Singular Integral Operators and Toeplitz Operators on Odd Spheres

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1. Introduction. Let C^n be the vector space of n -tuples of complex numbers with elements $z = (z_1, \dots, z_n)$ and the usual norm $|z| = (|z_1|^2 + \dots + |z_n|^2)^{1/2}$. We denote by B^{2n} the (real) $2n$ -dimensional open unit ball in C^n , $B^{2n} = \{z : |z| < 1\}$. Moreover, we let $S^{2n-1} = \{z : |z| = 1\}$ be the (real) $2n - 1$ dimensional unit sphere bounding B^{2n} and we write $\bar{B}^{2n} = \{z : |z| \leq 1\}$. Further, let $L^2(B^{2n})$ be the usual space of Lebesgue (volume) square-integrable complex-valued functions on B^{2n} and let $L^2(S^{2n-1})$ be the corresponding space of functions on S^{2n-1} , square-integrable with respect to the usual surface area measure.

In what follows, we will consider the subspaces $H^2(B^{2n})$, consisting of analytic functions on B^{2n} which are in $L^2(B^{2n})$, and $H^2(S^{2n-1})$, consisting of functions in $L^2(S^{2n-1})$ which are radial boundary-values of functions analytic in B^{2n} . For the basic properties of $H^2(S^{2n-1})$ including the fact that functions in $H^2(S^{2n-1})$ have unique analytic extensions to B^{2n} , see [3]. Let $P_n(\bar{P}_n)$ be the orthogonal projection from $L^2(S^{2n-1})$ ($L^2(B^{2n})$) onto $H^2(S^{2n-1})$ ($H^2(B^{2n})$). For ϕ, ψ respectively bounded measurable complex-valued functions on S^{2n-1} and B^{2n} we define the Toeplitz operators T_ϕ on $H^2(S^{2n-1})$ and $\tilde{T}_\psi$ on $H^2(B^{2n})$ by

$$T_\phi f = P_n(\phi f), \quad \tilde{T}_\psi g = \bar{P}_n(\psi g).$$

In Section 2 of this note, I describe the C^* -algebra, $\tau(S^{2n-1})$, generated by all Toeplitz operators on $H^2(S^{2n-1})$ associated with ϕ in $C(S^{2n-1})$, the sup norm algebra of all complex-valued continuous functions on S^{2n-1} . The analysis gives known results [1] in the case $n = 1$ and is directly related to a corresponding analysis of the C^* -algebra $\tau(B^{2n})$ generated by all Toeplitz operators on $H^2(B^{2n})$ associated with ψ in $C(\bar{B}^{2n})$, the sup norm algebra of all complex-valued continuous functions on $\bar{B}^{2n}$. The main work in the analysis of $\tau(B^{2n})$ was done by Venugopalkrishna [7] who showed (in a more general context) that commutators in $\tau(B^{2n})$ are compact operators and that if $\psi|_{S^{2n-1}} \equiv 0$, then $\tilde{T}_\psi$ is compact. We shall see that the compactness of commutators carries

over to $\tau(S^{2n-1})$. By adding a bit more, including the fact that $\tilde{T}_\psi$ compact implies $\psi|_{S^{2n-1}} \equiv 0$, we can completely analyze the structures of $\tau(S^{2n-1})$ and $\tau(B^{2n})$.

My interest in $\tau(S^{2n-1})$ is motivated by the close connection between $\tau(S^1)$ and the algebra of singular integral operators CZ (S^1) (here, CZ (S^{2n-1}) is the uniform closure of the algebra of Calderon–Zygmund operators of order zero on the compact manifold S^{2n-1} described in [4, p. 262]). In Section 3, I give a brief discussion of the exact relationship between $\tau(S^1)$ and CZ (S^1). In particular, it is well-known that P_1 is in CZ (S^1) [4, p. 272]. This fact is used to construct a faithful representation ρ from $\tau(S^1)$ into CZ (S^1). Finally, I combine the analysis of Section 2 with an index result of Venugopalkrishna [7] for $\tau(B^{2n})$ to conclude, via a special case of the Atiyah–Singer index theorem [4, 5], that for $n > 1$ there can be no such faithful representation of $\tau(S^{2n-1})$ into CZ (S^{2n-1}).

2. The algebras $\tau(S^{2n-1})$ and $\tau(B^{2n})$. In this section, we will use the natural orthonormal bases for $H^2(S^{2n-1})$ and $H^2(B^{2n})$ given respectively by

$$e_k = (2\pi^n)^{-1/2} \left(\frac{(n + |k| - 1)!}{k!} \right)^{1/2} z^k$$

$$\tilde{e}_k = (\pi^n)^{-1/2} \left(\frac{(n + |k|)!}{k!} \right)^{1/2} z^k$$

where $k = (k_1, \dots, k_n)$ is an n -tuple of non-negative integers and we take $|k| \equiv k_1 + k_2 + \dots + k_n$, $k! \equiv k_1! k_2! \dots k_n!$, $z^k = z_1^{k_1} z_2^{k_2} \dots z_n^{k_n}$. We will also make use of the fact that the reproducing kernels for $H^2(S^{2n-1})$ and $H^2(B^{2n})$ are given respectively (see [3]) by

$$K(\lambda, z) = \frac{(n-1)!}{2\pi^n} (1 - \bar{\lambda} \cdot z)^{-n}$$

$$\tilde{K}(\lambda, z) = \frac{n!}{\pi^n} (1 - \bar{\lambda} \cdot z)^{-n-1}$$

where $|z| \leq 1$, $|\lambda| < 1$, $\bar{\lambda} \equiv (\bar{\lambda}_1, \bar{\lambda}_2, \dots, \bar{\lambda}_n)$ and

$$\bar{\lambda} \cdot z = \bar{\lambda}_1 z_1 + \bar{\lambda}_2 z_2 + \dots + \bar{\lambda}_n z_n.$$

The structure of $\tau(S^{2n-1})$ and $\tau(B^{2n})$ is analyzed in the following sequence of Lemmas and Theorem 1.

Lemma 1. *The algebras $\tau(S^{2n-1})$ and $\tau(B^{2n})$ are irreducible.*

Proof. We will prove the lemma for $\tau(S^{2n-1})$. The proof for $\tau(B^{2n})$ is essentially the same. We assume there is a projection operator P which commutes with all the T_ϕ where ϕ are continuous on S^{2n-1} and extend analytically to B^{2n} . The set of all such ϕ is clearly dense in $H^2(S^{2n-1})$ (polynomials in the z_i will do). Now let $P1 = g$. Then $PT_\phi 1 = T_\phi P1$ so $P\phi = \phi g$ for a dense set of ϕ 's in $H^2(S^{2n-1})$. A standard argument now shows that $Pf = fg$ for all f in $H^2(S^{2n-1})$.

and that $|g(z)| \leq 1$ a.e. so g is the boundary-value function of a bounded analytic function in B^{2n} and $P = T_\sigma$. As usual, we denote the extension of g to an analytic function on B^{2n} by g as well. Now for $|\lambda| < 1$ we have

$$T_\sigma^* K(\lambda, \cdot) = \overline{g(\lambda)} K(\lambda, \cdot)$$

by standard properties of the kernel function. Since $P = T_\sigma$ is a projection, its spectrum must be contained in $\{0, 1\}$ and it follows that $g(B^{2n}) \subset \{0, 1\}$. Hence $g \equiv 0$ or $g \equiv 1$ and $P = 0$ or $P = I$.

Lemma 2. $\tilde{T}_\psi$ compact implies $\psi|_{S^{2n-1}} \equiv 0$ and T_ϕ compact implies $\phi \equiv 0$ on S^{2n-1} .

Proof. We will prove the lemma for $\tau(B^{2n})$. The proof for $\tau(S^{2n-1})$ is essentially the same. We assume that $\psi \in C(\bar{B}^{2n})$ and for some x_0 with $|x_0| = 1$, $\psi(x_0) \neq 0$. For a sequence of λ_m in B^{2n} with $\lambda_m \rightarrow x_0$ we consider the functions in $H^2(B^{2n})$

$$f_m(z) = \tilde{K}(\lambda_m, z)(\tilde{K}(\lambda_m, \lambda_m))^{-1/2}.$$

It is an easy computation that $\|f_m\| = 1$ and the f_m converge uniformly to zero in the complement of any open neighborhood of x_0 . It follows that $\|\tilde{T}_\psi f_m - \psi(x_0)f_m\| \rightarrow 0$. Now we can assume without loss of generality that $f_m \rightarrow g$ (weakly). If $\tilde{T}_\psi$ is compact then $\tilde{T}_\psi f_m \rightarrow \tilde{T}_\psi g$ and so we have $f_m \rightarrow g$ (strongly). Thus $\|g\| = 1$ and f_m converges to g in measure. But f_m converges to 0 pointwise a.e. so $g = 0$ a.e.. This is a contradiction.

Now let U be the unitary transformation from $H^2(S^{2n-1})$ onto $H^2(B^{2n})$ given by $U e_k = \tilde{e}_k$. We have

Lemma 3. $U^* \tilde{T}_\psi U = T_\phi + K(\psi)$ where $\phi = \psi|_{S^{2n-1}}$ and $K(\psi)$ is a compact operator.

Proof. Using approximation by polynomials in z and $\bar{z}$ and symmetry among the z_i , we see that it is enough to check that $U^* \tilde{T}_{z_1} U - T_{z_1}$ is compact. Letting $\delta_1 = (1, 0, \dots, 0)$, we see that

$$T_{z_1} - U^* \tilde{T}_{z_1} U = D S_{\delta_1}$$

where $S_{\delta_1} e_k = e_{k+\delta_1}$ and

$$D e_{k+\delta_1} = (k_1 + 1)^{1/2} \{ (n + |k|)^{-1/2} - (n + |k| + 1)^{-1/2} \} e_{k+\delta_1},$$

$$D e_k = 0 \quad \text{when} \quad k_1 = 0.$$

Thus, it suffices to show that D is compact. We write

$$\lambda(k) = (k_1 + 1)^{1/2} \{ (n + |k|)^{-1/2} - (n + |k| + 1)^{-1/2} \}.$$

To prove that D is compact it is enough to show that for any $\epsilon > 0$, $\lambda(k) < \epsilon$ for all but finitely many k . Fortunately, $k_1 \leq |k|$ so $\lambda(k) \rightarrow 0$ as $|k| \rightarrow \infty$ and we are done.

Theorem 1. *The algebras $\tau(S^{2n-1})$ and $\tau(B^{2n})$ both contain the full algebras of compact operators $\mathcal{K}, \tilde{\mathcal{K}}$ on their respective Hilbert spaces. Moreover, the mapping $A \rightarrow U^*AU$ gives a *-isomorphism between $\tau(B^{2n})$ and $\tau(S^{2n-1})$. We have $\tau(B^{2n}) = \{\tilde{T}_\psi + \tilde{K} \text{ for all } \psi \in C(\bar{B}^{2n}) \text{ and } \tilde{K} \in \tilde{\mathcal{K}}\}$ and $\tau(S^{2n-1}) = \{T_\phi + K \text{ for all } \phi \in C(S^{2n-1}) \text{ and } K \in \mathcal{K}\}$. The quotient $\tau(B^{2n})/\tilde{\mathcal{K}}$ is naturally identified with $C(S^{2n-1})$ by the map $\tilde{\sigma}(\tilde{T}_\psi + \tilde{K}) = \psi|_{S^{2n-1}}$ while the quotient $\tau(S^{2n-1})/\mathcal{K}$ is naturally identified with $C(S^{2n-1})$ by the map $\sigma(T_\phi + K) = \phi$.*

Proof. We note first that for ψ in a dense set of $C(\bar{B}^{2n})$, $(I - P)\psi P$ is a compact operator on $L^2(B^{2n})$ by a result of [7] mentioned previously. It follows easily that $\tilde{T}_{\psi_1}\tilde{T}_{\psi_2} - \tilde{T}_{\psi_1\psi_2}$ is compact for all ψ_i in $C(\bar{B}^{2n})$. This could also be shown by direct computation in the basis $\{\tilde{e}_k\}$. In particular, commutators in $\tau(B^{2n})$ are compact, and since $\tau(B^{2n})$ is easily seen to be non-commutative, $\tau(B^{2n})$ contains at least one non-trivial compact operator. It follows from Lemma 1 and a well-known result in C^* -algebras [2, p. 85] that $\tilde{\mathcal{K}} \subset \tau(B^{2n})$.

Now by Lemma 3, for ϕ in $C(S^{2n-1})$,

$$(*) \quad T_\phi = U^*(\tilde{T}_\psi + \tilde{K}(\psi))U$$

where ψ is any extension of ϕ to $C(\bar{B}^{2n})$. It follows that

$$T_{\phi_1}T_{\phi_2} - T_{\phi_1\phi_2}$$

is compact for all ϕ_i in $C(S^{2n-1})$ and another application of Lemma 1 allows us to conclude, as above, that $\mathcal{K} \subset \tau(S^{2n-1})$.

It is now immediate from Lemma 3 that $A \rightarrow U^*AU$ takes a set of generators of $\tau(B^{2n})$ into $\tau(S^{2n-1})$. Moreover, this map is onto a set of generators for $\tau(S^{2n-1})$ by (*). Hence, the mapping $A \rightarrow U^*AU$ is indeed a *-isomorphism between $\tau(B^{2n})$ and $\tau(S^{2n-1})$.

Now for ψ in $C(\bar{B}^{2n})$ we let $[\tilde{T}_\psi]$ be the class of $\tilde{T}_\psi$ in $\tau(B^{2n})/\tilde{\mathcal{K}}$. The map $\psi \rightarrow [\tilde{T}_\psi]$ is easily seen to be a *-homomorphism from $C(\bar{B}^{2n})$ into $\tau(B^{2n})/\tilde{\mathcal{K}}$. The range of this map is $\{[\tilde{T}_\psi] : \psi \in C(\bar{B}^{2n})\} = Q$ so by a standard result [2, p. 18], Q must be a closed *-subalgebra of $\tau(B^{2n})/\tilde{\mathcal{K}}$. But Q contains the quotient classes of all generators of $\tau(B^{2n})$ so $Q = \tau(B^{2n})/\tilde{\mathcal{K}}$ and it follows that

$$\tau(B^{2n}) = \{\tilde{T}_\psi + \tilde{K} : \psi \in C(\bar{B}^{2n}) \text{ and } \tilde{K} \in \tilde{\mathcal{K}}\}.$$

The same argument gives the corresponding result for $\tau(S^{2n-1})$.

It only remains to check the kernels of the homomorphisms $\psi \rightarrow [\tilde{T}_\psi]$ and $\phi \rightarrow [T_\phi]$. By Lemma 2 and its converse [7] we check easily that $\psi|_{S^{2n-1}} \leftrightarrow [\tilde{T}_\psi]$ and $\phi \leftrightarrow [T_\phi]$ are isomorphisms. This concludes the proof.

Letting M_r denote the algebra of $r \times r$ complex matrices and ${}_rA \equiv A \oplus A \cdots \oplus A$ (r times), we define $T_\phi f \equiv ({}_rP_n)\phi f$ for ϕ in $C(S^{2n-1}) \otimes M_r$ and f in ${}_rH^2(S^{2n-1}) = H^2(S^{2n-1}) \oplus \cdots \oplus H^2(S^{2n-1})$ (r times) and define $\tilde{T}_\psi$ similarly on ${}_rH^2(B^{2n})$. We now have

Corollary to Theorem 1. *The sequences*

$$\begin{aligned} 0 \rightarrow \mathcal{K} \otimes M_r &\rightarrow \tau(S^{2n-1}) \otimes M_r \xrightarrow{\sigma \otimes 1} C(S^{2n-1}) \otimes M_r \rightarrow 0 \\ 0 \rightarrow \tilde{\mathcal{K}} \otimes M_r &\rightarrow \tau(B^{2n}) \otimes M_r \xrightarrow{\tilde{\sigma} \otimes 1} C(S^{2n-1}) \otimes M_r \rightarrow 0 \end{aligned}$$

are exact. In particular, a $r \times r$ matrix with entries in $\tau(S^{2n-1})$ operating on $H^2(S^{2n-1})$ has the form $T_\phi + K$ where ϕ is an $r \times r$ matrix-valued continuous function on S^{2n-1} and K is compact. $T_\phi + K$ is a Fredholm operator [4, p. 119] if and only if ϕ has non-vanishing determinant. The corresponding result holds for $\tau(B^{2n})$ when $\psi|_{S^{2n-1}}$ has non-vanishing determinant. If U is taken as in Lemma 3 then $({}_rU^*)\tilde{T}_\psi({}_rU) = T_\phi + K(\psi)$ where $\phi = \psi|_{S^{2n-1}}$ and $K(\psi)$ is compact.

Proof. The exactness of the sequences is easy. The Fredholm criterion follows from Atkinson's theorem [4, p. 120] and the exactness. The last statement follows immediately from Lemma 3.

3. $\tau(S^{2n-1})$ and CZ (S^{2n-1}) . We begin by discussing the exact relationship between $\tau(S^1)$ and CZ (S^1) . It should be recalled that our CZ (S^{2n-1}) is the uniform closure of the algebra of Calderon-Zygmund operators of order zero considered in [4]. It follows that the symbol σ defined in [4, p. 262] extends to a *-homomorphism from CZ (S^1) onto $C(S^1 \times S^0)$ with kernel the full algebra of compact operators on $L^2(S^1)$ (here, $S^0 = \{-1, 1\}$ is the 0-sphere). The key fact is that P_1 is in CZ (S^1) and $\sigma(P_1)(z, 1) = 1$, $\sigma(P_1)(z, -1) = 0$.

Now define J on $L^2(S^1)$ by $(Jf)(z) = f(\bar{z})$. We make use of a natural unitary equivalence between $H^2(S^1) \oplus^\perp H^2(S^1)$ and $L^2(S^1)$ given for f, g in $H^2(S^1)$ and h in $L^2(S^1)$ by

$$V(f, g) = f(z) + Jzg(z), \quad V^*h = (P_1h, Jz(I - P_1)h).$$

Theorem 2. *The algebra CZ (S^1) consists precisely of all sums of the form $P_1\phi P_1 + (I - P_1)\psi(I - P_1) + K$ where ϕ, ψ are in $C(S^1)$ and K is compact. The above decomposition is unique. Moreover $V^*P_1\phi P_1V = T_\phi \oplus 0$, $V^*(I - P_1)\psi(I - P_1)V = 0 \oplus T_{\tilde{\psi}}$ where $\tilde{\psi}(z) = \psi(\bar{z})$, and the map $A \rightarrow V^*AV$ implements a *-isomorphism between CZ (S^1) and $\tau(S^1) \oplus \tau(S^1) + \mathcal{K}$ where $\mathcal{K}$ is the full algebra of compact operators on $H^2(S^1) \oplus^\perp H^2(S^1)$.*

Proof. Since P_1 is in CZ (S^1) with symbol described above, it is easy to check that

$$\begin{aligned} \sigma(P_1\phi P_1 + (I - P_1)\psi(I - P_1))(z, 1) &= \phi(z) \\ \sigma(P_1\phi P_1 + (I - P_1)\psi(I - P_1))(z, -1) &= \psi(z) \end{aligned}$$

so that σ maps elements of the given form in CZ (S^1) onto $C(S^1 \times S^0)$. It follows that every element of CZ (S^1) must have the given form. Uniqueness is also trivial by applying σ . The unitary equivalence is a direct computation.

Corollary to Theorem 2. The map ρ defined by

$$\rho(T_\phi + K) = V \begin{bmatrix} T_\phi + K & 0 \\ 0 & T_\phi + K \end{bmatrix} V^*$$

is a faithful representation of $\tau(S^1)$ into $\text{CZ}(S^1)$ and $\rho(\mathcal{K}) \subset \mathcal{K}'$ (here $\mathcal{K}'$ denotes the full algebra of compact operators on $L^2(S^{2n-1})$).

Proof. This follows immediately from Theorems 1 and 2.

It will be useful in what follows to notice that ρ extends in an obvious way to give a faithful representation $\rho_r = \rho \otimes 1$ of $\tau(S^1) \otimes M_r$ into $\text{CZ}(S^1) \otimes M_r$. Further ρ_r takes the compact operators $\mathcal{K}_r$ in $\tau(S^1) \otimes M_r$ to the compact operators $\mathcal{K}_r'$ in $\text{CZ}(S^1) \otimes M_r$ acting on $L^2(S^1)$.

We now consider the relationship between $\tau(S^{2n-1})$ and $\text{CZ}(S^{2n-1})$ for $n > 1$. From our definition of $\text{CZ}(S^{2n-1})$, it follows that the symbol σ of [4, p. 262] extends to a *-homomorphism from $\text{CZ}(S^{2n-1})$ onto $C[S^*(S^{2n-1})]$, the sup norm algebra of continuous complex-valued functions on the unit cosphere bundle of S^{2n-1} , with kernel the full algebra of compact operators $\mathcal{K}'$ on $L^2(S^{2n-1})$. Now for $n > 1$, $S^*(S^{2n-1})$ is connected. Since P_n and $I - P_n$ are both non-compact, we can use the fact that $C[S^*(S^{2n-1})]$ can not have a non-trivial idempotent to deduce that P_n is not in $\text{CZ}(S^{2n-1})$. The next result should be compared to the Corollary to Theorem 2.

Theorem 3. For $n > 1$, there is no faithful representation ρ of $\tau(S^{2n-1})$ into $\text{CZ}(S^{2n-1})$.

Proof. Using [6, p. 308], Theorem 1, and the above remarks about $\text{CZ}(S^{2n-1})$, we see that $\mathcal{K}$ and $\mathcal{K}'$ are the commutator ideals in the algebras $\tau(S^{2n-1})$ and $\text{CZ}(S^{2n-1})$. It follows that $\rho(\mathcal{K}) \subset \mathcal{K}'$. Moreover ρ induces faithful representations $\rho_r = \rho \otimes 1$ from $\tau(S^{2n-1}) \otimes M_r$ to $\text{CZ}(S^{2n-1}) \otimes M_r$ with $\rho_r(\mathcal{K}_r) \subset \mathcal{K}_r'$. Now recall that all rank one self-adjoint projections P, Q are mutually unitarily equivalent by unitaries of the form $W = I + K$ for K in $\mathcal{K}_r$. For $Q = W^*PW$, we have

$$\rho_r(Q) = \rho_r(W)^* \rho_r(P) \rho_r(W)$$

so $\rho_r(P)$ and $\rho_r(Q)$ are projections of the same rank, c_{ρ_r} . An easy argument now shows that c_{ρ_r} is independent of r and writing $c_{\rho_r} = c_\rho$

$$\text{rank } \rho_r(P) = c_\rho \text{ rank } P$$

for all finite-rank self-adjoint projections P . Since ρ_r is faithful, c_ρ must be a positive integer.

Next, we note that for A a Fredholm operator, the projection on the null-space of A , $N(A)$, has the form

$$P_{N(A)} = \frac{1}{2\pi i} \int_{|\lambda|=\epsilon} (\lambda - A^*A)^{-1} d\lambda$$

for λ in C^1 and $\epsilon > 0$ sufficiently small. Now it follows easily from the fact that $\rho_r(\mathcal{K}_r) \subset \mathcal{K}_r'$ that if A is Fredholm in $\tau(S^{2n-1}) \otimes M_r$ then $\rho_r(A)$ is Fredholm in $\text{CZ}(S^{2n-1}) \otimes M_r$. Hence we see, by an easy approximation argument, that $P_{N(A)}$ is in $\tau(S^{2n-1}) \otimes M_r$ and

$$\rho_r(P_{N(A)}) = P_{N(\rho_r(A))}$$

for all Fredholm A . Finally, noting that

$$\text{index } A = \text{rank } P_{N(A)} - \text{rank } P_{N(A)^*}$$

we conclude, for A Fredholm in $\tau(S^{2n-1}) \otimes M_r$, that

$$\text{index } \rho_r(A) = c_\rho \text{ index } A.$$

We now use the fact [7] that there are Fredholm operators in $\tau(B^{2n}) \otimes M_n$ with non-trivial index. Let $\tilde{T}_\psi$ be such an operator. It follows from Lemma 3 and the Corollary to Theorem 1 that if $\phi = \psi|_{S^{2n-1}}$ then T_ϕ in $\tau(S^{2n-1}) \otimes M_n$ is also a Fredholm operator with non-trivial index. Thus, $\rho_n(T_\phi)$ is Fredholm with non-zero index. This contradicts a special case of the Atiyah–Singer index theorem [4, 5] which implies that $\text{CZ}(S^{2n-1}) \otimes M_r$ contains Fredholm operators of non-trivial index only if $r \geq 2n - 1$. This concludes the proof.

It should be noted that for $n = 2$, it is easy to check that for ϕ given on S^3 by

$$\phi(z_1, z_2) = \begin{bmatrix} z_1 & -\bar{z}_2 \\ z_2 & \bar{z}_1 \end{bmatrix},$$

T_ϕ has index one on $H^2(S^3) \oplus H^2(S^3)$. In [7], a general index theorem is established for $\tau(B^{2n}) \otimes M_r$ which, by our earlier remarks, easily carries over to $\tau(S^{2n-1}) \otimes M_r$.

REFERENCES

1. L. A. COBURN, *The C*-algebra generated by an isometry I, II*; Bulletin AMS **73** (1967), 722–726; Trans. AMS **137** (1969), 211–217.
2. J. DIXMIER, *Les C*-algèbres et leurs représentations*, 1969.
3. A. KORANYI, *The Poisson integral for generalized half-planes and bounded symmetric domains*, Annals Math. **82** (1965), 332–350.
4. R. S. PALAIS, *Seminar on the Atiyah–Singer index theorem*, Annals of Mathematics Studies **57**, 1965.
5. R. T. SEELEY, *Integro-differential operators on vector bundles*, Trans. AMS **117** (1965), 167–204.
6. R. T. SEELEY, *The index of elliptic systems of singular integral operators*, J. Math. Analysis and Applications **7** (1963), 289–309.
7. U. VENUGOPALKRISHNA, *Fredholm operators associated with strongly pseudoconvex domains in C^n* , J. Functional Analysis **9** (1972), 349–373.

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