On the Essential Spectra

CHARLES F. AMELIN

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In [2], Bouldin asks the question: "Under what conditions does the Fredholm spectrum of a closed densely defined (linear) operator T on a Hilbert space $(\sigma_F(T))$ equal the Browder spectrum $(\sigma_B(T))$?" (We recall that $\sigma_F(T) = \{\lambda : T - \lambda \text{ is not Fredholm}\}$; $\sigma_B(T) = \{\lambda \in \sigma(T) : \lambda \text{ is not an isolated point of } \sigma(T) \text{ with } T - \lambda \text{ Fredholm}\}$). The answer given:

"Theorem 3. *If T is a closed densely defined (linear) operator which is reduced by each finite dimensional subspace $\ker(T - z)$ for any complex number z , then $\sigma_F(T) = \sigma_B(T) \dots$ "*

is seen to be inaccurate (as noticed by D. Lay and others also) by consideration of the unilateral shift. Recall now that $\omega(T) = \{\lambda : T - \lambda \text{ is not Fredholm; or, if } T - \lambda \text{ is Fredholm, then } \text{ind}(T - \lambda) \neq 0\}$. Clearly, $\sigma_F(T) \subset \omega(T) \subset \sigma_B(T)$. Let $\rho_F(T) = C - \sigma_F(T)$. The minimum modulus of T , $\gamma(T) = \inf \{ \|Tx\|/d(x, N(T)), x \in D(T), x \notin N(T) \}$. Let $\delta(\lambda, \lambda_0) = \sup \{ d(x_\lambda, N(T - \lambda_0)) : x_\lambda \in N(T - \lambda), \|x_\lambda\| = 1 \}$. T satisfies $K(\lambda_0)$ (the eigenspace gap condition; cf. Bouldin [1] and Gustafson [5]) if there exists a sequence $\lambda_n \rightarrow \lambda_0$ such that either (a) $\delta(\lambda_n, \lambda_0) \neq 0$ and $|\lambda_0 - \lambda_n|/\delta(\lambda_n, \lambda_0) \rightarrow 0$ or (b) $\text{nul}(T - \lambda_n) = 0$ and $\text{def}(T - \lambda_n) = 0$. The isolated (w.r.t. $\sigma(T)$) eigenvalues λ_0 of finite multiplicity (i.e., $\text{nul}(T - \lambda_0) < \infty$) are denoted by $\pi_{00}(T)$.

Gustafson's paper [5] contains five necessary and sufficient conditions for Weyl's theorem $(\sigma(T) - \pi_{00}(T) = \omega(T))$ to hold for a densely defined closed linear operator on a Banach space. In view of that article, the following theorem should be clear:

Theorem 1. *Let T be a closed, densely defined linear operator on a Banach space X . Then the following are equivalent:*

1. $\sigma_F(T) = \omega(T) = \sigma_B(T)$.
2. $\sigma(T) \cap \rho_F(T) \subset \text{bdy } \sigma(T)$.
3. $N(T - \lambda) - \bigcap_{n=1}^{\infty} R(T - \lambda)^n \neq \emptyset$ for every $\lambda \in \sigma(T) \cap \rho_F(T)$.
4. $T - \lambda$ has finite ascent and finite descent for every $\lambda \in \sigma(T) \cap \rho_F(T)$.
5. $\gamma(T - \lambda)$ is discontinuous for every $\lambda \in \sigma(T) \cap \rho_F(T)$.
6. $K(\lambda)$ holds for every $\lambda \in \sigma(T) \cap \rho_F(T)$.

Proof: Suppose that 1 is true; i.e., every $\lambda_0 \in \sigma(T) \cap \rho_F(T)$ is an isolated point of the spectrum. Then 2 holds since every isolated point λ_0 of the spectrum is a boundary point of the spectrum; 3 holds by lemma 8.1 of Gohberg and Krein [3]; 4 holds by Theorem 2.9 of Lay [7]; 5 holds because $\gamma(\lambda) = \|(\lambda - T)^{-1}\|^{-1} \rightarrow 0$ as $\lambda \rightarrow \lambda_0$ but $\gamma(\lambda_0) > 0$; 6 holds because b is true. On the other hand if 2 holds, then every $\lambda_0 \in \sigma(T) \cap \rho_F(T)$ is an isolated point of the spectrum by the stability of $\text{nul}(T - \lambda)$ for all λ in a sufficiently small deleted nbhd. of λ_0 . If 3 holds then by lemma 8.1 of Gohberg–Krein [3], $\sigma(T) \cap \rho_F(T)$ cannot contain any open set for which $\text{nul}(T - \lambda) > 0$; i.e., $\sigma(T) \cap \rho_F(T)$ consists only of isolated points. If 4 is true then by Theorem 2.1 of Lay [6], $\rho_F(T) \cap \sigma(T)$ consists only of isolated points of $\sigma(T)$. If 5 holds, then (by Oliver [8]) since $\gamma(T - \lambda) > 0$ for all λ close to λ_0 , $\text{nul}(T - \lambda) < \text{nul}(T - \lambda_0)$. Since this is true for every $\lambda_0 \in \rho_F(T) \cap \sigma(T)$, each such λ_0 is an isolated point of the spectrum. If 6 holds, then $\gamma(\lambda)$ is discontinuous at λ_0 or λ_0 is an isolated point of $\sigma(T)$.

For this paragraph and the following two corollaries we let T be a closed, densely defined linear operator on a Hilbert space. We note that $N(T - \lambda)$ reduces T iff $N(T - \lambda) \subset N(T - \lambda)^* = R(T - \lambda)^\perp$ which implies that $N(T - \lambda) \ominus R(T - \lambda) = N(T - \lambda)$. In view of theorem 1, (3), we have:

Corollary 1. *If for each $\lambda_0 \in \sigma(T) \cap \rho_F(T)$, $N(T - \lambda_0) \neq \{0\}$ and T is reduced by $N(T - \lambda_0)$, then $\sigma_F(T) = \sigma_B(T)$.*

A dual result can be formulated:

Corollary 2. *If for each $\lambda_0 \in \sigma(T) \cap \rho_F(T)$, $R(T - \lambda_0) \neq X$ and T is reduced by $R(T - \lambda_0)$, then $\sigma_F(T) = \sigma_B(T)$.*

Let $r(A)$ denote the spectral radius of A and $w(A) = \sup \{|\lambda| : \lambda \in W(A)\}$ the numerical range of A . Then we can note here that the hypothesis “Let A be a closed densely defined (linear) operator which is reduced by each finite dimensional subspace $\ker(A - z)$ ” of Bouldin’s [2] theorem 9 is not needed for the conclusion: “If A has no isolated eigenvalues of finite algebraic multiplicity then $r(A) \leq r(A + K)$ for every closed A -compact operator K . Furthermore if $w(A) = \|A\|$, then $\|A\| \leq \|A + K\|$ for every closed A compact operator K .” In fact, for any closed densely defined A with no isolated eigenvalues for which $A - \lambda$ is Fredholm then $r(A) \leq r(A + K)$ for every closed A -compact operator K . This follows because the boundary of the spectrum of such an A consists only of points in $\sigma_F(A)$ and $\sigma_F(A) = \sigma_F(A + K)$ for any A -compact K , also if λ_0 is an isolated point of $\sigma(T)$ for which $T - \lambda_0$ is Fredholm, then λ_0 has finite algebraic multiplicity by (e.g.) corollary 9.3 of Taylor [9]. The rest of the theorem follows.

Examples: Any linear operator T with $\text{meas } \sigma(T) = 0$ has the property that $\sigma_B(T) = \sigma_F(T)$. Every normal operator T , has $\sigma_B(T) = \sigma_F(T)$.

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University of Maryland

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