Almost Every Proper Isometry is a Shift

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An operator S on a Hilbert space $\mathcal{H}$ is called a shift if there is an orthonormal basis $\{e_n : n = 1, 2, \dots\}$ such that $Se_n = e_{n+1}$. More generally, an operator unitarily equivalent to the direct sum of α shifts is called a shift of multiplicity α , where α can be any positive cardinal number. It is known ([1]) that any isometry T of $\mathcal{H}$ onto a proper subspace can be approximated in norm by a shift, whose multiplicity is necessarily the corank of T . We will show that the set of shifts is a G_δ , hence justifying the title, and derive some consequences.

If $\mathcal{A}$ is a norm closed subset of bounded operators on the Hilbert space $\mathcal{H}$, we say almost every element of $\mathcal{A}$ has property P if $\{T \in \mathcal{A} : T \text{ has property } P\}$ contains a dense G_δ of $\mathcal{A}$. Let $\mathcal{U}$ denote the set of unitary operators on $\mathcal{H}$, $\mathcal{I}_\mathcal{L}$ the set of isometries of $\mathcal{H}$ onto the proper subspace $\mathcal{L}$, and $\mathcal{I}$ the set of proper isometries ($\mathcal{I} = \bigcup_{\mathcal{L}} \mathcal{I}_\mathcal{L}$). Also for T an isometry T_K denotes $\{T' : T' \text{ is an isometry and } T' - T \text{ is compact}\}$. Every element of T_K has the same corank as T .

Theorem 1. *If $\mathcal{H}$ is a Hilbert space of countably infinite dimension, then the set of shifts is a G_δ subset of $\mathcal{I}$, and hence almost every element of $\mathcal{I}$ and almost every element of T_K ($T \in \mathcal{I}$) is a shift.*

Proof. If $T \in \mathcal{I}$ and $\mathfrak{M} = T(\mathcal{H})^\perp$, then the spaces $T^n \mathfrak{M}$, $n = 0, 1, \dots$, are orthogonal and T is a shift if and only if they span $\mathcal{H}$. Now for $\epsilon > 0$ and $x \in \mathcal{H}$ let $S(\epsilon, x) = \{T \in \mathcal{I} : \|x - (I - TT^*)y_0 - \sum_{n=1}^k T^n(I - TT^*)y_n\| < \epsilon\}$, for some positive integer k and some $y_0, y_1, \dots, y_k \in \mathcal{H}$. $S(\epsilon, x)$ is obviously open. If $\{x_m : m = 1, 2, \dots\}$ is dense in $\mathcal{H}$, then the set of shifts is $\bigcap_{\epsilon, m=1}^\infty S(\epsilon, x_m)$. Since it was shown in [1] that shifts are dense in both $\mathcal{I}$ and T_K , the theorem follows.

Remark. The original motivation for this was to find out whether the map $U \rightarrow U^{-1}SU$, from $\mathcal{U}$ onto the set of shifts of fixed multiplicity, is open. An affirmative answer would have implied the theorem, but the answer is negative since S_K is separable whereas $\{U \in \mathcal{U} : U^{-1}SU \in S_K\}$ is not separable.

Corollary 1. *If $U_0 \in \mathcal{U}$ and $T \in \mathcal{I}$, then for almost every $U \in \mathcal{U}$ and for almost every $U \in U_{0K} UT$ is a shift.*

Proof. By changing notation (replace T by U_0T) and using the theorem, we can reduce this to the following proposition: there are elements U of I_K , arbitrarily close to I , such that UT is a shift.

Choose $1/2 > \epsilon > 0$. By [1] we can find a shift T_1 such that $\|T_1 - T\| < \epsilon$ and $T_1 \in T_K$. Let P and P_1 be the projections on the ranges of T and T_1 respectively and $Q = I - P$, $Q_1 = I - P_1$. Then T_1T^* is a partial isometry from $P\mathcal{H}$ to $P_1\mathcal{H}$, $\|T_1T^* - P\| < \epsilon$, and $T_1T^* - P$ is compact. We now wish to construct a partial isometry V from $Q\mathcal{H}$ to $Q_1\mathcal{H}$ such that $\|V - Q\|$ is small and $V - Q$ is compact. This is easy: simply let V be the partial isometry appearing in the polar decomposition of Q_1Q . (Note that $\|Q_1 - Q\| < 2\epsilon$.) We can then define $U = T_1T^* + V$. $UT = T_1$, and U can be made arbitrarily close to I by choosing ϵ small enough.

Corollary 2. *For almost every $U \in \mathfrak{U}$ and almost every $U \in U_{0K}$, TU is a shift. ($T \in \mathcal{S}$).*

Proof. TU is a shift if and only if $U(TU)U^{-1} = UT$ is a shift.

Corollary 3. *If $\mathcal{L}$ is a proper subspace of $\mathcal{H}$, then almost every element of $\mathcal{S}_{\mathcal{L}}$ is a shift and for $T \in \mathcal{S}_{\mathcal{L}}$ almost every element of $\mathcal{S}_{\mathcal{L}} \cap T_K$ is a shift.*

Proof. $T \in \mathcal{S}_{\mathcal{L}} \Rightarrow TU \in \mathcal{S}_{\mathcal{L}}$.

The next theorem answers a question of Halmos ([3]).

Theorem 2. *If $\mathcal{H}$ is infinite dimensional, $A : \mathcal{H} \rightarrow \mathcal{H}$ and $\|A\| \leq 1$, then there are shifts S_1 and S_2 of infinite multiplicity such that $A = S_1^*S_2$.*

Proof. We may assume $\mathcal{H}$ separable, since $\mathcal{H}$ is the direct sum of orthogonal, countably infinite-dimensional, reducing subspaces. Let T be an isometry with infinite corank and P the projection on its range. By [2], there is a unitary U on $\mathcal{H}$ which is a dilation of $TAT^*|_{P\mathcal{H}}$; i.e., $U \in \mathfrak{U}$ and $TAT^* = PUP$. Thus, $A = T^*(TAT^*)T = T^*PUPT = T^*UT = T_1^*T_2$, where $T_1 = T$, $T_2 = UT$, and T_1 and T_2 are isometries. For $V \in \mathfrak{U}$, $A = T_1^*V^*VT_2 = (VT_1)^*(VT_2)$. Therefore, in view of Corollary 1, we can take $S_1 = VT_1$ and $S_2 = VT_2$ for an appropriate V .

The shifts S_1 and S_2 just constructed both have the orthogonal dimension of $\mathcal{H}$ as their multiplicity. If $\mathcal{H}$ is non-separable this is the only possibility. From now on we assume $\mathcal{H}$ has countably infinite dimension and ask what can be the multiplicities of shifts S_1 and S_2 such that $A = S_1^*S_2$. Because of the proof above, it is sufficient to seek the coranks of isometries T_1 and T_2 such that $A = T_1^*T_2$. If T_1 has corank m and T_2 , n , and if either m or n is finite, then it is obvious that A is semi-Fredholm, and $\text{index } A = m - n$. Also, $\text{rank } (I - A^*A) = \text{rank } T_2^*(I - T_1T_1^*)T_2 \leq \text{rank } (I - T_1T_1^*) = m$. Similarly, $\text{rank } (I - AA^*) \leq n$. In general, if A is a contraction, if $\text{rank } (I - A^*A) = p$ and $\text{rank } (I - AA^*) = q$, and if p or q is finite, then A is semi-Fredholm and $\text{index } A = p - q$. In fact, the restriction of A to the kernel of $I - A^*A$ gives

rise to a partial isometry U whose initial space has codimension p and whose final space has codimension q ; and since p or q is finite, $A - U$ is compact. Thus, a necessary condition for the coranks m and n to occur as above, given A , is that $m = p + k$ and $n = q + k$ for some $k \geq 0$.

We will now show that this condition is also sufficient. We assume the contraction A is given and that p or q is finite. It is enough to find isometries T_1 and T_2 of coranks p and q respectively such that $A = T_1^*T_2$, since if V is any isometry of corank then also $A = (VT_1)^*(VT_2)$. We may assume $p \leq q$, since otherwise we could replace A by A^* .

Now if $A = WH$ is the polar decomposition of A , W can be "extended" to an isometry W_1 of corank $q - p$. Since the kernel of $I - H$ has dimension p , the formula for a unitary dilation of H given in [2] shows that H has a unitary dilation on a larger Hilbert space in which $\mathcal{H}$ has codimension p . Therefore, as in the proof of Theorem 2, we can write $H = R_1^*R_2$ where R_1 and R_2 are isometries of corank p . Thus, $A = W_1R_1^*R_2 = (W_1R_1^*W_1^*)W_1R_2 = (W_1R_1W_1^*)(W_1R_2)$. $W_1R_1W_1^*$ is a partial isometry from $W_1\mathcal{H}$ to a subspace of codimension p in $W_1\mathcal{H}$. We define an isometry T_1 of corank p by letting T_1 agree with $W_1R_1W_1^*$ on $W_1\mathcal{H}$ and letting T_1 be the identity on $(W_1\mathcal{H})^\perp$. If $T_2 = W_1R_2$, then T_1 and T_2 are as desired and $A = T_1^*T_2$. We therefore, have:

Theorem 3. *If $\mathcal{H}$ has countably infinite dimension, A is a contraction on $\mathcal{H}$, and $p = \text{rank}(I - A^*A)$ and $q = \text{rank}(I - AA^*)$, then m and n are the respective coranks of isometries T_1 and T_2 such that $A = T_1^*T_2$ if and only if $m = n = \infty$ or $m = p + k$ and $n = q + k$ for some non-negative integer k . T_1 and T_2 can be taken to be shifts if and only if in addition m and n are both positive.*

REFERENCES

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