Adjunction and Inversion of Invertible Hilbert-Space Operators

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1. Introduction. In this paper, we establish a duality between adjunction and inversion of bounded invertible operators on a fixed Hilbert space. The main objective is to give a perfect analogy between the similarity relation and the congruence relation. While there is rather rich information about the similarity relation in the literature, the duality really exposes the structure of the congruence relation.

Each invertible operator T is intimately associated with its adjoint T^* and its inverse T^{-1} . Indeed, adjunction and inversion are permutable order-2 anti-automorphisms on the multiplicative group of invertible operators. The interplay of these two actions gives rise to many significant results naturally. We call special attention to “conormal operators” (square roots of unitary operators), as many intrinsic properties of normal operators can be carried over to conormal operators analogously. The principal result is Theorem 3: Unitary equivalence agrees with the similarity relation on normal operators and with the congruence relation on conormal operators.

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2. Definitions and main results. All operators dealt with in this paper are invertible.

Definition 1. An operator T is *primary* if T is *hermitian* ($T = T^*$), or *involutory* ($T = T^{-1}$), or *unitary* ($T^* = T^{-1}$). (In particular, if T is a *symmetry* ($T = T^* = T^{-1}$).)

Definition 2. An operator T is *secondary* if T is *normal* ($T^*T = TT^*$), or *conormal* (T^2 is unitary), or *contranormal* (T^2 is positive). (See Remarks 1–3 for an alternative symmetric definition of a secondary operator.)

Definition 3. T is *similar* to S if $T = X^{-1}SX$ for certain invertible X . T is *congruent* to S if $T = X^*SX$ for certain invertible X .

Definition 4. A unitary operator U is *upper-cramped* if its spectrum lies in the closed upper half circle and -1 is not an eigenvalue.

Every secondary operator is a square root of a normal operator. The following Venn diagram can be deduced directly from definitions.

It is interesting to see that the symmetry of the above diagram applies as well to the algebraic structure of square roots of normal operators. Indeed, every secondary operator is the product of two permutable primary operators:

Theorem 1. An invertible operator T admits a factorization (i) $T = HU = UH$ iff T is normal, (ii) $T = VU = UV$ iff T is conormal, (iii) $T = HV = VH$ iff T is contranormal, (iv) $T = HUV$ (H, U, V are permutable) iff T is a square root of a normal operator where H is hermitian, V is involutory, U is unitary in all expressions. Moreover, the factorization is unique if we require further that in (i) either H is positive or U is upper-cramped, (ii) U is upper-cramped, (iii) H is positive, (iv) H is positive and U is upper-cramped.

We are going to give an analogy between the similarity relation and the congruence relation. The basic tool is Fuglede–Putnam’s theorem (see [3, Problem 152] for details) and its dual:

Theorem 2. (i) If T, S are normal and $T = X^{-1}SX$, then $T^* = X^{-1}S^*X$. (ii) If T, S are conormal and $T = X^*SX$, then $T^{-1} = X^*S^{-1}X$.

The significance of the similarity relation can be best revealed by its effect on normal operators (see [3, p. 99]). We establish the dual result for the congruence relation:

Theorem 3. (i) *If two normal operators are similar, then they are unitarily equivalent.* (ii) *If two conormal operators are congruent, then they are unitarily equivalent.*

Let T be an operator with a certain property p . It is often useful to determine whether p is invariant under the similarity (resp. congruence) relation; in other words, we investigate whether all operators in $\{X^{-1}TX \mid X \text{ is invertible}\}$ (resp. $\{X^*TX \mid X \text{ is invertible}\}$) have the property p in common. The following theorem gives full information about the invariant properties of a symmetry under the similarity relation and the congruence relation.

Theorem 4. *An invertible operator is (i) involutory iff it is similar to a symmetry, (ii) hermitian iff it is congruent to a symmetry.*

If they are constrained to be square roots of normal operators, all similar-to-primary operators and congruent-to-primary operators are just the secondary operators:

Theorem 5. *A square root of an invertible normal operator is (i) normal iff it is congruent to unitary, (ii) conormal iff it is similar to unitary, (iii) contranormal iff it is congruent to involutory, (iv) contranormal iff it is similar to hermitian.*

Theorem 6. *Every square root of an invertible normal operator is (i) similar to normal, and (ii) congruent to conormal.*

In general, the primary operators, via the similarity relation and the congruence relation, are extendable to operators admitting a factorization as the product of a positive operator and a primary operator:

Theorem 7.

(i) (Williams [9]) *An invertible operator is the product of a positive operator and a hermitian operator iff it is similar to hermitian.*

(ii) *An invertible operator is the product of a positive operator and an involutory operator iff it is congruent to involutory.*

(iii) *Every invertible operator is the product of a positive operator and a unitary operator. In fact, every invertible operator is similar to a congruent-to-unitary operator and congruent to a similar-to-unitary operator.*

3. Proofs. An upper-cramped unitary operator is somewhat like a positive operator. The following lemma is useful in some places.

Lemma. *If U_1 is unitary, then there exists a unique upper-cramped unitary U such that $U_1 = U^2$. Moreover, if any operator X permutes with U_1 , then X permutes with U as well.*

Proof. Apply the spectral theorem.

Proof of Theorem 1. (i) is an easy consequence of the polar decomposition. We shall prove (iv), then (ii) and (iii) follow automatically.

In fact (iv) is immediate from Radjavi and Rosenthal's representation [6, pp. 656–661] for a square root of a normal operator. We give an alternative short proof as follows:

Suppose T^2 is normal. By polar factorization, there exist uniquely a unitary U_1 and a positive P_1 , such that $T^2 = U_1 P_1 = P_1 U_1$. Let U be the upper-cramped unitary square root of U_1 , and P the positive square root of P_1 . From Fuglede's theorem, T , U , and P are permutable. Hence, $TP^{-1}U^{-1}$ is involutory, and $T = (TP^{-1}U^{-1})UP$ is the unique factorization as required.

Proof of Theorem 2(ii). We use operator matrices to denote

$$A = \begin{bmatrix} T & 0 \\ 0 & S \end{bmatrix}, \quad Z = \begin{bmatrix} 0 & X^{-1} \\ X & 0 \end{bmatrix}.$$

Then A is conormal, and $A = Z^*AZ$. The assertion $T^{-1} = X^*S^{-1}X$ is equivalent to $A^{-1} = Z^*A^{-1}Z$. Adjoint-inversion on

$$(1) \quad A = Z^*AZ$$

leads to $A^{*-1} = Z^{-1}A^{*-1}Z^{*-1}$, hence

$$(2) \quad A^{*-1}A = Z^{-1}A^{*-1}AZ.$$

By Theorem 1(ii), $A = UV = VU$, where U is unitary and V is involutory; so $A^{*-1}A = U^2(V^*V) = (V^*V)U^2$ is normal. Applying Fuglede's theorem to (2), we deduce that U^2 permutes with Z . Combining with (1), we get $AU^{-2} = Z^*AU^{-2}Z$, which leads to the desired conclusion since $A^{-1} = VU^{-1} = AU^{-2}$.

Proof of Theorem 3(ii). Suppose T , S are conormal, and $T = X^*SX$. By polar factorization, $X = PU$ where P is positive and U is unitary. So

$$(*) \quad T = U^*PSPU.$$

By the dual of Fuglede–Putnam's theorem (i.e., Theorem 2(ii)), $T^{-1} = U^*PS^{-1}PU$. Combining with (*), we get $P^{-2} = S^{-1}P^2S$, and thus $f(P^{-2}) = S^{-1}f(P^2)S$ for all polynomials f . Hence it follows that $P^{-1} = S^{-1}PS$. From (*) again, we obtain $T = U^*SU$ as required.

Proof of Theorem 4. The “if” parts are trivial. Conversely, (i) follows from the fact that $(I + VV^*)^{-1/2}V(I + VV^*)^{1/2}$ is a symmetry for every involutory V ; (ii) is a consequence of the polar decomposition $H = SP = P^{1/2}SP^{1/2}$ in which H is hermitian, P is positive, and S is a symmetry.

Proof of Theorem 5. The “only if” parts can be deduced from the proof of Theorem 6, which is concerned with square roots of normal operators. We proceed to prove the “if” parts.

(i) Suppose T^2 is normal and $T = X^*UX$ where U is unitary. By Theorem 1, $T = PM = MP$ where P is positive, M is conormal. Hence $PM = X^*UX$,

$M = P^{-1/2}X^*UXP^{-1/2}$. From Theorem 3(ii), M is unitary. Therefore, T is a permutable product of a positive operator and a unitary operator; i.e., T is normal.

(ii) Suppose T^2 is normal, and T is similar to unitary. Then T^2 is also similar to unitary. By Theorem 3(i), T^2 is unitary; i.e., T is conormal.

(iii) follows from a proof of the same pattern as (i).

(iv) is like (ii).

Proof of Theorem 6. (i) follows from more general results (see, e.g., [4, p. 285]). We give a short proof (provided by J. P. Williams) as follows:

Suppose T^2 is normal. From Theorem 1, $T = NV = VN$ where N is normal, V is involutory. Let $X = (I + VV^*)^{1/2}$, then $S = X^{-1}VX$ is a symmetry by direct verification. From Fuglede's Theorem, N permutes with V and V^* ; hence N permutes with X and S . Therefore, $X^{-1}TX = NS$ is normal as required.

(ii) Let T^2 be normal. From Theorem 1, $T = PM = MP$ where P is positive and M is conormal. Hence $T = P^{1/2}MP^{1/2}$ as required.

Proof of Theorem 7. (ii) If $T = X^*VX$ where V is involutory, then X^*X is positive, $X^{-1}VX$ is involutory and $T = (X^*X)(X^{-1}VX)$ as required. Conversely, if $T = PV$ where P is positive and V is involutory, then $P^{1/2}VP^{1/2}$ is involutory and $T = P^{1/2}(P^{1/2}VP^{1/2})P^{1/2}$ as required.

(iii) If $T = PU$ is the polar decomposition of T , then $P^{-1/2}TP^{1/2} = P^{1/2}UP^{1/2}$ and $P^{-1/2}TP^{-1/2} = P^{1/2}UP^{-1/2}$ as required.

4. Remarks.

Remark 1. It may seem by a first glance at the definitions that the notions of conormal operator and normal operator are not parallel. We improve this effect by introducing the notion of "cosquare": $T^{*-1}T$ is the *cosquare* of T (analogously, T^2 is the *square* of T). The following are restatements of definitions:

- (a) (i) T is hermitian iff the cosquare of T is I .
- (ii) T is involutory iff the square of T is I .
- (iii) T is unitary iff the square of T equals the cosquare of T .
- (b) (i) T is normal iff the cosquare of T is unitary.
- (ii) T is conormal iff the square of T is unitary.
- (iii) T is contranormal iff the square of T is positive iff the cosquare of T is positive.

(c) The square of T is normal iff the cosquare of T is normal. (This fact has been discovered by H. Radjavi. The "only if" part follows from Theorem 1. Conversely, if $T^{*-1}T$ is normal, then Fuglede–Putnam's theorem applied to the equality $T^*(T^{*-1}T) = (T^{*-1}T)^{*^{-1}}T^*$ will lead to the result.)

We have employed the notion of cosquares in the proof of Theorem 2(ii). Another application is the fact that a similar-to-unitary operator is always a cosquare. (In fact, if $S = X^{-1}UX$ where U is unitary, then $S = (X^*U_1X)^{*^{-1}}(X^*U_1X)$ for any unitary U_1 such that $U_1^2 = U$.) The following result is of

interest in itself: Let $S = T^{*-1}T$, then S is similar to unitary iff T is congruent to unitary. (See also [7, p. 182].) (Proof: The “if” part is trivial. Conversely, suppose $X^{-1}SX$ is unitary, i.e., $X^{-1}T^{*-1}TX = (X^*TX)^{*-1}(X^*TX)$ is unitary. From (b)(i) of the preceding paragraph, X^*TX is normal. Hence T is congruent to unitary by virtue of Theorem 5(i).)

Remark 2. In coordination with Remark 1, we define the *coproduct* of T and S as $T^{*-1}S$. We say T *copermutes* with S iff the coproduct of T and S equals the coproduct of S and T , i.e., $T^{*-1}S = S^{*-1}T$.

Obviously, every invertible operator permutes with its inverse and copermutes with its adjoint. It is pertinent to redefine a normal operator and a conormal operator as follows:

- (i) T is normal iff T permutes with T^* .
- (ii) T is conormal iff T copermutes with T^{-1} .

We restate Fuglede’s theorem: If T^* permutes with T , and T permutes with X , then T^* permutes with X . From Theorem 2(ii), the dual is also true: If T^{-1} copermutes with T , and T copermutes with X , then T^{-1} copermutes with X .

Remark 3. In terms of the magnitude of norms, we characterize normal operators and conormal operators as follows:

- (i) T is normal iff $\|Tx\| = \|T^*x\|$ for every vector x .
- (ii) T is conormal iff $\|Tx\| = \|T^{-1}x\|$ for every vector x .

Remark 4. A positive operator is always hermitian but hardly ever involutory. Nevertheless, the class of positive operators stands at the neutral position in Theorem 7 and Remark 1(b). The situation may be interpreted by the fact that the cosquare of every involutory operator, as well as the square of every hermitian operator, is positive.

Remark 5. Referring to Theorem 3, we might conjecture that if two operators are similar and congruent, then they are unitarily equivalent. The following example shows that the conjecture is false even for contranormal operators.

$$\text{Let } T = \begin{bmatrix} 2 & 2 & 0 & 0 \\ 0 & -2 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix}, \quad S = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & -2 \end{bmatrix}.$$

Then T, S are contranormal, and they are similar and congruent to each other. However, it is evident that $\|T\| > 2 = \|S\|$, so T is not unitarily equivalent to S .

Remark 6. Radjavi and Williams [7, Theorem 3] have proved that an operator is unitarily equivalent to its adjoint iff it is the product of a symmetry and a hermitian operator. It is not surprising that a similar proof leads to the

dual result: An operator is unitarily equivalent to its inverse iff it is the product of a symmetry and an involutory operator.

There is a common consequence of the above statements (Davis [2, Theorem 6.3]): A unitary operator is similar to its inverse iff it is the product of two symmetries.

Remark 7. From Theorem 1, we see that every secondary operator admits a factorization as the product of two permutable primary operators. It may be interesting to investigate the product of two (not necessarily permutable) primary operators. (See also Theorem 7 and Remark 6.) The following are Radjavi and Williams's conjecture and its duals:

An invertible operator is

- (i) similar to its adjoint iff it is a product of two hermitian operators (see [7]),
- (ii) similar to its inverse iff it is a product of two involutory operators,
- (iii) congruent to its adjoint iff it is a product of a hermitian operator and an involutory operator,
- (iv) congruent to its inverse iff it is a product of a hermitian operator and an involutory operator.

It is easy to see that the "if" parts are true. Conversely, making use of Theorem 3 and Remark 6, we obtain partial results: (i) and (ii) are true for similar-to-normal operators, (iii) and (iv) are true for congruent-to-conormal operators. In particular, the conjectures are true for square roots of normal invertible operators by virtue of Theorem 6.

Following is another pertinent conjecture: T^* is similar to T^{-1} iff T is a cosquare. Recently, the notion of cosquares has aroused much attention in matrix theory. The reader is referred to DePrima and Johnson's paper, *The range of $A^{-1}A^*$ in $GL(n, C)$* (to appear in Linear Algebra and Appl.).

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