A Characterization of Entire Functions of Exponential Type and M -Bounded Index

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1. Introduction. An entire function of bounded index N is of exponential type at most $N + 1$ [9] but an entire function of exponential type need not be of bounded index [9]. We define here functions of M -bounded index and show that a function of M -bounded index N is of exponential type at most $N + 1$ and conversely a function of exponential type is always of M -bounded index.

An entire function f is said to be of bounded index if there exists an integer N such that

$$\max_{0 \leq i \leq N} \left\{ \frac{|f^{(i)}(z)|}{i!} \right\} \leq \frac{|f^{(n)}(z)|}{n!}$$

for all n and z . The least such integer N is called the index of f ([4], [6]).

Write $M_n(r) = M(r, f^{(n)}) = \max_{|z|=r} \{|f^{(n)}(z)|\}$, where $f^{(0)}(z) = f(z)$, $M_0(r) = M(r)$.

An entire function f is said to be of M -bounded index if there exists an integer N such that

$$\max_{0 \leq i \leq N} \left\{ \frac{M_i(r)}{i!} \right\} \leq \frac{M_n(r)}{n!}$$

for all $r \geq 0$ and all n . The least such integer N is called the M -index of f . It is easy to see that an entire function f of index N is also of M -index not exceeding N . The following example shows that the M -index, which we shall write as $N(M)$, may be less than the index N :

Example. For $f(z) = ze^z$, $N = 2$, $N(M) = 1$.

We will see later that there are functions of M -index which are not of bounded index.

Write $T = \limsup_{r \rightarrow \infty} (\log M(r, f)/r)$. Functions for which $T < \infty$ are said to be of exponential type T . We state some properties of functions of M -bounded

index in Theorems 2 and 5 and in remarks following Theorem 4. In Theorem 1 we obtain some inequalities which we require in Theorem 2. In Theorems 3 and 4 we consider related questions on functions of exponential type.

2. Functions of exponential type.

Theorem 1. *Let $R > 0$ be given. An entire function f is of exponential type if and only if there exists a constant $K > 0$ such that for all $r \geq R$*

$$(2.1) \quad M_1(r) \leq KM(r).$$

$$(2.2) \quad \text{If } f \text{ is of exponential type } T, \text{ then } T \leq K.$$

Furthermore, given $\epsilon > 0$, there exists $R_0(\epsilon)$ such that

$$(2.3) \quad M_n(r) \leq (\exp((eT + \epsilon)n))M(r)$$

for $r \geq R_0(\epsilon)$ and $n = 1, 2, 3, \dots$.

Theorem 2. *An entire function f is of exponential type if and only if there exists an integer $N \geq 0$ such that f is of M -bounded index not exceeding N . If f is of exponential type T , then $T \leq N(M) + 1$. This upper bound on T is sharp.*

Let f be of exponential type T . Then the proof of (2.1) implies that there exists a constant K and a sequence $\{R_i\}_{i=0}^\infty$ such that for all $r \geq R_n$,

$$(2.4) \quad M_{n+1}(r) \leq KM_n(r), \quad n = 0, 1, \dots$$

The question arises: Can we find a K and an R such that

$$(2.5) \quad M_{n+1}(r) \leq KM_n(r), \quad \text{for all } r \geq R$$

and $n = 0, 1, 2, \dots$? The next theorem shows that the answer to this question is in the negative.

Theorem 3. *There exists an entire function f of exponential type T such that for each $K > 0$, $R > 0$, there exists an integer $p \geq 0$ and a number $R_p \geq R$ such that*

$$M_{p+1}(R_p) > KM_p(R_p).$$

If we consider, however, the class of functions of exponential type and having positive zeros, then (2.5) is satisfied. Note that if f belongs to this class, then the canonical product formed with the zeros of f is of genus zero.

Theorem 4. *Let $f(z) = e^{\alpha z + \beta} \prod_{n=1}^\infty (1 - z/a_n)$ where $0 < a_1 \leq a_2 \leq \dots$, $\sum (1/a_n) < \infty$ and $\alpha \leq 0$. Then there exists a K such that for all $r \geq 0$,*

$$(i) \quad M_{n+1}(r) \leq KM_n(r),$$

$$(ii) \quad \lim_{r \rightarrow \infty} \frac{M_{n+1}(r)}{M_n(r)} = |\alpha| = T, \quad n = 0, 1, 2, \dots$$

Remarks. We note that functions f of Theorem 4, and in fact functions of exponential type, are all of M -bounded index but not necessarily of bounded index, since f may have multiple zeros. Further, if f and g are of bounded index, then $f + g$ and f' may not be of bounded index ([8], [10]). But it is easy to see that if f and g are of M -bounded index, then so are $f + g$ and f' .

Recently Ekblaw [2] considered the space of entire functions whose topology is defined by the metric d as follows: Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$ and $g(z) = \sum_{n=0}^{\infty} b_n z^n$ be entire functions. Let $d(f, g) = \sup \{|a_0 - b_0|, |a_n - b_n|^{1/n}, n = 1, 2, 3, \dots\}$ (see [3]). Ekblaw proves that in the topology generated by d , the space of entire functions of bounded index is of the first category. We take a different metric and prove

Theorem 5. Let $A = \{f \mid f(z) = e^{\alpha z + \beta} \prod_{k=1}^{\infty} (1 - z/a_k); 0 < |a_1| \leq |a_2| \leq \dots, \sum_{k=1}^{\infty} (1/|a_k|) < \infty\}$. Let

$$\rho(f, g) = |\alpha_f - \alpha_g| + |\beta_f - \beta_g| + \sum_{k=1}^{\infty} |1/a(f)_k - 1/a(g)_k|$$

for $f, g \in A$. Then (A, ρ) is an incomplete metric space of the second category and is topologically complete.

3. Proofs. We require the following

Lemma [5]. Let f be an entire function of order ρ ($0 < \rho < \infty$) and type T ($0 < T < \infty$); then

$$(3.1) \quad 1 \leq \limsup_{r \rightarrow \infty} \frac{M_1(r)}{M(r) \rho T r^{\rho-1}} \leq e.$$

Proof of Theorem 1.

(a) Let $R > 0$ be given and let f be of exponential type T . Suppose first $T > 0$. Then by the lemma, given $\epsilon > 0$, there exist $R_0(\epsilon) > R$ such that for $r \geq R_0(\epsilon)$

$$(3.2) \quad M_1(r) \leq M(r)(eT + \epsilon).$$

Let

$$\sup \left\{ \frac{M_1(r)}{M(r)} \mid R \leq r \leq R_0(\epsilon) \right\} = K_1$$

and

$$K = \max(eT + \epsilon, K_1).$$

Then for $r \geq R$

$$(3.3) \quad M_1(r) \leq KM(r).$$

If $T = 0$, then by a theorem of Bernstein (see the reference in [5]), $M_1(r) \leq \epsilon M(r)$ and again (3.2) and (3.3) hold.

(b) Suppose now that there exist two positive numbers K and R such that $M_1(r) \leq KM(r)$ if $r \geq R$. Write $M'(r) = (d/dr)M(r)$ where it exists. Then

$$\log \frac{M(r)}{M(R)} = \int_R^r \frac{M'(t)}{M(t)} dt \leq \int_R^r \frac{M_1(t)}{M(t)} dt \leq K(r - R).$$

Hence, f is of exponential type T with $T \leq K$.

(c) Let f be of exponential type T . Then by (a) we have, for $d = eT + \epsilon$ and $r \geq R_0(\epsilon)$, that $M_1(r) \leq d M(r)$. Hence, for $r \geq R_0$, $r_1 > 0$, $\log (M(r_1 + r)/M(r)) < dr_1$. Now using Cauchy's integral formula, $M_n(r) \leq (n!/r_1^n)M(r_1 + r) \leq (n!/r_1^n)e^{dr_1}M(r)$. Let $r_1 = n$. Then for $r \geq R_0$ and $n = 1, 2, \dots$

$$(3.4) \quad M_n(r) \leq (\exp((eT + \epsilon)n))M(r).$$

The proof is complete.

Proof of Theorem 2.

(a) Let f be of exponential type T . Then by (3.4) $M_n(r) \leq k^n M(r)$, $k = \exp(eT + \epsilon)$, for $r \geq R_0(\epsilon)$, $n = 1, 2, \dots$. Clearly f is of M -bounded index on a bounded set (cf. [11], pp. 132–133). Hence, f is of M -bounded index.

(b) Let f be of M -bounded index $N(M)$. Then by an argument similar to that given in Theorem 1 of [9], f is of exponential type $T \leq N(M) + 1$.

(c) For the function $f(z) = \exp((N + 1)z)$, $T = N + 1$ and the M -index is N . Hence, the bound on T is sharp. There cannot be an inequality in the reverse direction. The function $f(z) = z^k e^z$ ($k \geq 2$, k an integer) is of M -index k and type $T = 1$.

Proof of Theorem 3. Let $f(z) = \sum_{n=1}^{\infty} a_n z^{2^n}$ where we choose positive a_n 's such that f is of exponential type T . Given $K > 0$, $R > 0$, let q be an integer such that $q > \max(5, K, R)$ and write $p = 2^q + 1$, $R_p = q$. Then

$$M_p(R_p) = \sum_{i=q+1}^{\infty} 2^i (2^i - 1) \cdots (2^i - p + 1) a_i R_p^{2^i - p}$$

and

$$\begin{aligned} M_{p+1}(R_p) &= \sum_{i=q+1}^{\infty} 2^i (2^i - 1) \cdots (2^i - p) a_i R_p^{2^i - p - 1} \\ &\leq \frac{2^{q+1} - p}{R_p} \sum_{i=q+1}^{\infty} 2^i (2^i - 1) \cdots (2^i - p + 1) a_i R_p^{2^i - p}. \end{aligned}$$

Thus

$$\frac{M_{p+1}(R_p)}{M_p(R_p)} \leq \frac{2^{q+1} - p}{R_p} = \frac{2^q - 1}{R_p} > q > K.$$

Proof of Theorem 4. We may suppose without loss of generality that $\beta = 0$. Let $\{a_n^{(k)}\}_{n=1}^{\infty}$ denote the zeros of $f^{(k)}$. Here $f^{(0)} = f$, $a_n^{(0)} = a_n$. By Laguerre's Theorem [1]

$$f^{(k)}(z) = f^{(k)}(0)e^{\alpha z} \prod_{n=1}^{\infty} \left(1 - \frac{z}{a_n^{(k)}}\right)$$

where $a_n^{(k)} \leq a_n^{(k+1)} \leq a_{n+1}^{(k)}$ for all n, k . Thus $\sum_{n=1}^{\infty} (1/a_n^{(k)}) \leq \sum_{n=1}^{\infty} (1/a_n) = A$, say. For $r \geq 0$ and $k = 0, 1, \dots$,

$$\begin{aligned} \frac{M_{k+1}(r)}{M_k(r)} &= \frac{|f^{(k+1)}(-r)|}{|f^{(k)}(-r)|} = \left| \alpha + \sum_{n=1}^{\infty} \frac{1}{-r - a_n^{(k)}} \right| \\ &= |\alpha| + \sum_{n=1}^{\infty} \frac{1}{r + a_n^{(k)}} \leq |\alpha| + \sum_{n=1}^{\infty} \frac{1}{a_n^{(k)}} \leq \alpha + A = K. \end{aligned}$$

Further,

$$\lim_{r \rightarrow \infty} \frac{M_{k+1}(r)}{M_k(r)} = |\alpha| + \lim_{r \rightarrow \infty} \sum_{n=1}^{\infty} \frac{1}{a_n^{(k)} + r} = |\alpha| = T$$

for all k . The theorem is proved.

To prove Theorem 5, two lemmas are required. Let $A^* = A \cup \{f \mid f(z) = e^{\alpha z + \beta} \prod_{k=1}^N (1 - z/a_k) \text{ where } N \text{ is a positive integer, } 0 < |a_1| \leq |a_2| \leq \dots \leq |a_N|, \alpha, \beta \in \mathbf{C}\} \cup \{f \mid f(z) = e^{\alpha z + \beta}, \alpha, \beta \in \mathbf{C}\}$. We make the agreement to write $f(z) = e^{\alpha z + \beta} \prod_{k=1}^N (1 - z/a_k)$ as $e^{\alpha z + \beta} \prod_{k=1}^{\infty} (1 - z/a_k)$, with $a_k = |a_k| = \infty$ for $k \geq N + 1$ and $1/a_k = 1/|a_k| = 0$ for $k \geq N + 1$, and $e^{\alpha z + \beta}$ as

$$e^{\alpha z + \beta} \prod_{k=1}^{\infty} (1 - z/a_k)$$

with all $a_k = \infty$ and $1/a_k = 0$. Then

$$A^* = \{f \mid f(z) = e^{\alpha z + \beta} \prod_{k=1}^{\infty} (1 - z/a_k)\}$$

where $a_k \in \mathbf{C}$ or $a_k = \infty$, $0 < |a_1| \leq |a_2| \leq \dots \leq \infty$, $\sum_{k=1}^{\infty} (1/|a_k|) < \infty$, and ρ as defined earlier is a metric on A^* .

Lemma 1. *The space (A^*, ρ) is a complete metric space.*

Proof. Let $\{f_n\}_{n=1}^{\infty}$, $f_n(z) = e^{\alpha_n z + \beta_n} \prod_{k=1}^{\infty} (1 - z/a_k^{(n)})$, be a Cauchy sequence in A^* . Then given $\epsilon > 0$ there is an $N = N(\epsilon)$ such that $\rho(f_m, f_n) < \epsilon$ if $m, n > N(\epsilon)$. That is,

$$|\alpha_m - \alpha_n| + |\beta_m - \beta_n| + \sum_{k=1}^{\infty} \left| \frac{1}{a_k^{(m)}} - \frac{1}{a_k^{(n)}} \right| < \epsilon \quad \text{if } m, n > N(\epsilon).$$

Hence, $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$, and for each k , $\{1/a_k^{(n)}\}_{n=1}^{\infty}$ are Cauchy sequences. Let $\lim_{n \rightarrow \infty} \alpha_n = \alpha$, $\lim_{n \rightarrow \infty} \beta_n = \beta$, and $\lim_{n \rightarrow \infty} (1/a_k^{(n)}) = 1/A_k$. Then for each k , $0 < |A_k| \leq \infty$. Since for all n and k , $|a_k^{(n)}| \leq |a_{k+1}^{(n)}|$, we have $\lim_{n \rightarrow \infty} |a_k^{(n)}| = |A_k| \leq \lim_{n \rightarrow \infty} |a_{k+1}^{(n)}| = |A_{k+1}|$. We now show that $\sum_{k=1}^{\infty} 1/|A_k| < \infty$. Observe that

$$\sum_{k=1}^{\infty} \frac{1}{|a_k^{(n)}|} - \sum_{k=1}^{\infty} \frac{1}{|a_k^{(m)}|} \leq \sum_{k=1}^{\infty} \left| \frac{1}{a_k^{(n)}} - \frac{1}{a_k^{(m)}} \right| < 1$$

whenever $m, n > N = N(1)$. Then, for $m = N + 1$, $\sum_{k=1}^{\infty} (1/|a_k^{(n)}|) < \sum_{k=1}^{\infty} (1/|a_k^{(m)}|) + 1 \equiv B$ for all $n > N$. Thus,

$$\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} (1/|a_k^{(n)}|) = \sum_{k=1}^{\infty} \lim_{n \rightarrow \infty} (1/|a_k^{(n)}|) = \sum_{k=1}^{\infty} (1/A_k) \leq B.$$

Hence, for $f(z) = e^{\alpha z + \beta} \prod_{k=1}^{\infty} (1 - z/A_k)$, we have $f \in A^*$.

We now show $f_n \rightarrow f$ with respect to ρ as $n \rightarrow \infty$. Let $\epsilon > 0$ be given. Then there exists an integer $N = N(\epsilon/3)$ such that $\rho(f_n, f_m) \leq \epsilon/3$ for all $m, n > N$. Clearly $|\alpha_n - \alpha_m| \leq \epsilon/3$, $|\beta_n - \beta_m| \leq \epsilon/3$ and $\sum_{k=1}^{\infty} |1/a_k^{(n)} - 1/a_k^{(m)}| \leq \epsilon/3$. Let $m > N$ be fixed; thus $\lim_{n \rightarrow \infty} |\alpha_n - \alpha_m| = |\lim_{n \rightarrow \infty} (\alpha_n - \alpha_m)| = |\alpha - \alpha_m| \leq \epsilon/3$, $\lim_{n \rightarrow \infty} |\beta_n - \beta_m| = |\beta - \beta_m| \leq \epsilon/3$ and $\lim_{n \rightarrow \infty} \sum_{k=1}^{\infty} |1/a_k^{(n)} - 1/a_k^{(m)}| = \sum_{k=1}^{\infty} \lim_{n \rightarrow \infty} |1/a_k^{(n)} - 1/a_k^{(m)}| = \sum_{k=1}^{\infty} |1/A_k - 1/a_k^{(m)}| \leq \epsilon/3$. Therefore $\rho(f, f_m) \leq \epsilon$ for any $m > N$. Thus, (A^*, ρ) is a complete metric space.

Lemma 2. For each positive integer N , let $P_N = \{f \mid f(z) = e^{\alpha z + \beta} \prod_{k=1}^M (1 - z/a_k) \text{ where } 0 < |a_1| \leq |a_2| \leq \dots \leq |a_M| < \infty, \alpha, \beta \in \mathbf{C}, M \leq N\} \cup \{f \mid f(z) = e^{\alpha z + \beta}, \alpha, \beta \in \mathbf{C}\}$. If $P = \bigcup_{N=1}^{\infty} P_N$, then P is a first category subset of A^* .

Proof. We show each P_N is a closed subset of A^* by showing $A^* \setminus P_N$ is an open subset of A^* . Let $f(z) = e^{\alpha z + \beta} \prod_{k=1}^{\infty} (1 - z/a_k) \in A^* \setminus P_N$. Then $|a_{N+1}| < \infty$. Choose ϵ such that $0 < \epsilon < 1/|a_{N+1}|$. If $g \in P_N$, then $\rho(f, g) \geq \sum_{k=N+1}^{\infty} (1/|a_k|) \geq 1/|a_{N+1}| > \epsilon$. So if $h \in A^*$ and $\rho(f, h) < \epsilon$, $h \in A^* \setminus P_N$. Hence, $A^* \setminus P_N$ is an open subset of A^* . We now show for each positive integer N , that P_N is nowhere dense in A^* . Let $\epsilon > 0$ and $g(z) \in P_N$ be given. Then choose $\delta > 0$ such that $\delta < \min \{\epsilon, 1/|a_1|, \dots, 1/|a_M|\}$, where $a_1, \dots, a_M$ are the finite zeros of $g(z)$ (if g has no zeros, choose $\delta < \epsilon$). Now for $f(z) = g(z) \prod_{k=1}^{\infty} (1 - \delta z/2^k)$ we have $f(z) \in A^* \setminus P_N$ and $\rho(f, g) = \sum_{k=1}^{\infty} (\delta/2^k) = \delta < \epsilon$. Thus, $P = \bigcup_{N=1}^{\infty} P_N$ is a countable union of nowhere dense subsets of A^* . Hence, P is a first category subset of A^* .

Proof of Theorem 5. To show (A, ρ) is an incomplete metric space, consider the sequence $\{f_n\}_{n=1}^{\infty}$, $f_n(z) = \prod_{k=1}^{\infty} (1 - z/(n+1)^k)$. Each $f_n \in A$ and $\rho(f_n, f_m) = \sum_{k=1}^{\infty} |1/(n+1)^k - 1/(m+1)^k| = |1/n - 1/m|$. Hence, $\{f_n\}_{n=1}^{\infty}$ is a Cauchy sequence in A . But for $g(z) \equiv 1 \notin A$, $\rho(f_n, g) = \sum_{k=1}^{\infty} (1/(n+1)^k) = 1/n$ and $f_n \rightarrow g$ as $n \rightarrow \infty$.

Using the Baire category theorem we have, by Lemma 1, (A^*, ρ) is a second category space. Now, $A^* = A \cup P$ and P is of the first category in A^* . Therefore A must be a second category subset of A^* and (A, ρ) is a second category space.

By Alexandroff's Theorem [7], any non-empty G_δ subset of a complete metric space is topologically complete, that is, the subset can be remetrized so as to be complete. (Remetrized meaning a new metric ρ' can be defined on A that induces the same topology as ρ , and (A, ρ') is complete.) Since $A = \bigcap_{N=1}^{\infty} A^* \setminus P_N$, it follows that A is a G_δ set.

4. Remarks and examples.

(i) Let f be of bounded index N . Denote by ν the central index of f . Then we have [9]

$$(4.1) \quad \limsup_{r \rightarrow \infty} \frac{\nu(r, f)}{r} \leq N + 1.$$

W. K. Hayman has recently proved [W. K. Hayman, Differential inequalities and local valency, to appear] the following sharp inequality:

$$(4.2) \quad |f(z)| \leq \exp((N + 1)|z|) \max_{0 \leq i \leq N} \frac{|f^{(i)}(0)|}{(N + 1)^i}.$$

It may be noted that there are functions f which satisfy (4.1) but not (4.2) and vice-versa. Furthermore there exist functions f which satisfy (4.1) and (4.2) for $N > N_0(f)$ but are of unbounded index.

Example 1. $f(z) = \exp((N + 1)z) + cz^{N+1}$, $c > 0$. Here (4.1) holds but not (4.2).

Example 2. $f(z) = 1 + \sum_{n=1}^{\infty} (z^{\lambda_n}/\lambda_n!)$, where $1 \leq \lambda_1 < \lambda_2 < \dots$, $\lambda_{n+1}/\lambda_n \rightarrow a > 1$. Here (4.2) holds, with $N = 0$, but not (4.1).

Example 3. Let $a_1 \geq 2$, $a_{k+1} \geq \max\{3(k+1)a_k, a_k^{(k+1)/k}\}$, $k \geq 1$, $\operatorname{Im} c \neq 0$, $\{1 + |c|/|1 - c| \leq e$, and

$$f(z) = \prod_{m=1}^{\infty} \left(1 - \frac{z}{a_m}\right)^m - c.$$

Then f is an entire function of zero order and has simple zeros only. Let

$$N_0 = \max \left\{ 1, \max_{|z|=1} \left| \frac{f(z)}{f(0)} - 1 \right| \right\}.$$

Then for any integer $N > N_0$, (4.1) and (4.2) are satisfied but f is not of bounded index.

(ii) Let f be of M -bounded index $N(M)$. Then we have

$$(4.3) \quad \limsup_{r \rightarrow \infty} \frac{\nu(r, f)}{r} \leq N(M) + 1.$$

The proof is similar to that of (4.1). Also following Hayman's proof of (4.2), we can show that

$$(4.4) \quad M(r, f) \leq \exp((N(M) + 1)r) \max_{0 \leq i \leq N(M)} \frac{|f^{(i)}(0)|}{(N(M) + 1)^i}.$$

However there exist functions satisfying (4.1) and (4.2) for some integer $N \geq 0$ but which are of M -bounded index exceeding N .

Example 4. $f(z) = (7/6)z^2 + 1$. Here (4.1) and (4.2) hold for $N = 0$ but $M_1(1) > M(1)$; in fact, $f(z)$ is of M -bounded index 2. $f(z)$ can also be replaced by the transcendental entire function $g(z) = f(z) + \sum_{n=4}^{\infty} (z^n/n!)$.

It is obvious that N , in Example 1, is not the index of the function f defined there. We have chosen N such that for f , (4.1) holds and not (4.2). A similar remark applies to the remaining examples 2 to 4.

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