S_1)^* S_1) = \ker (C - [D_1])$, so that $H_2 = (\ker S_1)^{\perp} = \text{cl} \{ \text{ran } (C - [D_1]) \}$. Hence, H_2 (and thus, E_1) is determined by C and D_1 . Noting that $(S_1)^* S_1 = P_1 (E_1)^* E_1 P_1$ and that $(E_1)^* E_1 \in \mathfrak{B}(H_1)$ is the orthogonal projection of H_1 onto H_2 , we have, since $H_2 = \text{cl}(\text{ran } P_1)$, that $(S_1)^* S_1 = (P_1)^2$, or,

$$P_1 = (S_1^* S_1)^{1/2} = (C - [D_1])^{1/2}.$$

We note for future reference that $S_1 (S_1)^* = E_1 (P_1)^2 (E_1)^* = (P_1)^2 | H_2$.

From iii.), $(S_1)^* D_2 = D_1 (S_1)^*$. If $X \in \mathfrak{B}(H_2)$ is any operator satisfying $(S_1)^* X = D_1 (S_1)^*$, then $(S_1)^* (D_2 - X) = 0$ and, since $\ker (S_1)^* = \{0\}$, $D_2 = X$. Note also that $S_1 (S_1)^* D_2 = S_1 D_1 (S_1)^*$ and hence, $D_2 = [S_1 (S_1)^*]^{-1} S_1 D_1 (S_1)^*$. The expression on the right in the last equation represents a bounded operator

even though $S_1(S_1)^*$ will not, in general, have a bounded inverse. Substitution from above yields

$$D_2 v = (C - [D_1])^{-1/2} D_1 (C - [D_1])^{1/2} v$$

for all $v \in H_2$.

A messy but rather easy use of induction completes the proof. We omit the details.

The formulas in Theorem 1.9 are much more manageable in the special case $\dim H_1 = \dim H_k$ for all $k \geq 1$. The operators E_k are unnecessary in this case, so that $S_k = P_k \geq 0$ for all $k \geq 1$. In this case, one obtains the formulas

$$P_k = \left(C - \sum_{i=1}^k [D_i] \right)^{1/2}$$

and

$$D_{k+1} = P_k^{-1} D_k P_k \quad \text{for all } k \geq 1.$$

The formulas in Theorem 1.9 are easy to handle only in special cases. An important observation, however, is that the structure of an abnormal operator T is determined by its action on the subspace $H_1(T)$. In case $H_1(T)$ is an infinite-dimensional subspace, then, in the absence of stronger hypotheses on $[T]$ and D_1 , nothing has been gained. If $\dim H_1(T) < \infty$, however, Theorem 1.9 asserts that the structure of (the abnormal part of) T is determined by two finite-dimensional operators.

§2. It is easily seen that the decomposition for operators given in Section 1 may be trivial. If T is normal, for instance, then $H = H_0(T)$. Even if T is abnormal, the decomposition will be trivial if $\ker [T] = \{0\}$, since $H = H_1(T)$ in this case. In this section we shall consider other conditions which imply that our decomposition is trivial.

Lemma 2.1: *Let $T \in \mathfrak{B}(H)$ and suppose that M is a subspace of H such that $TM \subseteq M$ and $M \subseteq \ker [T]$. Then $T|_M$ is hyponormal. If $T|_M$ is normal, then M reduces T .*

Proof: Write

$$T = \begin{bmatrix} A & B \\ 0 & C \end{bmatrix}$$

with respect to the decomposition $H = M \oplus M^\perp$. Then

$$[T] = \begin{bmatrix} [A] - BB^* & A^*B - BC^* \\ B^*A - CB^* & [C] + BB^* \end{bmatrix}.$$

The fact that $M \subseteq \ker [T]$ implies that M reduces $[T]$. Let $X = [T]|_{M^\perp}$. Then $[T]$ has the representation $[T] = 0 \oplus X$ relative to $H = M \oplus M^\perp$. Equating

corresponding entries in the two representations for $[T]$ yields $[A] = BB^* \geq 0$, so that $A = T|_M$ is hyponormal. If A is normal, then $BB^* = 0$. Thus $B = 0$ and M reduces T .

Theorem 2.2: *Let $T \in \mathfrak{B}(H)$ have compact real part. Then $H = H_0(T) \oplus H_1(T)$.*

Proof. We may as well assume that T is abnormal. Assume that $H \neq H_1(T)$ and let $A = T|(H_1(T))^\perp$. From Lemma 2.1, A is hyponormal. Since $\operatorname{Re} T$ is compact, so also is $\operatorname{Re} A$.

Putnam has shown ([6], p. 43) that if T is hyponormal and abnormal, then the measure of the spectrum of $\operatorname{Re} T$ is positive. Since a compact self-adjoint operator has countable spectrum, it follows that a hyponormal operator with compact real part is normal. In particular, A is normal. By Lemma 2.1 again, $[H_1(T)]^\perp$ reduces T and $T|(H_1(T))^\perp$ is normal, a contradiction, since T was assumed to be abnormal.

It is interesting to note what happens in case H is a finite-dimensional Hilbert space. Since every operator on a finite-dimensional space is compact, it follows from Theorem 2.2 that our decomposition is always trivial for finite-dimensional operators. Hence, non-trivial examples of our decomposition, much like non-unitary isometries, are purely infinite-dimensional phenomena.

As a consequence of Theorem 2.2 we obtain a simpler expression for the normal subspace of an operator having compact real part.

Corollary 2.3: *If $T \in \mathfrak{B}(H)$ has compact real part, then*

$$H_0(T) = \bigcap_{r=1}^{\infty} \ker (T^*T^r - T^rT^*).$$

Proof:

$$\begin{aligned} H_0(T) &= H \ominus H_1(T) \\ &= M_1(T) \\ &= \bigcap_{r=1}^{\infty} \ker (T^*T^r - T^rT^*). \end{aligned}$$

§3. Since our decomposition is trivial for operators on a finite-dimensional Hilbert space, we shall assume hereinafter that the underlying Hilbert space H is infinite-dimensional. Note that the easiest way to guarantee that the decomposition of an abnormal operator T is non-trivial is to assume that $\dim H_1(T) < \infty$. This ensures that $H \neq H_1(T)$, of course, but it also means that $H_k(T) \neq \{0\}$ for all $k \geq 1$, since, from Theorems 1.5 and 1.6, we have

$$\dim (H_1(T) \oplus \cdots \oplus H_k(T)) \leq k \dim H_1(T) < \infty.$$

The condition $\dim H_1(T) < \infty$ is difficult to verify in many cases. In this section we will show that this condition is easy to verify in case T is subnormal.

We recall that $T \in \mathfrak{B}(H)$ is subnormal if there exists a Hilbert space $K \supseteq H$ and a normal operator $N \in \mathfrak{B}(K)$ such that $NH \subseteq H$ and $T = N|_H$, in which case N is called a normal extension of T . We say that N is a minimal normal extension of T if the smallest subspace of K which contains H and reduces N is K itself. Halmos has shown that every subnormal operator has a minimal normal extension and that this extension is unique up to unitary equivalence. For proofs of these facts and an excellent discussion of subnormal operators, see [4], Chapter 16.

Lemma 3.1: *Let $T \in \mathfrak{B}(H)$ be subnormal. Then*

- i.) $\ker [T^r] \subseteq \ker ((T^*)^r T^s - T^s (T^*)^r)$ for $r, s > 0$.
- ii.) $\ker [T^r] = \bigcap_{s=1}^{\infty} \ker ((T^*)^r T^s - T^s (T^*)^r)$ for all $r > 0$.
- iii.) $\bigcap_{r=1}^{\infty} \ker [T^r] = H_0(T)$.

Proof: Let $N \in \mathfrak{B}(K)$, $K \supseteq H$, be the minimal normal extension of T . Since $NH \subseteq H$ and $T = N|_H$, we may write

$$N = \begin{bmatrix} T & X \\ 0 & Y \end{bmatrix}$$

with respect to the decomposition $K = H \oplus H^\perp$. Then

$$N^k = \begin{bmatrix} T^k & X_k \\ 0 & Y^k \end{bmatrix}$$

for all $k \geq 1$, where $X_1 = X$ and $X_{n+1} = TX_n + XY^n = T^n X_1 + X_n Y$ for all $n \geq 1$. Computing both $(N^*)^r N^s$ and $N^s (N^*)^r$ and equating corresponding entries yields

$$(1) \quad (T^*)^r T^s - T^s (T^*)^r = X_s X_r^*$$

and

$$(2) \quad (T^*)^r X_s = X_s (Y^*)^r$$

for all $r, s > 0$. Putting $r = s$ in (1) and recalling that $\ker T = \ker (T^* T)$ for $T \in \mathfrak{B}(H)$ yields $\ker [T^r] = \ker (X_r (X_r)^*) = \ker (X_r)^*$ for all $r \geq 1$. Thus, $\ker ((T^*)^r T^s - T^s (T^*)^r) = \ker (X_s (X_r)^*) \subseteq \ker (X_r)^* = \ker [T^r]$, and i.) holds.

From i.), the intersection, taken over $s \geq 1$, of the subspaces $\ker ((T^*)^r T^s - T^s (T^*)^r)$ contains $\ker [T^r]$ for $r \geq 1$. Since the reverse containment is trivial, the two sets are equal. Part iii.) is an immediate consequence of part ii.) and Lemma 1.3.

Recall that if $T \in \mathfrak{B}(H)$, then

$$M_k(T) = \bigcap_{r=1}^k \bigcap_{s=1}^{\infty} \ker ((T^*)^r T^s - T^s (T^*)^r).$$

If T is subnormal, then, applying Lemma 3.1, we obtain

$$M_k(T) = \bigcap_{r=1}^k \ker [T^r].$$

In particular, if $T \in \mathfrak{B}(H)$ is subnormal, then $H_1(T) = (M_1(T))^\perp = \text{cl}(\text{ran } [T])$. This shows that if T is subnormal and abnormal and if $H \neq \text{cl}(\text{ran } [T])$ (in particular, if $[T]$ has finite rank), then our decomposition for T will be non-trivial.

It follows from Theorem 1.5 and the remarks above that if T is subnormal, then $M_k(T)$ is invariant under T . Actually, a stronger statement is at hand. Taking adjoints in equation (2) in the proof of Lemma 3.1 and putting $r = 1$, we get $(X_s)^*T = Y(X_s)^*$ for all $s > 0$. Thus, $\ker [T^r] = \ker (X_r)^*$ is invariant under T for all $r \geq 1$. In case $r = 1$ this observation is due to Stampfli ([7]).

If T is subnormal and abnormal and if $H \neq \text{cl}(\text{ran } [T])$, or equivalently, if $\ker [T] \neq \{0\}$, then it follows from Theorem 1.9 that the structure of T is determined (up to unitary equivalence) by $[T]$ and $T^*[H_1(T)]$. In the special case in which $[T]$ is of finite rank, the structure of the abnormal part of T is determined by two matrices. In case T is abnormal and $[T]$ is of rank one, there are two constants which are a complete set of unitary invariants for T .

Proposition 3.2: *Let $T \in \mathfrak{B}(H)$ be subnormal with one-dimensional self-commutator. Let U denote the unilateral shift on ℓ^2 . Then there exist scalars s_1, d_1 ($s_1 > 0$) such that T is unitarily equivalent to the direct sum of a normal operator and $s_1U + d_1I$.*

Proof: We may as well assume that T is abnormal. We have $\dim(H_1(T)) = \dim(\text{cl}(\text{ran } [T])) = 1$. It follows from Theorem 1.5 that $\dim(H_k(T)) \leq 1$ for all $k \geq 1$. Since H is the direct sum of the spaces $H_k(T)$, $k \geq 1$, and since H is infinite-dimensional, we must have $\dim(H_k(T)) = 1$ for $k \geq 1$. From Theorem 1.9, T is unitarily equivalent to a matrix with scalars d_i on the main diagonal, positive scalars s_i on the first subdiagonal, and zeros in the other entries. Further, since $\bar{s}_k d_{k+1} = d_k \bar{s}_k$ and since $s_k > 0$, $k \geq 1$, we have $d_k = d_1$ for all $k \geq 1$. From Theorem 1.8 again we have $0 = [d_k] = s_{k-1} \bar{s}_{k-1} - \bar{s}_k s_k = |s_{k-1}|^2 - |s_k|^2$ for all $k \geq 2$. We then have $s_k = s_1$ for all $k \geq 1$ and we conclude that T is unitarily equivalent to $s_1U + d_1I$.

A careful examination of the proof of Proposition 3.2 shows that the result holds if we assume only that $T \in \mathfrak{B}(H)$ satisfies $\dim(H_1(T)) = 1$. Since one may conclude from this that T is subnormal, the apparent generalization is really an artificial one. Finally, we note that Proposition 3.2 has been obtained independently by K. Clancey ([2]).

Recall that $T \in \mathfrak{B}(H)$ is quasinormal if T commutes with T^*T , or, equivalently, if $T^*[T] = 0 = [T]T$. Thus, if T is quasinormal, then $T^*x = 0$ for every $x \in \text{cl}(\text{ran } [T])$. The following lemma was first proved by A. Brown in [1].

Lemma 3.3: *If $T \in \mathfrak{B}(H)$ is quasinormal, then T is subnormal.*

Proof: Write $x \in H$ as $x = x_1 + x_2$, where $x_1 \in \ker [T]$ and $x_2 \in \text{cl}(\text{ran } [T])$. Then

$$\begin{aligned} \langle [T]x, x \rangle &= \langle [T]x, x_1 \rangle + \langle [T]x, x_2 \rangle \\ &= \langle x, [T]x_1 \rangle + \langle x, [T]x_2 \rangle \\ &= \langle x_1, [T]x_2 \rangle + \langle x_2, [T]x_2 \rangle \\ &= \langle x_2, T^*Tx_2 \rangle - \langle x_2, TT^*x_2 \rangle \\ &= \|Tx_2\|^2 \geq 0. \end{aligned}$$

Thus, T is hyponormal. Note that $\text{cl}(\text{ran } [T]^{1/2}) = \text{cl}(\text{ran } [T])$, so that $T^*[T]^{1/2} = 0 = [T]^{1/2}T$. A direct computation shows that the operator X defined on $H \oplus H$ by

$$X = \begin{bmatrix} T & [T]^{1/2} \\ 0 & T^* \end{bmatrix}$$

is normal. Hence, T is subnormal.

Theorem 3.4: (A. Brown, [1]) Let $T \in \mathfrak{B}(H)$ be quasinormal. Put $R = \text{cl}(\text{ran } [T]^{1/2})$ and $C = [T]|_R$. Then T is unitarily equivalent to the direct sum of a normal operator with the operator defined on $R \oplus R \oplus \cdots$ by the matrix of operators $\{T_{i,j}\}$ with $T_{i+1,i} = C^{1/2}$ for $i \geq 1$, $T_{i,i} = 0$ if $i \neq j + 1$.

Proof: We may as well assume that T is both abnormal and in standard form. Since T is subnormal, we have $H_1 = R$, and since $T^*[T] = 0$, we have $T^*|_{H_1} = 0$, or, in the notation of section 1, $D_1 = 0$. Since $(S_i)^*D_{i+1} = D_i(S_i)^*$ and $\ker (S_i)^* = \{0\}$ for all $i \geq 1$, we have $D_i = 0$ for all $i \geq 1$.

We next observe that since $D_i = 0$ for $i \geq 1$, the inclusion $\ker S_i \subset \ker T$ holds for all $i \geq 1$. But T is both abnormal and hyponormal, and hence, $\ker T = \{0\}$. Thus $\ker S_i = \{0\}$ for all $i \geq 1$, and, since $H_{i+1} = \text{cl}(S_i H_i)$, we have $\dim H_i = \dim H_1$ for all $i \geq 2$. We have shown that T is a matrix of operators on $R \oplus R \oplus \cdots$ whose only non-zero entries are the non-negative operators $S_i = P_i$, $i \geq 1$, which appear on the first subdiagonal.

From part ii.) of Theorem 1.9, we have $0 = [D_i] = (S_{i-1})^2 - (S_i)^2$ for all $i \geq 1$, and hence, $S_i = S_1$ for all $i \geq 2$. From part i.) of Theorem 1.9, $0 = [D_1] = C - (S_1)^2$. Hence $S_i = C^{1/2}$ for all $i \geq 1$.

If $V \in \mathfrak{B}(H)$ is isometric, then $V^*V = I$ and V is quasinormal. Recall that if V is isometric and if P denotes the orthogonal projection on $(VH)^\perp$, then $VV^* = I - P$, so that $[V] = I - (I - P) = P$. This implies that $H_1(V) = \text{cl}(\text{ran } [V]) = (VH)^\perp$.

Corollary 3.5: (von Neumann-Halmos) Every isometry $V \in \mathfrak{B}(H)$ is unitarily equivalent to the direct sum of a unitary operator with a unilateral shift of multiplicity $\dim (VH)^\perp$.

Proof: A normal isometry is unitary, since $V^*V = I = VV^*$. Noting that $[V]H_1(V)$ is the identity operator on $(VH)^\perp$ and applying Theorem 3.4, we see that V is unitarily equivalent to the direct sum of a unitary operator and a matrix of operators on $R \oplus R \oplus \cdots$ with identity operators on the first sub-diagonal and zeros elsewhere, *i.e.*, a unilateral shift of multiplicity $\dim R = \dim (VH)^\perp$.

We note that if T is quasinormal, then a complete set of unitary invariants for $[T]$ is a complete set of unitary invariants for the abnormal part of T . In case V is an isometry, the fact that $[V] = I$ on $H_1(V) = \text{ran } [V] = (VH)^\perp$ means that the scalar $\dim (\text{ran } [V]) = \dim (VH)^\perp$ is a complete set of unitary invariants for the abnormal part of V .

§4. We conclude with a simple application of our decomposition theorems to the study of quasitriangular operators. Recall that $T \in \mathfrak{B}(H)$ is triangular if there exists an increasing sequence $\{E_k\}$ of projections of finite rank such that $\{E_k\} \rightarrow I$ strongly as $k \rightarrow \infty$ and such that $TE_k - E_kTE_k = 0$ for all k . We say that T is quasitriangular if there exists an increasing sequence $\{E_k\}$ of projections of finite rank such that $\{E_k\} \rightarrow I$ strongly as $k \rightarrow \infty$ and $\|TE_k - E_kTE_k\| \rightarrow 0$ as $k \rightarrow \infty$. It is clear that every triangular operator is quasitriangular. We note that the study of quasitriangular operators was initiated by Halmos ([5]). We shall use the facts, first proved in [5], that every normal operator is quasitriangular and that the direct sum of two quasitriangular operators is quasitriangular.

Proposition 4.1: *Let $T \in \mathfrak{B}(H)$ be abnormal with $\dim (H_1(T)) < \infty$. Then T^* is a triangular operator.*

Proof: Let E_k denote the orthogonal projection of H onto $[M_k(T)]^\perp$ for all $k \geq 1$. From Theorem 1.5 the sequence $\{E_k\}$ is an increasing sequence of projections of finite rank, and since $TM_k(T) \subseteq M_k(T)$ for all $k \geq 1$, we also have $T^*E_k - E_kT^*E_k = 0$ for all $k \geq 1$. The abnormality of T implies that the intersection of the subspaces $M_k(T)$, $k \geq 1$, is the zero subspace, or, equivalently, that E_k tends strongly to I as k tends to infinity.

The preceding proposition, together with Lemma 1.3 and Halmos' results, yield the following:

Corollary 4.2: *Let $T \in \mathfrak{B}(H)$ satisfy $\dim (H_1(T)) < \infty$. Then T^* is quasitriangular.*

Corollary 4.3: *Suppose that $T \in \mathfrak{B}(H)$ is subnormal and that its self-commutator has finite rank. Then T^* is quasitriangular.*

The answer to the following question is apparently unknown.

Question: If T is subnormal and has compact (or, trace-class) self-commutator, is T^* quasitriangular?

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