Direct and Inverse Theorems for Multidimensional Spline Approximation

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1. Introduction. In two recent papers Scherer [9, 10] has exploited a general approximation theoretic result of Butzer and Scherer [2, 3] to obtain direct and inverse theorems as well as saturation results for spline approximation of continuous and of L_p functions.

The purpose of this paper is to apply the Butzer–Scherer result to obtain direct and inverse theorems as well as saturation theorems for multidimensional spline approximation of functions in certain Sobolev spaces. In order to employ the Butzer–Scherer Theorem, a Jackson-type estimate and a Bernstein-type inequality are obtained. For convenience, we restrict the discussion to the case of two dimensions; the extension to higher dimensions is straightforward.

2. Multidimensional Jackson-type estimates. In this section we give an estimate for how well functions in the Sobolev spaces $W^{k,p}(H)$, $1 \leq p \leq \infty$, can be approximated by splines, where H is the rectangle $H = [a, b] \times [c, d]$.

Let $m_1, n_1 \geq 0$ and $\Delta_1 = \{a = x_0 < x_1 < \cdots < x_{n_1+1} = b\}$. We define $S_{m_1, \Delta_1} = \{f \in H_p^{m_1}[a, b] : f^{(m_1+1)}(x) = 0 \text{ for } x \in (x_i, x_{i+1}), i = 0, 1, \dots, n_1\}$, where for $m_1 > 0$

$$H_p^{m_1}[a, b] = \{f \in C^{m_1-1}[a, b] : f^{(m_1-1)} \text{ is absolutely continuous, } f^{(m_1)} \in L_p[a, b]\},$$

and

$$H_p^0[a, b] = L_p[a, b].$$

For $m_2, n_2 \geq 0$ and $\Delta_2 = \{c = y_0 < y_1 < \cdots < y_{n_2+1} = d\}$, S_{m_2, Δ_2} is defined similarly. We are interested in the class of tensor product splines

$$S_{m_1, \Delta_1} \otimes S_{m_2, \Delta_2} = \{s(x, y) = s_1(x)s_2(y) : s_1 \in S_{m_1, \Delta_1}, s_2 \in S_{m_2, \Delta_2}\}$$

defined on H . We note that

$$S_{m_1, \Delta_1} \otimes S_{m_2, \Delta_2} \subset W^{m, p}(H).$$

Let $\bar{\Delta}_1 = \max_{0 \leq i \leq n_1} (x_{i+1} - x_i)$, $\bar{\Delta}_2 = \max_{0 \leq i \leq n_2} (y_{i+1} - y_i)$, $\underline{\Delta}_1 = \min_{0 \leq i \leq n_1} (x_{i+1} - x_i)$, $\underline{\Delta}_2 = \min_{0 \leq i \leq n_2} (y_{i+1} - y_i)$, $\bar{\Delta} = \max(\bar{\Delta}_1, \bar{\Delta}_2)$, and $\underline{\Delta} = \min(\underline{\Delta}_1, \underline{\Delta}_2)$. Set $m = \min(m_1, m_2)$. The following theorem is contained in de Boor and Fix [4].

Theorem 2.1. Suppose $1 \leq p \leq \infty$ and $f \in W^{k, p}(H)$, $1 \leq k \leq m + 1$. Then

$$(2.1) \quad d_p(f) = \inf_{g \in S_{m_1, \Delta_1} \otimes S_{m_2, \Delta_2}} \|f - g\|_p \leq C(\bar{\Delta})^k \|f\|_{W^{k, p}(H)},$$

where C is a constant independent of f and $\bar{\Delta}$. (Throughout this paper C always denotes a constant; we permit it to change its value from paragraph to paragraph.)

3. A Bernstein-type estimate.

Theorem 3.1. Let $0 \leq m$, $1 \leq k \leq m + 1$ and $1 \leq p \leq \infty$. If $s \in S_{m_1, \Delta_1} \otimes S_{m_2, \Delta_2}$ then

$$(3.1) \quad \|s\|_{W^{k, p}(H)} \leq C \underline{\Delta}^{-k} \|s\|_{L_p(H)}.$$

Proof: We consider first the case of $k = 1$. On each rectangle $[x_i, x_{i+1}] \times [y_i, y_{i+1}]$ the spline s is such that for each fixed y it is a polynomial of degree m_1 in x . Applying the Markoff inequality (see [10], [11]) we have for each y

$$\int_{x_i}^{x_{i+1}} |s^{(1,0)}(x, y)|^p dx \leq C \underline{\Delta}_1^{-p} \int_{x_i}^{x_{i+1}} |s(x, y)|^p dx.$$

Integrating with respect to y over $[y_i, y_{i+1}]$ and summing over (i, j) yields

$$(3.2) \quad \|s^{(1,0)}\|_{L_p(H)} \leq C \underline{\Delta}_1^{-1} \|s\|_{L_p(H)}.$$

Similarly, (3.2) holds for $s^{(0,1)}$ with $\underline{\Delta}_1$ replaced by $\underline{\Delta}_2$, and (3.1) follows for $k = 1$ and any $m \geq 0$.

Suppose now that $\tilde{m} \geq 1$ and $2 \leq \tilde{k} \leq m + 1$. We prove the result for $(\tilde{m}, \tilde{k})$ under the assumption that it holds for $\tilde{m} - 1$ and all $1 \leq k \leq \tilde{m}$ as well as for $\tilde{m}$ and all $1 \leq k \leq \tilde{k} - 1$. Since the $m \geq 0$, $k = 1$ cases were verified above, this will complete the proof of the theorem by induction.

Given $s \in S_{m_1, \Delta_1} \otimes S_{m_2, \Delta_2}$ with $\tilde{m} = \min(m_1, m_2)$, the functions $s^{(0,1)}$ and $s^{(1,0)}$ are splines in the classes $S_{m_1, \Delta_1} \otimes S_{m_2-1, \Delta_2}$ and $S_{m_1-1, \Delta_1} \otimes S_{m_2, \Delta_2}$ respectively. Then the minimum degrees of the splines $s^{(0,1)}$ and $s^{(1,0)}$ are either $\tilde{m}$ or $\tilde{m} - 1$. Thus if $0 \leq q \leq \tilde{k}$,

$$\begin{aligned} \|s^{(q, \tilde{k}-q)}\|_{L_p(H)} &\leq \|s^{(1,0)}\|_{W^{\tilde{k}-1, p}(H)} + \|s^{(0,1)}\|_{W^{\tilde{k}-1, p}(H)} \\ &\leq C \underline{\Delta}^{1-\tilde{k}} (\|s^{(1,0)}\|_{L_p(H)} + \|s^{(0,1)}\|_{L_p(H)}) \\ &\leq 2C \underline{\Delta}^{-\tilde{k}} \|s\|_{L_p(H)}. \end{aligned}$$

The proof for $p = \infty$ is similar except that the Markoff inequality needed in this case is *e.g.*

$$\sup_{x_i \leq x \leq x_{i+1}} |s^{(1,0)}(x, y)| \leq C \Delta_1^{-1} \sup_{x_i \leq x \leq x_{i+1}} |s(x, y)|$$

(see [9, 11]).

4. The Butzer-Scherer Theorem. Our main direct and inverse theorems follow from a result of Butzer-Scherer [2, 3], as extended by Scherer [9, 10]. For convenience we state it without proof.

Let X be a Banach space, and for $0 < t < \infty$ let P_t be a family of linear subspaces of X such that for some $t_0 > 0$

$$(4.1) \quad P_t = P_{t_0}, \quad t \geq t_0,$$

$$(4.2) \quad P_{2t} \subset P_t, \quad \text{all } t > 0.$$

Suppose that for each $0 < t < \infty$ and each $f \in X$ there exists $p_t^* \in P_t$ with

$$(4.3) \quad E_t(f) = \inf_{p \in P_t} \|f - p\|_X = \|f - p_t^*\|_X$$

and that

$$(4.4) \quad \text{for each } f \in X, \quad \lim_{t \rightarrow 0+} E_t(f) = 0.$$

Now suppose Y is a Banach space continuously imbedded in X and with $P_t \subset Y$, $0 < t < \infty$. Suppose for some constant $\sigma > 0$

$$(4.5) \quad \text{for each } f \in Y, \quad E_t(f) \leq Ct^\sigma \|f\|_Y$$

and

$$(4.6) \quad \text{for each } p \in P_t, \quad \|p\|_Y \leq Ct^{-\sigma} \|p\|_X.$$

We remark that (4.4-6) are generalized forms of the classical Weierstrass, Jackson, and Bernstein theorems, respectively.

Theorem 4.1. *Let the above hypotheses hold. Then for every $f \in X$ and any $\theta > 0$, as $t \rightarrow 0+$*

$$(4.7) \quad E_t(f) = O(t^\theta)$$

is equivalent to the following:

If Y is such that $\sigma < \theta$, then

$$(4.8) \quad f \in Y \quad \text{and} \quad \|f - p_t^*(f)\|_Y = O(t^{\theta-\sigma});$$

if Y is such that $\theta < \sigma$, then

$$(4.9) \quad \|p_t^*(f)\|_Y = O(t^{\theta-\sigma});$$

if Y is such that $\theta < \sigma$, then

$$(4.10) \quad K(t^\sigma, f; X; Y) = \inf_{g \in Y} (\|f - g\|_X + t^\sigma \|g\|_Y) = O(t^\theta).$$

The K -functional in (4.10) is related to the K -method of Peetre for generating an intermediate Banach space between Y and X ; see *e.g.* [1, p. 166]. It is a monotone increasing, continuous function of t .

5. Direct and Inverse Theorems. In this section we apply Theorem 4.1 to certain classes of tensor product splines on $H = [a, b] \times [c, d]$. Let $m_1, m_2 \geq 0$ be integers and let $m = \min(m_1, m_2)$. Suppose $0 < t_0 < \infty$ and that for $t \in [0, t_0]$ there exist integer-valued functions $p(t)$, $q(t)$ and vector-valued functions

$$(5.1) \quad \begin{aligned} \xi_1(t) &= (x_0(t), x_1(t), \dots, x_{p(t)}(t)) \\ \xi_2(t) &= (y_0(t), y_1(t), \dots, y_{q(t)}(t)) \end{aligned}$$

with the properties

$$(5.2) \quad \begin{aligned} a &= x_0(t) < \dots < x_{p(t)}(t) = b \\ c &= y_0(t) < \dots < y_{q(t)}(t) = d. \end{aligned}$$

Suppose further that there exist positive constants α, β with

$$(5.3) \quad \alpha t \leq \Delta_i(t) \leq \bar{\Delta}_i(t) \leq \beta t, \quad i = 1, 2$$

where

$$\begin{aligned} \Delta_1(t) &= \min (x_{i+1} - x_i), & \Delta_2(t) &= \min (y_{i+1} - y_i) \\ \bar{\Delta}_1(t) &= \max (x_{i+1} - x_i), & \bar{\Delta}_2(t) &= \max (y_{i+1} - y_i). \end{aligned}$$

Finally, setting $\Delta_1(t) = \{x_0(t), \dots, x_{p(t)}(t)\}$ and $\Delta_2(t) = \{y_0(t), \dots, y_{q(t)}(t)\}$, assume that for $t \in (0, t_0]$,

$$(5.4) \quad \Delta_i(2t) \subset \Delta_i(t), \quad i = 1, 2.$$

We are interested in the class of splines

$$(5.5) \quad S_t = S(m_1, m_2, \xi_1, \xi_2, t) = S_{m_1, \Delta_1(t)} \otimes S_{m_2, \Delta_2(t)}.$$

For $f \in L_p(H)$, $1 \leq p < \infty$, or $f \in C(H)$ if $p = \infty$, define

$$(5.6) \quad E_t(f) = \inf_{s \in S_t} \|f - s\|_p.$$

Theorem 5.1. Suppose $f \in L_p(H)$, $1 \leq p < \infty$, and that $0 < \theta$ and $0 < k \leq m$. Then as $t \rightarrow 0+$

$$(5.7) \quad E_t(f) = O(t^\theta)$$

is equivalent to the following:

If $\theta > k$

$$(5.8) \quad f \in W^{k,p}(H) \text{ and } \|f - s^*(f)\|_{W^{k,p}} = O(t^{\theta-k}),$$

where $s^*(f)$ is the best approximation of f in S_t with respect to the L_p norm;

If $\theta < k$

$$(5.9) \quad \|s^*(f)\|_{W^{k,p}} = O(t^{\theta-k});$$

If $\theta < k$

$$(5.10) \quad K(t^k; f, L_p(H), W^{k,p}(H)) = O(t^\theta),$$

where K is defined in (4.10). If $f \in C(H)$, the above equivalences hold for $p = \infty$, with $L_p(H)$ replaced by $C(H)$ in (5.10).

Proof. Consider first the case $1 \leq p < \infty$. We apply Theorem 4.1 with $X = L_p(H)$, $Y = W^{k,p}(H)$ and $\sigma = k$. Clearly Y is continuously imbedded in X . For $0 < t \leq t_0$ let $P_t = S_t$ and for $t \geq t_0$ let $P_t \equiv S_{t_0}$. Since $k \leq m$, the class of splines P_t is a subspace of $W^{k,p}(H)$ for all t . Since $\Delta_i(2t) < \Delta_i(t)$, $i = 1, 2$, the relation (4.2) holds. Since S_t is a finite-dimensional subspace of $L_p(H)$, (4.3) also holds. The Weierstrass convergence (4.4) is a consequence of Theorem 5.1 of [7]. Finally, the Jackson and Bernstein estimates (4.5) and (4.6) are just Theorems 2.1 and 3.1, respectively.

The proof of $p = \infty$ is identical if we set $X = C(H)$ and use Theorem 4.2 of [7].

It follows from Theorem 5.1 that for all choices of meshes $\Delta_1(t)$ and $\Delta_2(t)$ satisfying properties (5.1–4), the corresponding spline classes give the same asymptotic order of approximation to a prescribed f in $L_p(H)$ or $C(H)$. More precisely, we have

Corollary 5.2. Suppose $0 < t_0 < \infty$ and that for $t \in (0, t_0]$, $p(t)$, $q(t)$, $x(t)$, $y(t)$, $\Delta_1(t)$, $\Delta_2(t)$ are as above and that (5.1–4) are satisfied. Suppose $p^*(t)$, $q^*(t)$, $x^*(t)$, $y^*(t)$, $\Delta_1^*(t)$, $\Delta_2^*(t)$ are another set of functions also satisfying (5.1–4), with a corresponding class of splines S_t^* defined as in (5.5). Suppose $f \in L_p(H)$, $1 \leq p < \infty$, or $f \in C(H)$ if $p = \infty$. Then

$$(5.11) \quad E_t(f) = O(t^\theta) \quad \text{if and only if}$$

$$(5.12) \quad E_t^*(f) = O(t^\theta)$$

as $t \rightarrow 0+$ for all $0 < \theta < m$.

Proof. For $0 < \theta < k = m$, both (5.11) and (5.12) are equivalent to (5.10).

The assertion (5.10) of Theorem 5.1 can be translated in terms of the behavior of the k -th modulus of continuity in $L_p(H)$ or $C(H)$. Given f defined on H , we define for $1 \leq p \leq \infty$

$$\omega_k(t, f, p) = \sup_{0 \leq |h_1|, |h_2| \leq t} \|\Delta_{h_1, h_2}^k f\|_p,$$

where for $(h_1, h_2) \in E^2$,

$$\Delta_{h_1, h_2}^k f(x, y) = \sum_{l=0}^k (-1)^{k-l} \binom{k}{l} f(x + lh_1, y + lh_2),$$

and where the p -norm in the above is taken over all points $(x, y) \in H$ such that $(x + lh_1, y + lh_2)$ is also in H (cf. [11, p. 113 or 1, p. 257]).

We need a lemma relating ω_k to the K -functional.

Lemma 5.3. *Let l be any positive integer and $1 \leq p < \infty$. Then there exists a constant $C = C_{l,p}$ such that for any $f \in L_p(H)$ and sufficiently small t ,*

$$(5.13) \quad \frac{1}{C} K(t^l, f, L_p(H), W^{l,p}(H)) \leq \min(1, t^l) \|f\|_p + \omega_l(t, f, p) \\ \leq CK(t^l, f, L_p(H), W^{l,p}(H)).$$

Proof. With H replaced by E^2 the inequality (5.13) was proved in [1, p. 258]. The result for a rectangle H does not follow trivially from the E^2 case by extending the function to be zero outside. (For the analogous problem in 1-dimension, see [10]). This difficulty was overcome in [10] by considering a partition of unity $1 = z_1(x) + z_2(x)$ where $z_1(x), z_2(x)$ are nonnegative, belong to $C^\infty[a, b]$ and have supports on $[a, b - \delta]$ and $[a + \delta, b]$ —then extensions of f could be used. To prove (5.13) for H we may follow [10]. Taking a similar partition $z_3(y), z_4(y)$ of $[c, d]$, we may consider the product $1 = (z_1(x) + z_2(x)) \cdot (z_3(y) + z_4(y)) = \sum_{i=1}^4 \phi_i(x, y)$. Using this partition the ideas of [10] carry over directly.

Corollary 5.4. *Let $0 < \theta < k \leq m$ and suppose $f \in L_p(H)$, $1 \leq p < \infty$. Then as $t \rightarrow 0+$, (5.7) is equivalent to*

$$(5.14) \quad \omega_k(t, f, p) = O(t^\theta).$$

Proof. We need only show the equivalence of (5.10) and (5.14). This is an immediate consequence of Lemma 5.3.

The assertion of Corollary 5.4 that (5.14) implies (5.7) is also true for $k = m + 1$ by the following lemma.

Lemma 5.5. *Suppose $f \in L_p(H)$, $1 \leq p < \infty$, or $f \in C(H)$ and $p = \infty$. Then for $t > 0$*

$$E_t(f) \leq C[t^{m+1} \|f\|_p + \omega_{m+1}(t, f, p)].$$

Proof. Suppose $1 \leq p < \infty$. Let $g \in W^{m+1,p}(H)$. By Theorem 2.1,

$$E_t(f) \leq E_t(f - g) + E_t(g) \leq \|f - g\|_p + C(\beta t)^{m+1} \|g\|_{W^{m+1,p}} \\ \leq \max(1, (\beta)^{m+1} C) K(t^{m+1}, f, L_p, W^{m+1,p})$$

By (5.13) we have the desired inequality.

The converse, that (5.7) implies (5.14) for $k = m + 1$ can be proved with certain further restrictions on the character of the meshes on which S_t are defined (see Theorem 6.2).

6. Saturation. Before giving a saturation result (Theorem 6.3), we must first introduce some special meshes for H .

Let

$$\begin{aligned}
 (6.1) \quad x_i^* &= a + it, & i &= 0, 1, \dots, p^*, & x_{p^*+1}^* &= b \\
 \bar{x}_i &= a - (t/2) + it, & i &= 1, 2, \dots, p^*, & \bar{x}_{p^*+1} &= b, \quad \bar{x}_0 = a \\
 y_j^* &= c + jt, & j &= 0, 1, \dots, q^*, & y_{q^*+1}^* &= d \\
 \bar{y}_j &= c - (t/2) + jt, & j &= 1, 2, \dots, q^*, & \bar{y}_{q^*+1} &= d, \quad \bar{y}_0 = c,
 \end{aligned}$$

where $p^*(t)$ and $q^*(t)$ are chosen such that

$$\begin{aligned}
 \frac{t}{2} &\leq b - x_{p^*}^* < \frac{3t}{2} \\
 \frac{t}{2} &\leq d - y_{q^*}^* < \frac{3t}{2}.
 \end{aligned}$$

Then

$$\begin{aligned}
 t &\leq b - \bar{x}_{p^*} < 2t \\
 t &\leq d - \bar{y}_{q^*} < 2t,
 \end{aligned}$$

and the meshes $\Delta_1^*(t) = \{x_i^*\}_{i=0}^{p^*+1}$ and $\Delta_2^*(t) = \{y_j^*\}_{j=0}^{q^*+1}$ satisfy (5.2)–5.4 with $\alpha = \frac{1}{2}$ and $\beta = 2$. These meshes can be used to define four classes of splines:

$$\begin{aligned}
 S_t^1 &= S_{m_1, \Delta_1^*} \otimes S_{m_2, \Delta_2^*}, \\
 S_t^2 &= S_{m_1, \Delta_1^*} \otimes S_{m_2, \bar{\Delta}_2}, \\
 S_t^3 &= S_{m_1, \bar{\Delta}_1} \otimes S_{m_2, \Delta_2^*}, \\
 S_t^4 &= S_{m_1, \bar{\Delta}_1} \otimes S_{m_2, \bar{\Delta}_2}.
 \end{aligned}$$

Define

$$E_i^i(f) = \inf_{s \in S_{t^i}} \|f - s\|_p, \quad i = 1, \dots, 4.$$

Lemma 6.1. Suppose $f \in L_p(H)$, $1 \leq p < \infty$, or $f \in C(H)$ and $p = \infty$. Then for $t > 0$.

$$(6.2) \quad \omega_{m+1}(t, f, p) \leq C \sup_{0 < u \leq t} \max_{1 \leq i \leq 4} E_u^i(f).$$

Proof. First consider $1 \leq p < \infty$. Let $(h_1, h_2) \in E^2$ and set $h = \max(h_1, h_2)$. Suppose $w = (m+1)h < \min(b-a, d-c)$, and consider a subdivision of H into subrectangles defined by the points

$$\{a + iw\}^{N_1}_{i=0} \cup \{b\} \quad \text{and} \quad \{c + jw\}^{N_2}_{j=0} \cup \{d\},$$

where $N_1 = [(b-a)/w]$, $N_2 = [(d-c)/w]$. Suppose these subrectangles are numbered $\{R_{ij}\}_{i=0, j=0}^{N_1, N_2}$. Now, for all $(x, y) \in R_{i,j}$, if both of the points (x, y) and $(x + (m+1)h_1, y + (m+1)h_2)$ lie in H , they will lie in a rectangle $Q(i, j, h_1, h_2)$ consisting of the union of R_{ij} with three adjacent $R_{r,\mu}$'s.

For any such Q , one of the four classes of splines S_{2w}^i , $1 \leq i \leq 4$, will have no knots in the interior of Q . Suppose that $\sigma = \sigma(i, j, h_1, h_2)$ is such that S_{2w}^σ is the corresponding class. Then for any $s \in S_{2w}^\sigma$,

$$(6.3) \quad \Delta_{h_1, h_2}^{m+1} s(x, y) = 0$$

for all $(x, y) \in R_{ij}$. But then

$$\iint_{R_{ij}} |\Delta_{h_1, h_2}^{m+1} f(x, y)|^p dx dy \leq \sum_{l=1}^4 \iint_{R_{ij}} |\Delta_{h_1, h_2}^{m+1} (f - s^l(f))(x, y)|^p dx dy,$$

where $s^l(f)$ is the best approximation of f in S_{2w}^l . Summing over i, j yields

$$\|\Delta_{h_1, h_2}^{m+1} f\|_p^p \leq 2^{p(m+1)} \sum_{i=1}^4 (E_{2w}^i)^p \leq 4 \cdot 2^{(m+1)p} (\max_{1 \leq i \leq 4} E_{2w}^i)^p,$$

which can be rewritten

$$\|\Delta_{h_1, h_2}^{m+1} f\|_p \leq C \max_{1 \leq i \leq 4} E_{2h(m+1)}^i.$$

Taking the supremum over $h \leq t/2(m+1)$ on the right and then on the left yields

$$\omega_{m+1}\left(\frac{t}{2(m+1)}, f, p\right) \leq C \sup_{0 < u \leq t} \max_{1 \leq i \leq 4} E_u^i(f).$$

Using the fact that for any integer μ , $\omega_{m+1}(\mu\tau) \leq \mu^{m+1} \omega_{m+1}(\tau)$, we get (6.2) for $1 \leq p < \infty$.

The proof of (6.2) for $p = \infty$ is quite similar. From (6.3) we have

$$|\Delta_{h_1, h_2}^{m+1} f(x, y)| \leq \sum_{l=1}^4 |\Delta_{h_1, h_2}^{m+1} (f - s^l(f))(x, y)|$$

for all $(x, y) \in R_{ij}$. Taking the supremum as (x, y) varies over H yields $\omega_{m+1}(t, f, \infty) \leq 4 \max_{1 \leq i \leq 4} 2^{m+1} E_{2(m+1)h}^i(f)$. As before, taking the supremum over $h \leq t/2(m+1)$ first on the right, then on the left, and using $\omega_{m+1}(\mu\tau) \leq \mu^{m+1} \omega_{m+1}(\tau)$ yields (6.2) for $p = \infty$.

Combining Lemmas 5.5 and 6.1 we obtain

Theorem 6.2. *Let $0 < \theta \leq m+1$, and suppose $f \in L^p(H)$, $1 \leq p < \infty$, or $f \in C(H)$ and $p = \infty$. Then as $t \rightarrow 0+$, $\max_{1 \leq i \leq 4} E_t^i(f) = O(t^\theta)$ is equivalent to $\omega_{m+1}(f, t, p) = O(t^\theta)$.*

We also have the following saturation result.

Theorem 6.3. *Let $f \in L_p(H)$, $2/m < p < \infty$, or $f \in C(H)$ and $p = \infty$. If*

$$(6.4) \quad \max_{1 \leq i \leq 4} E_t^i(f) = O(t^{m+1}),$$

then for all $(x, y) \in H$, f is a polynomial of degree m in x for all y and a polynomial of degree m in y for all x .

Proof. First, note that (6.4) implies $E^1(f) = 0(t^{m+1})$, so by Theorem 5.1, $f \in W^{m,p}(H)$. By the Sobolev imbedding theorem, $f \in C(H)$ as $mp > 2$. By Lemma 6.1, (6.4) implies $\omega_{m+1}(t, f, p) = o(t^{m+1})$, and by the properties of modulus of smoothness (cf. [11, p. 104], this can hold iff $\Delta_{h_1, h_2}^{m+1} f(x, y) = 0$ for all $(x, y) \in H$ and $(h_1, h_2) \in E^2$. In particular, choosing $h_2 = 0$ implies the $(m+1)$ st difference of f in the x -direction is zero and hence f is a polynomial of degree m in x . Similarly, $h_1 = 0$ leads to the other assertion.

7. Extensions to non-rectangular regions. Suppose Ω is a bounded open set in E^2 . Then certainly there is a closed rectangle H with $\bar{\Omega} \subset \text{interior } H$. With appropriate restrictions on Ω , Theorem 2.1 is also valid for $f \in W^{k,p}(\Omega)$ where the norms $L_p(H)$ and $W^{k,p}(H)$ are replaced by $L_p(\Omega)$ and $W^{k,p}(\Omega)$ (see [4]). This is so, for example, if Ω is regular [6], and in particular if it is convex.

The Bernstein estimate of Theorem 3.1 cannot be extended so easily to non-rectangular Ω . Let H and Δ_1, Δ_2 be as in sections 2 and 3. Suppose each of the rectangles $R_{ij} = [x_i, x_{i+1}] \times [y_j, y_{j+1}]$, $i = 0, 1, \dots, n_1$; $j = 0, 1, \dots, n_2$ is subdivided into a further finite collection of rectangles, $R_{ij} = \bigcup_{\nu, \mu} R_{ij\nu\mu}$. Then the narrowest $R_{ij\nu\mu}$ has width proportional to Δ_1 and the shortest $R_{ij\nu\mu}$ has height proportional to Δ_2 . With only minor modifications of the proof of Theorem 3.1 we obtain the following extension.

Theorem 7.1. *Let Ω be the union of some subcollection of the $R_{ij\nu\mu}$ making up H . Let $1 \leq k \leq m$ and $1 \leq p \leq \infty$. If $s \in S_{m_1, \Delta_1} \otimes S_{m_2, \Delta_2}$, then*

$$\|s\|_{W^{k,p}(\Omega)} \leq C \Delta^{-k} \|s\|_{L_p(\Omega)}.$$

The difficulty with the Bernstein estimate makes extension of the results of section 5 problematical. The trouble is that we want to deal with a *fixed* Ω while for asymptotic results the meshes are changing with t . It would not seem unreasonable to expect most of these results to be valid for rather general non-rectangular regions Ω , however.

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Theorem 2.1 as stated by de Boor and Fix requires $p > z$. The result as stated here holds, however, as shown in a forthcoming paper on B -spline approximation methods by Lyche and Schumaker.

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