Last Exit Decompositions and Distributions

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1. Introduction. During the early years of modern Markov process theory great emphasis was placed on the study of first entrance times and distributions. This was only natural in view of their connection with harmonic measures and “reduites” in potential theory, and also, because of the central role played by the strong Markov property during those early years. However, in recent years it has become increasingly clear that last exit times and distributions are just as important not only probabilistically but also potential theoretically. It was Chung who insisted on the importance of last exit times, and he demonstrated their power especially within the framework of Markov chains.

In a recent paper, [4], we pointed out that there was a close relationship between the balayage of additive functionals and last exit distributions. In particular we were able to derive the “last exit decomposition” for a finite set. In the present paper we extend these results and methods to finely closed nearly Borel sets.

In Section 2 we develop a theory of “adapted” balayage of raw (*i.e.*, non-adapted) additive functionals on a set F . This has the same relationship to the ordinary balayage as the notion of well measurable dual projection has to previsible dual projection in the general theory of processes. These results are then applied in Section 3 to obtain “last exit decompositions”. Actually there are two such decompositions. If F is a finely closed nearly Borel set and

$$(1.1) \quad L^t = \sup \{s \leq t : X_s \in F\},$$

then, roughly speaking, one is a decomposition according to the position of $X(L^t)$ and the other is a decomposition according to the position of $X(L^t -)$. In both cases the essential ingredients of the decomposition are an additive functional (adapted in the first case, natural in the second) and an entrance law for the process (X, T_F) —that is, X is killed when it first hits F . The main results of Section 3 are summarized in Theorems 1.3 and 1.8 of this introduction. When F is finite (and each point in F is regular for itself) Theorem 1.8 reduces to the familiar “first entrance–last exit decomposition”. See [4], [7], [12], and [13]. A somewhat more general result is contained in (3.30).

With the time L' one associates two σ -fields $\mathfrak{F}(L' -)$ and $\mathfrak{F}(L')$. See (1.10) and (1.11) for the definitions. Given the results of Section 3, it is very easy to obtain certain conditional distributions given these fields. The main results are contained in Theorems 1.12 and 1.13 of this introduction. See, also, (4.8), (4.11), and (4.12).

If $L = \sup \{s : X_s \in F\}$ is the last exit time from F , then $L = \lim_{t \rightarrow \infty} L'$. If F is transient in the sense that $L < \infty$, then by letting $t \rightarrow \infty$ in our results about L' we obtain information about the "post L process". Section 5 contains a sampling of such results.

Recently Pittenger and Shih [8] have obtained results very similar to Theorems 1.12 and 1.13. However, their methods are very different from ours. In particular we feel that we obtain more information about the various quantities involved and that our methods are simpler and more probabilistic. Pittenger and Shih deal with an axiomatically defined system (L') rather than with (1.1), but our methods clearly carry over to this situation. More importantly, Pittenger and Shih allow the basic process X to have branch points while we assume that X is a Hunt process with a reference measure for simplicity. Undoubtedly, our methods extend to more general processes but we have not pursued this point.

For the convenience of the reader we shall now describe in some detail the main results of Sections 3 and 4. We assume throughout this paper that $X = (\Omega, \mathfrak{F}, \mathfrak{F}_t, X_t, \theta_t, P^x)$ is a Hunt process with state space $(E, \mathcal{E})$, and that there exists a σ -finite reference measure ξ . See [1] for all unexplained notation and terminology. It is convenient to assume that ξ is in fact excessive. (This last assumption is harmless in that a 1-excessive reference measure certainly exists, and the appropriate arguments could be made on the 1-subprocess, with the same conclusions. These arguments occur only in Section 2.) We fix once and for all a finely closed nearly Borel set F . We let $T = T_F = \inf \{t > 0 : X_t \notin F\}$, $T' = T(t) = t + T \circ \theta_t = \inf \{s > t : X_s \notin F\}$, and $L' = L(t) = \sup \{s \leq t : X_s \in F\}$, where, by convention the infimum of the empty set is infinite and the supremum of the empty set is zero. We let (Q_t) and (V^α) denote respectively the semigroup and the resolvent of the process (X, T_F) . That is, for $f \in b\mathcal{E}$

$$(1.2) \quad Q_t f(x) = E^x \{f(X_t); t < T\}; \quad V^\alpha f(x) = E^x \int_0^\infty e^{-\alpha t} f(X_t) dt.$$

Recall that a family $\{\gamma_t(\cdot)\}_{t \geq 0}$ of finite measures on E is called an entrance law for (Q_t) provided $\gamma_t Q_s = \gamma_{t+s}$ for all $t, s \geq 0$. In particular, each γ_t is carried by $E - F^r$ where F^r is the set of regular points for F . The entrance law (γ_t) is *integrable* provided $\int_0^\infty \gamma_t(1) dt < \infty$. We are now in position to state our first result.

(1.3) Theorem. *For each x in E there exists an integrable entrance law $\{\gamma_t(x, \cdot)\}$ for (Q_t) such that $(t, x) \rightarrow \gamma_t(x, h)$ is $\mathcal{B} \times \mathcal{E}$ measurable for each $h \in b\mathcal{E}$ where $\mathcal{B}$ is the σ -algebra of Borel subsets of $\mathbf{R}^+ = [0, \infty)$. In addition, there exists an adapted*

additive functional, $\tilde{A}$, with finite one potential such that if Z is a bounded well measurable process and h is in $b\mathcal{E}$, then

$$(1.4) \quad E^x\{Z_{L(t)}h(X_t); 0 < L(t) < t\} = E^x \int_{(0,t)} Z_s \gamma_{t-s}(X_s, h) d\tilde{A}_s.$$

In Theorem 1.3, $\tilde{A}$ is the adapted 1-balayage (defined in Section 2) of $t \rightarrow t \wedge \zeta$ on the set F . If $x \in E - F^r$, then $\gamma_t(x, h) = Q_t h(x)/V^1 1(x)$ and it is only for $x \in F^r$ that $\gamma_t(x, \cdot)$ may be a nontrivial entrance law. An immediate consequence of (1.3) is the “last exit” decomposition (3.19).

We turn now to the second main result of Section 3. Roughly speaking this amounts to taking the “previsible dual projection” of Theorem 1.3. We begin by writing

$$\tilde{A} = C + D + \tilde{B}$$

where C is a continuous additive functional (CAF), D is a purely discontinuous natural additive functional (NAF), and $\tilde{B}$ is a purely discontinuous quasi-left-continuous (qlc) additive functional. This decomposition of $\tilde{A}$ is unique. We let B be the natural (or previsible) dual projection of $\tilde{B}$, that is, B is the unique NAF having the same 1-potential as $\tilde{B}$. Since $\tilde{B}$ is qlc it follows that B is a CAF. Finally let $A = C + D + B$. Then A is the natural dual projection of $\tilde{A}$ and is, in fact, the ordinary 1-balayage of $t \rightarrow t \wedge \zeta$ on F .

It is known that

$$(1.5) \quad \tilde{B}_t = \sum_{s \leq t} G(X_{s-}, X_s)$$

for some nonnegative $\mathcal{E} \times \mathcal{E}$ measurable G vanishing on the diagonal of $E \times E$. In the present situation $G(x, y) = 0$ if $y \in F^r$. See (3.7) for another form of $\tilde{B}$. In the special, but important, case when F is finely perfect one may take $G(x, y) = 1_F(x)V^1 1(y)$. Let $n(x, dy)$ be a kernel and H a CAF such that (n, H) is a Lévy system for X . See [6]. It then follows from (1.5) that

$$B_t = \int_0^t nG(X_s) dH_s$$

where $nG(x) = \int n(x, dy)G(x, y)$.

We introduce the following condition on the set F .

(1.6) *The continuous part, $B + C$, and the discontinuous part, D , of A are carried by disjoint Borel sets.*

It turns out that the following condition is equivalent to (1.6).

(1.7) *There exists a Borel measurable φ with $0 \leq \varphi \leq 1$ such that $B = \varphi * A$.*

Here $\varphi * A$ denotes the functional $(\varphi * A)_t = \int_{(0,t)} \varphi(X_s) dA_s$. If F is projective, then $D = 0$ and so (1.6) holds. Recall that F is *admissible* provided $T_F = \inf\{t > 0 : X_{t-} \in F\}$ almost surely on $\{T_F < \zeta\}$. It was shown in [3] that if F

is admissible then D is carried by $E - F^r$ and $B + C$ is carried by F^r , so that (1.6) holds for admissible F . It is known that any finely perfect set is admissible, and that if X satisfies Hunt's hypothesis (B)—in particular under duality assumptions—every nearly Borel set is admissible. See (4.13) for an example of a finely closed set F that does not satisfy (1.6).

(1.8) Theorem. Assume that (1.6) holds. Using the above notation define (set $0/0 = 0$)

$$\bar{n}(x, dy) = \frac{\varphi(x)G(x, y)}{nG(x)} n(x, dy),$$

$$r(t, x, \Phi) = (1 - \varphi(x)) \int \Phi(x, y) \gamma_t(x, dy) + \int \bar{n}(x, dy) \int \Phi(y, z) \gamma_t(y, dz)$$

for $\Phi \in b(\mathcal{E} \times \mathcal{E})$. Then if Y is a bounded previsible process and $\Phi \in b(\mathcal{E} \times \mathcal{E})$ one has

$$(1.9) \quad E^x \{ Y_{L(t)} \Phi[X_{L(t)}, X_t]; 0 < L(t) < t \} = E^x \int_{(0, t)} Y_s r(t-s, X_s, \Phi) dA_s.$$

This leads to the “last exit decomposition” (3.24). It is this formula which reduces to the usual first entrance last exit decomposition (3.30) when F is finite and each point in F is regular. See [4], [7], [12], and [13].

If $R \geq 0$ is $\mathcal{F}$ measurable, then one associates with R two σ -algebras $\mathcal{F}(R-)$ and $\mathcal{F}(R)$ defined as follows.

(1.10) $\mathcal{F}(R-)$ is the σ -algebra generated by $\mathcal{F}_0$ and all sets of the form $\Lambda \cap \{R > t\}$, $\Lambda \in \mathcal{F}_t$, $t \geq 0$.

(1.11) $\mathcal{F}(R)$ is the σ -algebra generated by $\mathcal{F}(R-)$ and all sets of the form $\Lambda \cap \{R = T\}$, $\Lambda \in \mathcal{F}_T$, T a stopping time.

If R is a stopping time, then these σ -algebras reduce to the usual fields denoted by $\mathcal{F}_{R-}$ and $\mathcal{F}_R$. To the best of our knowledge $\mathcal{F}(R)$ was first introduced by Pittenger and Shih [8]. Lemma 4.1 contains a useful characterization of $\mathcal{F}(R-)$ and $\mathcal{F}(R)$.

In the results of Section 4, ζ plays no role and it is convenient to assume that $\zeta = \infty$. (This can always be accomplished by considering E_Δ as the state space of the process X .) Therefore in the next two theorems it is assumed that $\zeta = \infty$.

(1.12) Theorem. For $t > 0$, $x \in E$, and $h \in b\mathcal{E}$ define $k(t, x, h) = \gamma_t(x, h)/\gamma_t(x, 1)$ where $0/0 = 0$. Then

$$E^x \{ h(X_t) \mid \mathcal{F}(L') \} = k(t - L(t), X_{L(t)}, h)$$

on $\{0 < L(t) < t\}$.

(1.13) Theorem. For $t > 0$, $x \in E$, and $\Phi \in b(\mathcal{E} \times \mathcal{E})$ define $K(t, x, \Phi) = r(t, x, \Phi)/r(t, x, 1)$ where $0/0 = 0$. If (1.6) holds, then

$$E^x\{\Phi[X_{L(t)}, X_t] \mid \mathfrak{F}(L^t-)\} = K(t - L(t), X_{L(t)-}, \Phi)$$

on $\{0 < L(t) < t\}$.

In particular when $\Phi(x, y) = f(x)$ or $\Phi(x, y) = h(y)$ we obtain the conditional distributions of $X_{L(t)}$ or X_t given $\mathfrak{F}(L^t-)$. See (4.8). We emphasize the fact that $k(t, x, h)$ (respectively $K(t, x, \Phi)$) is a kernel in the sense that $(t, x) \rightarrow k(t, x, h)$ (respectively $(t, x) \rightarrow K(t, x, \Phi)$) is $\mathfrak{B} \times \mathfrak{E}$ measurable for $h \in b\mathfrak{E}$ (respectively $\Phi \in b(\mathfrak{E} \times \mathfrak{E})$) and that for fixed (t, x) , $k(t, x, \cdot)$ (respectively $K(t, x, \cdot)$) is a subprobability measure on $(E, \mathfrak{E})$ (respectively $(E \times E, \mathfrak{E} \times \mathfrak{E})$). The relationship between k and K is given in (4.11). Finally Example (4.13) shows that Theorem 1.13 may fail if (1.6) does not hold.

Throughout this paper we use the notation and terminology of [1]. In particular the state space E of our process X is a locally compact space with a countable base, and $\mathfrak{E}$, $\mathfrak{E}^*$, and $\mathfrak{E}''$ denote respectively the σ -algebras of Borel, universally measurable, and nearly Borel subsets of E . We write $f \in b\mathfrak{E}$ ($f \in \mathfrak{E}_+$) to mean that f is a bounded (nonnegative) Borel function, and similarly for $\mathfrak{E}^*$ and $\mathfrak{E}''$. We shall often, but not always, omit the phase "almost surely" where it is obviously required.

It is a pleasure for us to acknowledge our debt to J. Azema. Even though this paper contains no explicit references to Azema's work, his ideas have greatly influenced our point of view and our approach to the problems treated here.

2. Adapted balayage of additive functionals. As mentioned in the introduction, we shall assume throughout this paper that $X = (\Omega, \mathfrak{F}, \mathfrak{F}_t, X_t, \theta_t, P^x)$ is a Hunt process, and that there exists a σ -finite reference measure, ξ , on its state space $(E, \mathfrak{E})$. See [1]. It is convenient to assume that ξ is in fact excessive. Let ζ denote the lifetime of X .

In applying the general theory of processes in the present situation, it should be remarked that a process Y over $(\mathfrak{F}_t)$ is called previsible if, for every initial measure μ , there exists a process Y^μ which is previsible over $(\Omega, \mathfrak{F}_t^\mu, P^\mu)$ such that Y and Y^μ are P^μ -indistinguishable. The same modifications are made for well measurable processes. We shall ignore the trivialities of making such modifications in all subsequent arguments. Since X is a Hunt process, accessibility and previsibility are identical conditions because $(\mathfrak{F}_t)$ contains no time of discontinuity. With regard to the general theory of processes we adopt the terminology of [2]. In particular if $A = (A_t)$ is an increasing process ([2], p. 86) there exists a unique increasing well measurable process A^1 such that

$$E \int_0^\infty Z_s dA_s = E \int_0^\infty Z_s dA_s^1$$

for all nonnegative well measurable Z . The process A^1 is called the *well measurable dual projection* of A . The previsible dual projection is defined in a similar manner. See Section V-3 of [2].

We shall use the term *raw additive functional* (RAF) for a process on $(\Omega, \mathfrak{F})$ which possesses all properties of an additive functional except that it need not be adapted. (Perhaps the term additive functional is more appropriate here, with adapted additive functional being used for the more familiar objects. It is now clear that non-adapted functionals are very useful and not at all exotic.) If A is a RAF of X having finite α -potential $u_A^\alpha(x) = E^x \int_0^\infty e^{-\alpha t} dA_t$, there exists a unique NAF of X having the same α -potential. This NAF is independent of α and is indistinguishable from the previsible dual projection of the increasing process A relative to $(\Omega, \mathfrak{F}_t^\mu, P^\mu)$ for each μ . We call it the *natural dual projection* of A . The first business of this section is to show the existence of an AF, A' of X which is the well measurable dual projection of A relative to $(\Omega, \mathfrak{F}_t^\mu, P^\mu)$ for each μ .

If A is a RAF and $f \in b\mathcal{E}_+^*$, we let $f * A$ denote the RAF given by $(f * A)_t = \int_0^t f(X_s) dA_s$.

(2.1) Definition. If A is a RAF of X which is finite on $[0, \zeta)$, let $A_t^n = \int_0^t 1_{(X_s = X_{s-})} dA_s$ and $A_t^{\alpha} = \int_0^t 1_{(X_s \neq X_{s-})} dA_s$. The α -bipotential operator $\tilde{U}_A^\alpha$ of A is defined on $(\mathcal{E}^* \times \mathcal{E}^*)_+$ by setting, for $G \in (\mathcal{E}^* \times \mathcal{E}^*)_+$

$$(2.2) \quad \tilde{U}_A^\alpha G(x) = E^x \int_0^\infty e^{-\alpha t} G(X_{t-}, X_t) dA_t.$$

Finally, define the positive functional $\tilde{\nu}_A$ on $(\mathcal{E}^* \times \mathcal{E}^*)_+$ by

$$(2.3) \quad \tilde{\nu}_A(G) = \lim_{\alpha \rightarrow \infty} \int \alpha \tilde{U}_A^\alpha G(x) \xi(dx).$$

As in [10], the limit term in (2.3) increases with α . For $f \in \mathcal{E}_+^*$, let $\nu_A(f) = \tilde{\nu}_A(f \otimes 1)$ where $(f \otimes g)(x, y) = f(x)g(y)$. If $A = A^\alpha$ and A has finite α -potential u_A^α for some $\alpha \geq 0$, then u_A^α is obviously a regular α -potential so the natural dual projection B of A is a CAF. Since $s \rightarrow f(X_{s-})$ is previsible one has $\tilde{U}_A^\alpha(f \otimes 1)(x) = \tilde{U}_B^\alpha(f \otimes 1)(x) = U_B^\alpha f(x)$, and by [9], Theorem III.1 ν_A may be identified with a σ -finite measure on $\mathcal{E}^*$. It follows that $\tilde{\nu}_A$ may be identified with a σ -finite measure on $\mathcal{E}^* \times \mathcal{E}^*$ if $A = A^\alpha$.

(2.4) Proposition. If A and A' are CAF's of X , both finite on $[0, \zeta)$, and $\nu_A = \nu_{A'}$, then $A = A'$. If B and B' are AF's of X with finite α -potential for some $\alpha \geq 0$, and if $B = B^\alpha$ and $B' = (B')^\alpha$, then $\tilde{\nu}_B = \tilde{\nu}_{B'}$ implies $B = B'$.

Proof. The first assertion is Remark 3 of [9], II.1. For $f \in b\mathcal{E}_+^*$, the hypotheses imply $\tilde{\nu}_{f*B} = \tilde{\nu}_{f*B'}$ and consequently $\nu_{f*B} = \nu_{f*B'}$. It follows from the first assertion that $f * B$ and $f * B'$ have the same natural dual projection. Therefore $U_B^\alpha f(x) = U_{B'}^\alpha f(x)$ for all $x \in E$ and all $f \in b\mathcal{E}_+^*$ and so $B = B'$ by (IV-2.12) of [1].

We define now a canonical measure $\tilde{\nu}$ on $\mathcal{E} \times \mathcal{E}$ in the following way. One may cover $(E \times E)_0 = \{(x, y) \in E \times E: x \neq y\}$ by a countable union of Borel product sets of the form $J \times K$ having the following properties:

- (2.5) (i) J and K have disjoint compact closures and J is finely closed;
(ii) $\sup \{E^x e^{-T_K} : x \in J\} < 1$.

To obtain such a covering, firstly obtain a cover by sets $J \times K$ where J and K are disjoint compact sets and then write $J = \bigcup_{n=1}^{\infty} J \cap \{x : E^x e^{-T_K} \leq 1 - 1/n\}$ to obtain sets satisfying (2.5). If J and K satisfy (2.5) then, since $\inf \{t > 0 : X_{t-} \in J\} \geq T_J$ and $T_J \neq T_K$, we have

$$T_{J \times K} = \inf \{t > 0 : (X_{t-}, X_t) \in J \times K\} \geq T_J + T_K \circ \theta_{T_J}$$

and therefore $\sup E^x [\exp(-T_{J \times K})] \leq \sup E^x \{\exp(-T_J) E^{X(T_J)} [\exp(-T_K)]\} < 1$. Letting $T = T_{J \times K}$ and $T^{n+1} = T^n + T \circ \theta_{T^n}$ we see that $\sup E^x \sum e^{-T^n} < \infty$ so $A_t(J \times K) = \sum_{s \leq t} 1_{J \times K}(X_{s-}, X_s)$ defines an AF, $A(J \times K)$ with bounded one potential. One may define then a σ -finite measure $\tilde{\nu}^{J \times K}$ on $E \times E$ by $\tilde{\nu}^{J \times K} = \tilde{\nu}_{A(J \times K)}$. Obviously $\tilde{\nu}^{J \times K}$ is carried by $J \times K$. If Γ is a Borel set contained in $J \times K$, one has

$$\tilde{U}_{A(J \times K)}^{\alpha} 1_{\Gamma}(x) = E^x \sum 1_{\Gamma}(X(T^n-), X(T^n)) e^{-\alpha T^n} = E^x \sum e^{-\alpha R^n},$$

where $R = \inf \{t > 0 : (X_{t-}, X_t) \in \Gamma\}$ and R^n is its n -th iterate. Consequently $\tilde{\nu}^{J \times K}(\Gamma) = \tilde{\nu}^{J' \times K'}(\Gamma)$ if $\Gamma \subset (J \times K) \cap (J' \times K')$. Thus there exists a σ -finite measure $\tilde{\nu}$ on $E \times E$ which does not charge the diagonal and whose restriction to $J \times K$ is $\tilde{\nu}^{J \times K}$ for all products $J \times K$ satisfying (2.5). A simple argument shows that if $G \in (\mathcal{E}^* \times \mathcal{E}^*)_+$ vanishes on the diagonal and if $A_t(G) = \sum_{s \leq t} G(X_{s-}, X_s)$ defines an AF, $A(G)$, then $d\tilde{\nu}_{A(G)} = G d\tilde{\nu}$. For a set $\Gamma \in \mathcal{E} \times \mathcal{E}$, the symbols $A(\Gamma)$ and $A(1_{\Gamma})$ are used interchangeably.

(2.6) Proposition. *If B is a RAF of X having finite α -potential for some $\alpha \geq 0$, then there exists a unique AF, B' , of X having the same α -potential operator as B . In fact, B' is the well measurable dual projection of the increasing process B relative to $(\Omega, \mathfrak{F}_t^{\mu}, P^{\mu})$ for every initial measure μ . We call B' the adapted dual projection of B .*

Proof. It is sufficient to give proofs separately for the cases $B = B^n$ and $B = B^q$. Suppose firstly that $B = B^n$. We define B' to be the natural dual projection of B . It must be shown that for every positive well measurable Z and all $x \in E$

$$(2.7) \quad E^x \int_0^{\infty} Z_t dB_t = E^x \int_0^{\infty} Z_t dB'_t.$$

It is known ([2], V, T19) that there exists a positive accessible, hence previsible, process Y such that $\{(t, \omega) : Y_t(\omega) \neq Z_t(\omega)\}$ is contained in the union of graphs of a countable number of totally inaccessible stopping times. Neither B nor B' charges totally inaccessible stopping times and so Z may be replaced by Y in each side of (2.7), proving the result in the case $B = B^n$.

Assume now that $B = B^q$. One has $\tilde{\nu}_B \ll \tilde{\nu}$, where $\tilde{\nu}$ is defined in the paragraph preceding (2.6), because if $J \times K$ satisfies (2.5) and $\Gamma \subset J \times K$ with $\tilde{\nu}(\Gamma) = 0$,

then $u_{A(\Gamma)}^\alpha = 0$ so $T_\Gamma = \inf \{t > 0 : (X_{t-}, X_t) \in \Gamma\} = \infty$ almost surely, and this implies $\tilde{\nu}_B(\Gamma) = 0$. We may write then $d\tilde{\nu}_B = Gd\tilde{\nu}$ where $G \in (\mathcal{E} \times \mathcal{E})_+$ is finite and vanishes on the diagonal. Let $\Gamma_n = \{(x, y) : G(x, y) \leq n\}$. For a fixed product set $J \times K$ satisfying (2.5), $\Lambda = \Lambda_n = \Gamma_n \cap (J \times K)$ has the property that

$$E^x \sum_s e^{-s} G(X_{s-}, X_s) 1_\Lambda(X_{s-}, X_s) \leq n E^x \sum_s e^{-s} 1_{J \times K}(X_{s-}, X_s) < \infty$$

for all $x \in E$. It follows that $A_t(G1_\Lambda) = \sum_{s \leq t} (G1_\Lambda)(X_{s-}, X_s)$ defines an AF and $d\tilde{\nu}_{A(G1_\Lambda)} = G1_\Lambda d\tilde{\nu} = 1_\Lambda d\tilde{\nu}_B$. Thus $A(G1_\Lambda)$ and $1_\Lambda * B$ have the same associated measure. If $H \in b(\mathcal{E} \times \mathcal{E})_+$, $H * A(G1_\Lambda)$ and $H * (1_\Lambda * B)$ have the same associated measure and therefore have identical natural dual projections. This means that for every bounded previsible process Y , and $x \in E$

$$E^x \sum_s e^{-\alpha s} Y_s G(X_{s-}, X_s) (H1_\Lambda)(X_{s-}, X_s) = E^x \int_0^\infty e^{-\alpha s} Y_s (H1_\Lambda)(X_{s-}, X_s) dB_s$$

and the right side is finite. As $n \rightarrow \infty$, $\Lambda = \Lambda_n \rightarrow J \times K$ so

$$E^x \sum_s e^{-\alpha s} Y_s H(X_{s-}, X_s) G(X_{s-}, X_s) = E^x \int_0^\infty e^{-\alpha s} Y_s H(X_{s-}, X_s) dB_s < \infty$$

whenever $H \in b(\mathcal{E} \times \mathcal{E})_+$ vanishes off $J \times K$. This formula remains true if H vanishes off a finite union of sets $J \times K$ satisfying (2.5) and so it is true for all $H \in b(\mathcal{E} \times \mathcal{E})_+$, upon passage to the limit. The right side remains finite and consequently $A_t(G) = \sum_{s \leq t} G(X_{s-}, X_s)$ defines an AF with finite α -potential. We define B' to be $A(G)$. Taking $H = 1 \otimes f$ we find that for all positive previsible processes Y , $f \in b\mathcal{E}_+$ and $x \in E$

$$(2.8) \quad E^x \int_0^\infty Y_t f(X_t) dB_t = E^x \int_0^\infty Y_t f(X_t) dB'_t.$$

To complete the proof, we must show that (2.7) holds. If Γ is a Borel set in $E \times E$ having compact closure disjoint from the diagonal, it follows easily from (2.8) that for all $x \in E$, all $f \in b\mathcal{E}_+$ and all bounded positive previsible Y ,

$$E^x \int_0^\infty e^{-\alpha t} Y_t 1_\Gamma(X_{t-}, X_t) f(X_t) dB_t = E^x \int_0^\infty e^{-\alpha t} Y_t 1_\Gamma(X_{t-}, X_t) f(X_t) dB'_t,$$

and each integral is finite. Letting $T = T_\Gamma = \inf \{t > 0 : (X_{t-}, X_t) \in \Gamma\}$ and letting T^n be the n th iterate of T , we obtain

$$E^x \sum_{n=1}^\infty e^{-\alpha T^n} Y(T^n) f \circ X(T^n) \Delta B(T^n) = E^x \sum_{n=1}^\infty e^{-\alpha T^n} Y(T^n) f \circ X(T^n) \Delta B'(T^n).$$

Replacing Y by $Y1_{(T^{n-1}, T^n]}$ we find from this last equation that for all stopping times R of the form T_Γ^n for some $n \geq 1$ and some Γ ,

$$(2.9) \quad E^x[e^{-\alpha R} Y_R f \circ X_R \Delta B_R] = E^x[e^{-\alpha R} Y_R f \circ X_R \Delta B'_R].$$

It is known ([2], IV, T21) that the restriction of $\mathfrak{F}_{R-}$ to $\{R < \infty\}$ is exactly $\{Y_R 1_{\{R < \infty\}} : Y \text{ previsible}\}$ and the restriction of $\mathfrak{F}_R$ to $\{R < \infty\}$ is exactly $\{Z_R 1_{\{R < \infty\}} : Z \text{ well measurable}\}$. Moreover it is well known (and follows easily from the proof of [1], IV, (4.2)) that $\mathfrak{F}_R = \mathfrak{F}_{R-} \vee \sigma(X_R)$. The content of (2.9) is therefore

$$E^x[e^{-\alpha R} \Delta B_R Z_R] = E^x[e^{-\alpha R} \Delta B_R' Z_R]$$

for all R of the form T_{Γ}^n and all positive well measurable Z . This equality implies that

$$E^x \int_0^\infty e^{-\alpha t} Z_t 1_{\Gamma}(X_{t-}, X_t) dB_t = E^x \int_0^\infty e^{-\alpha t} Z_t 1_{\Gamma}(X_{t-}, X_t) dB_t',$$

and letting Γ increase to $(E \times E)_0$ we obtain an equation equivalent to (2.7).

Fix now a set $F \in \mathcal{E}^n$ and let $T = T_F$, $T^t = \inf \{s > t : X_s \in F\} = t + T \circ \theta_t$, $L^s = \sup \{t \leq s : X_t \in F\}$, and $\ell^s = \sup \{t < s : X_t \in F\}$. For typographical convenience, T^t is sometimes written $T(t)$, and so on. The reader is referred to [4] for a discussion of these processes, including the simple, but useful, change of variable formula

$$(2.10) \quad \int_0^\infty g(s) da(T^s) = \int_0^\infty g(\ell(t)) 1_{\{\ell(t) > 0\}} da_t,$$

where g is positive and Borel on $[0, \infty)$, and a is an increasing function on $[0, \infty)$.

(2.11) Definition. If A is a RAF of X having finite α -potential, the raw α -balayage of A on F is the RAF, $\bar{A}_F^\alpha$ defined by

$$\bar{A}_F^\alpha(t) = \int_{(T, T^t]} e^{-\alpha(s-\ell(s))} dA_s.$$

The fact that $\bar{A}_F^\alpha$ is a RAF is verified by an easy computation. If $\alpha = 0$, the definition reduces to $\bar{A}_F^0(t) = A_{T(t)} - A_T$. See [4] for further discussion. If B^1 and B^2 are the increasing processes defined by $dB_t^1 = e^{-\alpha t} dA_t$ and $dB_t^2 = e^{-\alpha t} d\bar{A}_F^\alpha(t)$, then by (2.10)

$$\bar{A}_F^\alpha(t) = \int_{(T, T^t]} e^{-\alpha(s-\ell(s))} dA_s = \int_{(T, T^t]} e^{\alpha \ell(s)} 1_{\{\ell(s) > 0\}} dB_s^1 = \int_{(0, t]} e^{\alpha u} dB_{T(u)}^1.$$

and so $B_t^2 = B_{T(t)}^1 - B_T^1$. This together with another application of (2.10) leads to the following useful formula for calculating integrals with respect to the raw α -balayage of A on F , $\bar{A}_F^\alpha$. If Z is a positive measurable process, then

$$(2.12) \quad \int_0^\infty Z_t e^{-\alpha t} d\bar{A}_F^\alpha(t) = \int_0^\infty Z_{\ell(s)} 1_{\{\ell(s) > 0\}} e^{-\alpha s} dA_s.$$

If A is continuous, $\ell(s)$ may be replaced by $L(s)$ in (2.12).

The α -balayage (or natural α -balayage), A_F^α , of A on F is just the natural dual projection of $\bar{A}_F^\alpha$ since (2.12) shows $E^x \int_0^\infty e^{-\alpha t} d\bar{A}_F^\alpha(t) = E^x \int_T^\infty e^{-\alpha s} dA_s = P_F^\alpha u_A^\alpha(x)$.

(2.13) Definition. Let A be a RAF of X having finite α -potential. The adapted α -balayage, $\tilde{A}_F^\alpha$, of A on F is defined to be the adapted dual projection of $\tilde{A}_F^\alpha$.

It is immediate from (2.12) and (2.6) that for every positive well measurable process Z , one has, for all $x \in E$,

$$(2.14) \quad E^x \int_0^\infty e^{-\alpha t} Z_t d\tilde{A}_F^\alpha(t) = E^x \int_0^\infty e^{-\alpha s} Z_{\ell(s)} 1_{\{\ell(s) > 0\}} dA_s.$$

If A is continuous, $\ell(s)$ may be replaced by $L(s)$ in (2.14) and it is in this form that (2.14) will be used in the next section.

We remark that the "natural" formulation of (2.14) is given by (2.11) of [4].

3. Last exit decompositions. This section is devoted to the proof of Theorems 1.3 and 1.8. In the course of the proof we shall obtain considerably more information about the ingredients of these theorems than it was possible to summarize in Section 1. We fix once and for all a finely closed nearly Borel set F . Then (Q_t) and (V^α) defined in (1.2) are the semigroup and resolvent of the process (X, T_F) . Since T_F is an exact terminal time, $V^\alpha f$ is Borel measurable for $f \in \mathcal{E}_+^*$ by (V-1.24) of [1]. For continuous bounded f , $t \mapsto Q_t f$ is right continuous and standard Laplace inversion formulas shown that $Q_t f$ is Borel measurable. Therefore $Q_t f \in b\mathcal{E}$ if $f \in b\mathcal{E}$. As before we let $T = T_F$, $T(t) = t + T \circ \theta_t$, $L(t) = \sup \{s \leq t : X_s \in F\}$, and $\ell(t) = \sup \{s < t : X_s \in F\}$. Since F is finely closed $L(t)$ is the right continuous inverse of $T(t)$ and $\ell(t)$ is the left continuous inverse of $T(t)$.

We now introduce some notation that will be used throughout this section.

(3.1) For $h \in b\mathcal{E}_+$, the following AF 's are defined:

- (i) $\Lambda_t(h) = \int_0^t h(X_s) ds$;
- (ii) $\tilde{A}^\alpha(h)$ is the adapted α -balayage of $\Lambda(h)$ on F ;
- (iii) $A^\alpha(h)$ is the α -balayage of $\Lambda(h)$ on F ;
- (iv) $\tilde{A}^\alpha(h) = C^\alpha(h) + D^\alpha(h) + \tilde{B}^\alpha(h)$ where $C^\alpha(h)$ is a CAF, $D^\alpha(h)$ is a purely discontinuous NAF, and $\tilde{B}^\alpha(h) = (\tilde{A}^\alpha(h))^a$;
- (v) $B^\alpha(h)$ is the natural projection of $\tilde{B}^\alpha(h)$, so $A^\alpha(h) = C^\alpha(h) + D^\alpha(h) + B^\alpha(h)$;
- (vi) $\tilde{A}, A, C, D, \tilde{B}$ and B are abbreviations for $\tilde{A}^1(1), A^1(1), C^1(1), D^1(1), \tilde{B}^1(1)$ and $B^1(1)$ respectively.

The fundamental dependence on the parameter α which will be exploited is given by the following result.

(3.2) Lemma. With notation as in the preceding paragraph one has, for all $\alpha > 0, \beta > 0$ and $h \in b\mathcal{E}_+$

$$(3.3) \quad \tilde{A}^\alpha(h) - \tilde{A}^\beta(h) = (\beta - \alpha)\tilde{A}^\alpha(V^\beta h).$$

Proof. Let $I = \tilde{A}^\alpha(h)$, $J = \tilde{A}^\beta(h)$ and $H = \tilde{A}^\alpha(V^\beta h)$. We shall prove (3.3) by showing that for all $f \in b\mathcal{E}_+$,

$$U_I^{\alpha+\beta}f = U_J^{\alpha+\beta}f + (\beta - \alpha)U_H^{\alpha+\beta}f.$$

Notice that I , J and H each have finite $(\alpha + \beta)$ -potential. Because of (2.14) we have, since $L(t)$ and $X(L(t))$ are $\mathfrak{F}_t$ -measurable,

$$\begin{aligned} U_H^{\alpha+\beta}f(x) &= E^x \int_0^\infty e^{-(\alpha+\beta)t} f(X_t) d\tilde{A}_t^\alpha(V^\beta h) \\ &= E^x \int_0^\infty e^{-\alpha t} e^{-\beta L(t)} f \circ X_{L(t)} 1_{\{L(t) > 0\}} V^\beta h(X_t) dt \\ &= E^x \int_0^\infty e^{-\alpha t} e^{-\beta L(t)} f \circ X_{L(t)} 1_{\{L(t) > 0\}} \left(\int_0^T e^{-\beta u} h(X_u) du \right) \circ \theta_t dt \\ &= E^x \int_0^\infty e^{-(\alpha+\beta)t} e^{-\beta L(t)} f \circ X_{L(t)} 1_{\{L(t) > 0\}} \int_0^\infty 1_{\{t, T(t)\}}(u) e^{-\beta u} h(X_u) du dt. \end{aligned}$$

However, $L(t) > 0$ and $t < u \leq T(t)$ if and only if $0 < \ell(u) \leq t < u$, so inverting the order of integration leads to

$$U_H^{\alpha+\beta}f(x) = E^x \int_0^\infty e^{-\beta u} h(X_u) 1_{\{\ell(u) > 0\}} \int_{\ell(u)}^u e^{-(\alpha+\beta)t} e^{-\beta L(t)} f(X_{L(t)}) dt du.$$

Since $\ell(u) = L(u)$ for all but countably many $u > 0$ and $L(t) = L(u)$ for $(L(u), u]$ we find

$$\begin{aligned} t \varepsilon (\beta - \alpha) U_H^{\alpha+\beta}f(x) &= E^x \int_0^\infty e^{-\beta u} h(X_u) 1_{\{L(u) > 0\}} e^{-\beta L(u)} f(X_{L(u)}) [e^{-(\alpha+\beta)u} - e^{-(\alpha+\beta)L(u)}] du \\ &= E^x \int_0^\infty e^{-\alpha u} h(X_u) 1_{\{L(u) > 0\}} e^{-\beta L(u)} f(X_{L(u)}) du \\ &\quad - E^x \int_0^\infty e^{-\beta u} h(X_u) 1_{\{L(u) > 0\}} e^{-\alpha L(u)} f(X_{L(u)}) du \\ &= U_I^{\alpha+\beta}f(x) - U_J^{\alpha+\beta}f(x) \end{aligned}$$

once again by (2.14), completing the proof.

(3.4) Lemma. For $h \in b\mathcal{E}_+$, $C^\alpha(h)$ is carried by $F^r = \{x : P^x(T_r = 0) = 1\}$ and

$$(3.5) \quad C^\alpha(h) - C^\beta(h) = (\beta - \alpha)C^\alpha(V^\beta h).$$

On the other hand $D^\alpha(h)$ and $\tilde{B}^\alpha(h)$ are carried by $E - F^r$ and

$$(3.6) \quad D_t^\alpha(h) = \sum_{0 < s \leq t} V^\alpha h(X_s) 1_{\{L(s) = s, X_s - = X_s\}},$$

$$(3.7) \quad \tilde{B}_t^\alpha(h) = \sum_{0 < s \leq t} V^\alpha h(X_s) 1_{\{L(s) = s, X_s - \neq X_s\}}.$$

Proof. For any stopping time R , $1_{[R]}$, where $[R]$ is the graph of R , is well measurable so for all positive well measurable Z we have, by (2.14)

$$\begin{aligned} E^x[e^{-\alpha R} Z_R \Delta \tilde{A}_R^\alpha(h)] &= E^x \int_0^\infty e^{-\alpha t} Z_t 1_{[R]}(t) d\tilde{A}_t^\alpha(h) \\ &= E^x \int_0^\infty e^{-\alpha t} Z_{L(t)} 1_{[R]}(L(t)) 1_{\{L(t) > 0\}} h(X_t) dt \\ &= E^x \int_0^\infty Z_R 1_{\{L(t) = R > 0\}} e^{-\alpha t} h(X_t) dt. \end{aligned}$$

On $\{L(R) < R\}$, $L(t) = R > 0$ is impossible, while on $\{L(R) = R\} \in \mathcal{F}_R$, $\{t : L(t) = R\} = [R, R + T \circ \theta_R]$ so

$$\begin{aligned} E^x[e^{-\alpha R} Z_R \Delta \tilde{A}_R^\alpha(h)] &= E^x \left[Z_R 1_{\{L(R) = R > 0\}} e^{-\alpha R} \left(\int_0^T e^{-\alpha t} h(X_t) dt \right) \circ \theta_R \right] \\ &= E^x[Z_R 1_{\{L(R) = R > 0\}} e^{-\alpha R} V^\alpha h(X_R)]. \end{aligned}$$

Hence $\Delta \tilde{A}_R^\alpha(h) = V^\alpha h(X_R) 1_{\{L(R) = R > 0\}}$ for all stopping times R , and (3.6) and (3.7) follow immediately. Since $V^\alpha h(x)$ vanishes for $x \notin F^r$, $D^\alpha(h)$ and $\tilde{B}^\alpha(h)$ must be carried by $E - F^r$.

For each ω let Γ denote the closure of $\{t : X_t(\omega) \in F\}$. Letting $\bar{\Lambda}^\alpha(h)$ be the raw α -balayage of $\Lambda(h)$ on F , we have, by (2.12), for all positive well measurable Z ,

$$\begin{aligned} \int_0^\infty e^{-\alpha t} Z_t 1_\Gamma(t) d\bar{\Lambda}_t^\alpha &= \int_0^\infty e^{-\alpha t} Z_{L(t)} 1_\Gamma \circ L(t) 1_{\{L(t) > 0\}} h(X_t) dt \\ &= \int_0^\infty e^{-\alpha t} Z_{L(t)} 1_{\{L(t) > 0\}} h(X_t) dt \end{aligned}$$

since $L(t) \in \Gamma$ for each t . Hence

$$\int_0^\infty e^{-\alpha t} Z_t 1_\Gamma(t) d\bar{\Lambda}_t^\alpha(h) = \int_0^\infty e^{-\alpha t} Z_t d\bar{\Lambda}_t^\alpha(h).$$

By [2], VI, T2, $I_\Gamma(t)$ is well measurable, so for all $x \in E$

$$E^x \int_0^\infty e^{-\alpha t} Z_t 1_\Gamma(t) d\tilde{A}_t^\alpha(h) = E^x \int_0^\infty e^{-\alpha t} Z_t d\tilde{A}_t^\alpha(h).$$

However, Γ differs from $\{t : X_t(\omega) \in F^r\}$ by only a countable set a.s., so the continuous part, $C^\alpha(h)$, of $\tilde{A}^\alpha(h)$ must be carried by F^r . Finally, (3.5) follows from (3.2) and the fact that $C^\alpha(h) = 1_{F^r} * \tilde{A}^\alpha(h)$.

(3.8) Remark. One may write ([5], XV-T67), $F = F^p \cup F^s$ where $F^p \cap F^s = \emptyset$, F^s is semipolar and F^p is the largest finely perfect set contained in F . Since $C^\alpha(h)$ is carried by F^r and the fine support of $C^\alpha(h)$ is finely perfect, $C^\alpha(h)$ must in fact be carried by F^p .

Let $v^\alpha(x, h) = V^\alpha h(x)/V^1 1(x)$. It is obvious from (3.6) and (3.7) that

$$(3.9) \quad D^\alpha(h) = v^\alpha(\cdot, h) * D$$

and

$$(3.10) \quad \tilde{B}^\alpha(h) = v^\alpha(\cdot, h) * \tilde{B}.$$

We also have the relation

$$(3.11) \quad v^\alpha(x, h) = \int_0^\infty e^{-\alpha t} q_t(x, h) dt$$

where $q_t(x, h) = Q_t h(x)/V^1 1(x)$, Q_t being the semigroup for (X, T_r) . The important observation here is that $q_t(x, h)$ is, for each fixed x , an entrance law for the semigroup Q_t , and $q_t(x, \cdot)$ is carried by $E - F^r$. For each fixed $h \in b\mathcal{E}_+$, $q_t(x, h)$ is measurable relative to $\mathcal{B} \times \mathcal{E}$, where $\mathcal{B}$ is the Borel σ -field on $[0, \infty)$.

(3.12) Lemma. *There exist positive functions $w^\alpha(x, h)$ for $\alpha > 0$, $x \in E$, $h \in b\mathcal{E}_+$ such that*

- (i) $w^\alpha(x, h) = 0$ if $x \notin F^r$;
- (ii) $w^\alpha(x, h) = w^\beta(x, h) + (\beta - \alpha)w^\alpha(x, V^\beta h)$;
- (iii) $C^\alpha(h) = w^\alpha(\cdot, h) * C$;
- (iv) For fixed $\alpha > 0$ and $x \in E$, $w^\alpha(x, \cdot)$ is a measure and $w^\alpha(x, 1) \leq 1 + \alpha^{-1}$;
- (v) For fixed $h \in b\mathcal{E}_+$, $(\alpha, x) \rightarrow w^\alpha(x, h)$ is in $\mathcal{B} \times \mathcal{E}$.

Proof. We shall assume that the measure $\nu = \nu_C$ defined following (2.3) is finite. To obtain the general case, C (respectively $C^\alpha(h)$) should be replaced in the following argument by $f * C$ (respectively $f * C^\alpha(h)$) for a fixed $f \in b\mathcal{E}_+$, $f > 0$, such that $\nu_C(f) < \infty$. (Notice that (3.5) holds with $C^\alpha(h)$ replaced by $f * C^\alpha(h)$). One has (recall that $C = C^1(1)$)

$$\begin{aligned} (1 + \alpha^{-1})C^1(1) &= C^1[(1 - \alpha V^\alpha 1 + V^\alpha 1) + (\alpha V^\alpha 1 + \alpha^{-1}(1 - \alpha V^\alpha 1))] \\ &= C^\alpha(1) + C^1(\alpha V^\alpha 1 + \alpha^{-1}(1 - \alpha V^\alpha 1)), \end{aligned}$$

using (3.5). Hence, by Motoo's theorem ([1], V-2.6), $C^\alpha(1) = g * C$ for some $g \in b\mathcal{E}_+$ with $0 \leq g \leq 1 + \alpha^{-1}$. If $h \in \mathcal{E}$ satisfies $0 \leq h \leq 1$ we have $C^\alpha(h) + C^\alpha(1 - h) = C^\alpha(1)$ so once again, by Motoo's theorem, $C^\alpha(h) = k * C$ for some $k \in \mathcal{E}$ with $0 \leq k \leq (1 + \alpha^{-1})$. It follows that $\nu_{C^\alpha(h)}$ is a finite measure if $h \in b\mathcal{E}_+$. For $h \in b\mathcal{E}$, let $C^\alpha(h) = C^\alpha(h^+) - C^\alpha(h^-)$. Standard arguments show that $h \rightarrow C^\alpha(h)$ is linear in h and that $C^\alpha(h)$ is uniquely determined by the finite signed measure $\nu_{C^\alpha(h)} = \nu_{C^\alpha(h^+)} - \nu_{C^\alpha(h^-)}$. Let $\{h_n : n \geq 1\}$ be a linearly independent sequence of $\mathbf{C}_K^+$ functions generating a vector space $\mathcal{H}$ which is uniformly dense in $\mathbf{C}_0$ and let $\mathcal{H}_r$ denote the set of rational linear combinations of the h_n . Let $\|h\|$ denote the uniform norm of h . For each rational $\alpha > 0$ and each h_n choose a fixed $\mathcal{E}$ -measurable function $x \rightarrow w^\alpha(x, h_n)$ satisfying $0 \leq w^\alpha(x, h_n) \leq \|h_n\| (1 + \alpha^{-1})$ and such that (3.12 iii) holds with $h = h_n$. Since $C^\alpha(h)$ is carried by the Borel set F^r , we may assume that (i) holds for $h = h_n$.

For rational $\alpha > 0$ and $h \in \mathcal{H}_r$ we define $w^\alpha(x, h)$ by linearity. Obviously (i) and (iii) hold then for α rational and $h \in \mathcal{H}_r$. Noticing that for $f, f' \in b\mathcal{E}$, $f * C = f' * C$ if and only if $f = f'$ a.e. (ν), we find

$$(3.13) \quad |w^\alpha(x, h)| \leq \|h\| (1 + \alpha^{-1})$$

for ν -a.a. x , for each rational $\alpha > 0$ and $h \in \mathcal{H}_r$. Let Γ_0 be a fixed set in $\mathcal{E}$ such that $\nu(\Gamma_0) = 0$ and such that (3.13) holds identically for $x \notin \Gamma_0$, α rational and $h \in \mathcal{H}_r$. Replace now $w^\alpha(x, h)$ for α rational and $h \in \mathcal{H}_r$ by $1_{E-\Gamma_0}(x)w^\alpha(x, h)$ so that (i), (iii) and (3.13) hold identically. For each fixed rational α and $x \in E$, $w^\alpha(x, \cdot)$ can then be extended in a unique way to be a measure on $\mathcal{E}$, and this extension is also denoted $w^\alpha(x, h)$ for $h \in b\mathcal{E}$. Clearly (3.12 i) and (3.13) hold for the new $w^\alpha(x, h)$. By (2.4), (3.12 iii) holds if and only if $\nu_{C^\alpha(h)}(dx) = w^\alpha(x, h) \nu(dx)$. Since $C^\alpha(h) = 1_{F^r} * \tilde{A}^\alpha(h)$, (2.3) and (2.12) show that for $f \in b\mathcal{E}_+$, if $0 \leq k_n \uparrow k$, then $\nu_{C^\alpha(h_{k_n})}(f) \uparrow \nu_{C^\alpha(h)}(f)$. It follows that for fixed $f \in b\mathcal{E}_+$, $k \rightarrow \nu_{C^\alpha(h)}(f)$ is a measure, and hence (3.12 iii) holds for α rational and $h \in b\mathcal{E}_+$.

Because of (3.5), $w^\alpha(x, h) = w^\beta(x, h) + (\beta - \alpha)w^\alpha(x, V^\beta h)$ a.e. (ν) for each rational α and β , and each $h \in b\mathcal{E}_+$. One may choose $\Gamma \in \mathcal{E}$ such that $\nu(\Gamma) = 0$ and such that (ii) holds for all $h \in \mathcal{H}_r$, α and β rational, whenever $x \notin \Gamma$. Each side of (ii) is a measure in h so in fact, if $x \notin \Gamma$, (ii) holds for all $h \in b\mathcal{E}$ and α, β rational. We once again modify $w^\alpha(x, h)$ to be identically zero if $x \in \Gamma$, and the modified $w^\alpha(x, h)$ satisfies (i), (ii), (iii) and (3.13) identically, for α, β rational. For fixed $h \in b\mathcal{E}$, since $\beta \|V^\beta h\| \leq \|h\|$, (ii) implies that for all $x \in E$ and all rational α and β ,

$$|w^\alpha(x, h) - w^\beta(x, h)| \leq |1 - \alpha/\beta| (1 + \alpha^{-1}) \|h\|.$$

Thus $\alpha \rightarrow w^\alpha(x, h)$ is uniformly continuous on the rationals $\geq \delta$ for all $\delta > 0$. We denote by $w^\alpha(x, h)$ the continuous extension to all real $\alpha > 0$. The properties (i), (ii), (iii), (iv) are easily verified. The property (v) holds for $h \in \mathcal{H}_r$, hence for all $h \in b\mathcal{E}$, completing the proof.

Since $V^\beta(h1_{F^r}) = 0$, (3.12 ii) shows that $w^\alpha(x, h1_{F^r}) = w^\beta(x, h1_{F^r})$ for all α and $\beta > 0$. Therefore we may write

$$w^\alpha(x, h) = u^\alpha(x, h) + s(x, h)$$

where $u^\alpha(x, h) = w^\alpha(x, h1_{E-F^r})$ and $s(x, h) = w^\alpha(x, h1_{F^r})$ for all $\alpha > 0$. Since $V^\beta(h1_{E-F^r}) = V^\beta h$ vanishes on F^r we have, by (3.12 ii), that for all α and $\beta > 0$, $x \in E$ and $h \in b\mathcal{E}$,

$$(3.14) \quad u^\alpha(x, h) - u^\beta(x, h) = (\beta - \alpha)u^\alpha(x, V^\beta h).$$

Since $\beta V^\beta 1$ increases to 1_{E-F^r} as $\beta \rightarrow \infty$, we have

$$\begin{aligned} u^\alpha(x, 1) - \lim_{\beta \rightarrow \infty} u^\beta(x, 1) &= \lim_{\beta \rightarrow \infty} u^\alpha(x, \beta V^\beta 1) - \lim_{\beta \rightarrow \infty} \alpha u^\alpha(x, V^\beta 1) \\ &= u^\alpha(x, 1_{E-F^r}) = u^\alpha(x, 1). \end{aligned}$$

Thus $\lim_{\beta \rightarrow \infty} u^\beta(x, 1) = 0$ for all x . By (3.14) and [4], Theorem 6.9, there exists

for each $x \in E$ an entrance law $\eta_t(x, \cdot)$ for the semigroup (Q_t) such that for all $\alpha > 0$, $x \in E$ and $h \in b\mathcal{E}_+$

$$(3.15) \quad u^\alpha(x, h) = \int_0^\infty e^{-\alpha t} \eta_t(x, h) dt.$$

Notice that $\eta_t(x, h) = 0$ if $x \notin F^r$, and the measure $\eta_t(x, \cdot)$ lives on $E - F^r$. If h is continuous, $t \rightarrow \eta_t(x, h)$ is right continuous and the standard inversion formulas for Laplace transforms show that $(t, x) \rightarrow \eta_t(x, h)$ is in $\mathcal{B} \times \mathcal{E}$ since $(\alpha, x) \rightarrow u^\alpha(x, h)$ is in $\mathcal{B} \times \mathcal{E}$. It follows that $(t, x) \rightarrow \eta_t(x, h)$ is in $\mathcal{B} \times \mathcal{E}$ for every $h \in b\mathcal{E}$.

Let $\gamma_t(x, \cdot)$ denote the entrance law $\eta_t(x, \cdot) + q_t(x, \cdot)$ for (Q_t) . It is evident that $\gamma_t(x, \cdot) = \eta_t(x, \cdot)$ if $x \in F^r$ and that $\gamma_t(x, \cdot) = q_t(x, \cdot)$ if $x \notin F^r$. Moreover from the measurability properties of η and q it is clear that $\gamma_t(x, h)$ is a *kernel* in the sense that $h \rightarrow \gamma_t(x, h)$ is a measure for fixed $t > 0$ and $x \in E$ and that $(t, x) \rightarrow \gamma_t(x, h)$ is $\mathcal{B} \times \mathcal{E}$ measurable for fixed $h \in b\mathcal{E}$. The word "kernel" for a function of t, x , and h will be used in this sense throughout this paper. Observe that the Laplace transform of $\gamma_t(x, h)$ is $u^\alpha(x, h)1_{F^r}(x) + v^\alpha(x, h)1_{E-F^r}(x)$. However, if h vanishes on F^r , we have

$$\begin{aligned} \tilde{A}^\alpha(h) &= C^\alpha(h) + D^\alpha(h) + \tilde{B}^\alpha(h) \\ &= u^\alpha(\cdot, h) * C + v^\alpha(\cdot, h) * (D + \tilde{B}) \\ &= (u^\alpha(\cdot, h) + v^\alpha(\cdot, h)) * \tilde{A} \end{aligned}$$

since $1_{F^r} * (\tilde{B} + D) = 1_{E-F^r} * C = 0$. By (2.14), for every bounded positive well measurable Z , $\alpha > 0$ and $x \in E$

$$\begin{aligned} E^x \int_0^\infty e^{-\alpha t} Z_{L(t)} 1_{\{L(t) > 0\}} h(X_t) dt &= E^x \int_0^\infty e^{-\alpha s} Z_s d\tilde{A}_s^\alpha(h) \\ &= E^x \int_0^\infty e^{-\alpha s} Z_s (u^\alpha(X_s, h) + v^\alpha(X_s, h)) d\tilde{A}_s \\ &= E^x \int_0^\infty e^{-\alpha s} Z_s \int_0^\infty e^{-\alpha u} \gamma_u(X_s, h) du d\tilde{A}_s \\ &= E^x \int_0^\infty Z_s \int_s^\infty e^{-\alpha t} \gamma_{t-s}(X_s, h) dt d\tilde{A}_s. \end{aligned}$$

Upon inversion of the order of integration we obtain

$$(3.16) \quad \int_0^\infty e^{-\alpha t} E^x [Z_{L(t)} 1_{\{L(t) > 0\}} h(X_t)] dt = \int_0^\infty e^{-\alpha t} E^x \int_0^t Z_s \gamma_{t-s}(X_s, h) d\tilde{A}_s dt.$$

Here, as is standard, $\int_0^t = \int_{(0, t]}$ and $\int_0^{t-} = \int_{(0, t)}$.

In order to complete the proof of Theorem 1.3 we must show that for all bounded positive well measurable Z , $t > 0$, and $h \in b\mathcal{E}_+$

$$(3.17) \quad E^x[Z_{L(t)} \mathbf{1}_{\{0 < L(t) < t\}} h(X_t)] = E^x \int_0^{t-} Z_s \gamma_{t-s}(X_s, h) d\tilde{A}_s.$$

Assume firstly that h is the restriction to $E - F^r$ of a bounded positive continuous function. Then $u \rightarrow \gamma_u(x, h)$ is right continuous on $(0, \infty)$, $u \rightarrow \gamma_u(x, 1)$ is decreasing, and so $0 \leq \gamma_u(x, h) \leq \|h\| \gamma_u(x, 1)$. Fix $r > 0$, $\epsilon > 0$, and let $\Gamma_n = \{x : \gamma_\epsilon(x, 1) \leq n\}$. Clearly $\bigcup \Gamma_n = E$. Replacing Z_t in (3.16) by

$$e^{-t} Z_t \mathbf{1}_{(0, r)}(t) \mathbf{1}_{\Gamma_n} \circ X_t$$

for a bounded positive well measurable Z , we find, inverting Laplace transforms,

$$(3.18) \quad E^x[e^{-L(t)} Z_{L(t)} \mathbf{1}_{\Gamma_n} \circ X(L(t)) \mathbf{1}_{\{0 < L(t) < r\}} h(X_t)] \\ = E^x \int_0^{r-} e^{-s} Z_s \mathbf{1}_{\Gamma_n} \circ X_s \gamma_{t-s}(X_s, h) d\tilde{A}_s$$

for almost all $t > r$. Since for sufficiently small δ , $L(s) = L(t)$ and $X_s \notin F^r$ when $s \in [t, t + \delta]$ and $0 < L(t) < r < t$, the left side of (3.18) is right continuous in t for $t > r$. For $t > r + \epsilon$, $\gamma_{t-s}(X_s, h) \leq \gamma_\epsilon(X_s, 1) \|h\| \leq n \|h\|$ for $X_s \in \Gamma_n$, $0 \leq s \leq r$. Because Z is bounded and $E^x \int_0^r e^{-s} d\tilde{A}_s < \infty$, the right continuity of $\gamma_u(x, h)$ in u shows that the right side of (3.18) is right continuous in t for $t > r + \epsilon$. Letting $n \rightarrow \infty$, (3.18) holds with $\mathbf{1}_{\Gamma_n}$ replaced by 1 for all $t > r + \epsilon$ and hence for all $t > r$. For fixed $t > 0$, let $r \uparrow t$ and replace Z_s by $e^s Z_s$ to obtain (3.17) for h the restriction of a bounded positive continuous function to $E - F^r$. Since each side of (3.17) is a measure in h , (3.17) holds whenever $h \in b\mathcal{E}_+$ vanishes on F^r . Finally, the fact that $\gamma_u(x, \cdot)$ is carried by $E - F^r$, and that $X_t \notin F^r$ a.s. on $\{0 < L(t) < t\}$ establishes (3.17) for every $h \in b\mathcal{E}_+$ and bounded positive well measurable Z . This completes the proof of Theorem 1.3.

Taking $Z = 1$ in (3.17) and noting that $\{L(t) = 0\} = \{t < T_F\}$ we obtain the following last exit decomposition for the distribution of X_t :

$$(3.19) \quad E^x\{h(X_t)\} = Q_t h(x) + E^x\{h(X_t); L(t) = t\} + E^x \int_0^{t-} \gamma_{t-s}(X_s, h) d\tilde{A}_s,$$

for all $h \in b\mathcal{E}$, $x \in E$ and $t > 0$.

Under certain conditions on F , $\{L^t = t\} = \{X_t \in F\}$ almost surely, and the second term on the right side of (3.19) may be replaced by $E^x\{h(X_t) : X_t \in F\} = E^x[(h1_F)(X_t)]$. If, for example, F is projective, the raw α -balayage of $\Lambda(1)$ cannot have mass at the previsible time $t > 0$ so $P^x\{L^t = t, X_t \notin F\} = 0$. Thus $\{L^t = t\} = \{X_t \in F\}$ a.s. if F is projective. In fact, in this case, $\{L^t = t\} = \{X_t \in F^r\}$ almost surely.

We now turn to the proof of Theorem 1.8. Taking $Z_t = Y_t f(X_t)$, where Y is a positive previsible process and $f \in b\mathcal{E}_+$, (3.17) gives

$$E^x[Y_{L(t)} f(X_{L(t)}) h(X_t) \mathbf{1}_{\{0 < L(t) < t\}}] \\ = E^x \int_0^{t-} Y_s f(X_s) \gamma_{t-s}(X_s, h) d(C_s + D_s) + E^x \int_0^t Y_s f(X_s) q_{t-s}(X_s, h) d\tilde{B}_s,$$

since $\tilde{B}$ does not charge the fixed time t . It is known [6] (and it follows easily from the discussion in Section 2) that $\tilde{B}$ must have the form $\tilde{B}_t = \sum_{0 < s \leq t} G(X_{s-}, X_s)$ for some $G \in (\mathcal{E} \times \mathcal{E})_+$ which vanishes on the diagonal. By (3.7), we may assume $G(x, y) = 0$ if $y \notin F^x$. Let $n(x, dy)$ be a kernel and H a CAF of X such that (n, H) is a Lévy system for the process. It is immediate from the properties of a Lévy system and (VII-T17) of [5] that for a positive previsible process Y , a positive Borel function g on $[0, \infty)$, and $\Phi \in (\mathcal{E} \times \mathcal{E})_+$ one has

$$E^x \int_0^\infty Y_s g(s) \Phi(X_{s-}, X_s) d\tilde{B}_s = E^x \int_0^\infty Y_s g(s) n(\Phi G)(X_s) dH_s$$

where $n(\Phi G)(x) = \int n(x, dy) \Phi(x, y) G(x, y)$. It follows that if $\Psi \in (\mathcal{B} \times \mathcal{E} \times \mathcal{E})_+$ and we write $\Psi_s(x, y) = \Psi(s, x, y)$, then

$$E^x \int_0^\infty Y_s \Psi_s(X_{s-}, X_s) d\tilde{B}_s = E^x \int_0^\infty Y_s n(\Psi_s G)(X_s) dH_s.$$

Therefore

$$\begin{aligned} E^x \int_0^t Y_s f(X_s) q_{t-s}(X_s, h) d\tilde{B}_s \\ = E^x \int_0^t Y_s \int n(X_s, dy) G(X_s, y) f(y) q_{t-s}(y, h) dH_s. \end{aligned}$$

However, $dB_s = nG(X_s) dH_s$ so

$$\begin{aligned} E^x \int_0^t Y_s f(X_s) q_{t-s}(X_s, h) d\tilde{B}_s \\ = E^x \int_0^t Y_s \left[\left(\int n(X_s, dy) G(X_s, y) f(y) q_{t-s}(y, h) \right) / nG(X_s) \right] dB_s. \end{aligned}$$

The term $\int n(x, dy) G(x, y) f(y) q_{t-s}(y, h) / nG(x) = \rho_u(x, f, h)$ is defined to be 0 whenever $nG(x) = 0$, and is, for each fixed $x \in E$ and $f \in \mathcal{B}\mathcal{E}_+$, an entrance law for (Q_t) . One has now, for positive previsible Y

$$\begin{aligned} (3.20) \quad E^x [Y_{L(t)} f \circ X_{L(t)} h(X_t) 1_{\{0 < L(t) < t\}}] \\ = E^x \int_0^{t-} Y_s f(X_s) \gamma_{t-s}(X_s, h) d(C_s + D_s) + E^x \int_0^t Y_s \rho_{t-s}(X_s, f, h) dB_s. \end{aligned}$$

So far we have not invoked the condition (1.7). First of all, making use of Lemma 3.4 and Motoo's absolute continuity theorem, (V-2.5) of [1], it is easy to see that (1.7) is equivalent to the existence of $\psi \in \mathcal{E}$ with $0 \leq \psi \leq 1$ such that $D = \psi * A$. But this implies $(1 - \psi) * D = \psi * (C + B) = 0$, and from this it is easily seen that ψ may be replaced by $1_{\{\psi > 0\}}$. As a result (1.7) is equivalent to (1.6) as stated in Section 1.

For F satisfying (1.7) one has $C + D = (1 - \varphi) * A$ so (3.20) leads, for positive previsible Y and $\Phi \in (\mathcal{E} \times \mathcal{E})_+$, to

$$(3.21) \quad E^x[Y_{L(t)}\Phi(X_{L(t)}, X_t)1_{\{0 < L(t) < t\}}] = E^x \int_0^{t-} Y_s r(t-s, X_s, \Phi) dA_s$$

where the kernel r is defined by

$$(3.22) \quad r(u, x, \Phi) = \int \gamma_u(x, dy)\Phi(x, y)(1 - \varphi(x)) + \int \bar{n}(x, dy) \int q_u(y, dz)\Phi(y, z)$$

and

$$(3.23) \quad \bar{n}(x, dy) = n(x, dy)G(x, y)\varphi(x)/nG(x),$$

the right side of (3.23) being defined to be zero if $nG(x) = 0$. Since $G(x, y) = 0$ if $y \notin F^r$, $\bar{n}(x, \cdot)$ is carried by $E - F^r$ and consequently both integrals on the right side of (3.22) are only over $E - F^r$. This establishes Theorem 1.8.

Notice that for fixed $f \in b\mathcal{E}_+$ and $x \in E$, $r(t, x, f \otimes h)$ is an entrance law for (Q_t) , and (3.21) leads to the following formula which is analogous to (3.19)

$$(3.24) \quad E^x\{h(X_t)\} = Q_t h(x) + E^x\{h(X_t) : L(t) = t\} \\ + E^x \int_0^{t-} r(t-s, X_s, 1 \otimes h) dA_s$$

for $t > 0$ and $h \in b\mathcal{E}$.

In the special, but important, case that F is finely perfect the function $G(x, y)$ appearing in the above formulas may be taken to satisfy

$$(3.25) \quad G(x, y) = 1_F(x)V^1 1(y).$$

See the proof of Proposition 3.8 of [3]. In this case (3.22) simplifies considerably.

We close this section by showing that under special assumptions (3.24) reduces to the familiar last exit decomposition. Suppose that F is projective. Then A is continuous, and, as pointed out in the discussion following (3.19), $\{L^t = t\} = \{X_t \in F^r\}$. We suppose that each point $b \in F$ is regular, *i.e.*, $P^b\{T_{\{b\}} = 0\} = 1$. In particular F is finely perfect. For each $b \in F$ let J^b be a local time at b . Finally we assume that $(b, t, \omega) \rightarrow J_t^b(\omega)$ is (jointly) measurable and that there exists a measure m (necessarily carried by F) such that

$$(3.26) \quad A_t = \int J_t^b m(db).$$

This last assumption is certainly satisfied if F is finite. More generally, using the measure ν_A , the argument on page 282 of [11] is easily adapted to show that (3.26) always holds for such a set F (each $b \in F$ is regular) provided $(x, y) \rightarrow E^x \exp(-T_{\{y\}})$ is $\mathcal{E} \times \mathcal{E}$ measurable. Substituting (3.26) into (3.24) and using the fact that J^b is carried by $\{b\}$ one obtains

$$(3.27) \quad E^x\{h(X_t)\} = Q_t h(x) + E^x\{h(X_t); X_t \in F\} \\ + \int m(db) \int_0^t r(t-s, b, 1 \otimes h) dA_s^{x, b}$$

where $M_s^{x,b} = E^x(J_s^b)$ is a continuous increasing function of s for each fixed x and b . Using the strong Markov property (I-8.16) of [1], we have

$$(3.28) \quad M_s^{x,b} = \int_F \int_0^\infty M_{s-\tau}^{a,b} H^x(d\tau, da)$$

where

$$(3.29) \quad H^x(d\tau, da) = P^x[T_F \in d\tau, X(T_F) \in da]$$

is the joint distribution of T_F and $X(T_F)$ under P^x . In view of (3.28) we may write (3.27) in the form

$$(3.30) \quad E^x\{h(X_t)\} = Q_t h(x) + E^x\{h(X_t); X_t \in F\} \\ + \int_0^t \int_F \int_F H^x(d\tau, da) \int_\tau^t r(t-s, b, 1 \otimes h) d_s M_{s-\tau}^{a,b} m(db).$$

If we define

$$e_t(x, \Gamma) = H^x((t, \infty), \Gamma) = P^x[T_F > t; X(T_F) \in \Gamma],$$

then one easily checks that for each fixed $\Gamma \in \mathcal{E}$, $(t, x) \rightarrow e_t(x, \Gamma)$ is an *exit* law for (Q_t) , that is

$$Q_t e_s(\cdot, \Gamma) = e_{t+s}(\cdot, \Gamma).$$

Consequently (3.30) is the first entrance last exit decomposition relative to F . When F is finite this reduces to the familiar result from Markov chain theory. See [4], [7], [12], and [13].

It is also worthwhile to examine (3.19) when F is a finite finely perfect set. However, we leave this as an exercise for the interested reader.

4. Conditional distributions. In this section we shall prove Theorems 1.12 and 1.13, and, in addition, certain closely related results. As mentioned in Section 1, ζ does not play any role in this development and so we shall simplify our notation by assuming that $\zeta = \infty$. (This can always be accomplished by considering E_Δ as the new state space—if X is a Hunt process on E it is also a Hunt process on E_Δ .) Thus we assume that X is a Hunt process on E with a Markovian transition function, that is, $P_t(x, E) = 1$ for all t and x , and that there exists a σ -finite excessive reference measure ξ for X . We fix a finely closed nearly Borel set F and recall that $L^t = L(t) = \sup \{s \leq t : X_s \in F\}$. These assumptions and notations will be in force throughout this section without special mention.

We need some preliminary remarks about the two σ -algebras $\mathfrak{F}(R-)$ and $\mathfrak{F}(R)$ defined in (1.10) and (1.11) for arbitrary positive random variables R . As mentioned in Section 1, it is immediate that if R is a stopping time, then $\mathfrak{F}(R-)$ and $\mathfrak{F}(R)$ reduce to the usual fields $\mathfrak{F}_{R-}$ and $\mathfrak{F}_R$ associated with a stopping time. It is also easy to see that $\mathfrak{F}(R-)$ contains all sets of the form $\Lambda \cap \{R > T\}$,

$\Lambda \in \mathcal{F}_T$, T a stopping time. The following characterization of $\mathcal{F}(R-)$ and $\mathcal{F}(R)$ is extremely useful. Its proof is a routine modification of the corresponding proof when R is a stopping time and, consequently, is omitted. See, for example, IV-T21 of [2]. We omit the qualifying phrase “almost surely” where it is obviously required.

(4.1) Lemma. *Let $R \geq 0$ be $\mathcal{F}_\infty$ measurable. Then if $\Lambda \in \mathcal{F}_\infty$, $\Lambda \cap \{R = \infty\} \in \mathcal{F}(R-)$. An $\mathcal{F}_\infty$ measurable random variable Z is $\mathcal{F}(R-)$ (respectively $\mathcal{F}(R)$) measurable if and only if there exists a previsible (respectively well measurable) process Y such that $Z = Y_R$ on $\{R < \infty\}$.*

In particular X_{R-} (respectively X_R) is $\mathcal{F}(R-)$ (respectively $\mathcal{F}(R)$) measurable on $\{R < \infty\}$. Of course, R is $\mathcal{F}(R-)$ measurable.

Using the notation of Section 3 we define for $t > 0$ and $x \in E$

$$(4.2) \quad \begin{aligned} k(t, x, h) &= \frac{\gamma_t(x, h)}{\gamma_t(x, 1)} \quad \text{if } \gamma_t(x, 1) > 0, \\ &= 0 \quad \text{if } \gamma_t(x, 1) = 0. \end{aligned}$$

Clearly $k(t, x, h)$ is a kernel. Let Z be a bounded well measurable process. If h is a bounded Borel function and $t > 0$, then $s \rightarrow k(t - s, X_s, h)1_{\{s < t\}}$ is well measurable. Consequently (3.17) and (4.2) yield

$$(4.3) \quad \begin{aligned} E^x\{Z_{L(t)}k(t - L(t), X_{L(t)}, h); 0 < L(t) < t\} \\ &= E^x \int_0^{t-} Z_s k(t - s, X_s, h) \gamma_{t-s}(X_s, 1) d\tilde{A}_s \\ &= E^x \int_0^{t-} Z_s \gamma_{t-s}(X_s, h) d\tilde{A}_s \\ &= E^x\{Z_{L(t)}h(X_t); 0 < L(t) < t\}. \end{aligned}$$

But by (4.1) this is just the statement that on $\{0 < L(t) < t\}$

$$(4.4) \quad E^x\{h(X_t) \mid \mathcal{F}(L^t)\} = k(t - L(t), X_{L(t)}, h)$$

proving (1.12). If H is a bounded $\mathcal{F}_\infty$ measurable random variable and if $\bar{H}(x) = E^x(H)$, then it follows at once from (4.4) and $\mathcal{F}(L^t) \subset \mathcal{F}_t$ (see the corollary of Lemma 4.5 of [8]) that

$$(4.5) \quad E^x\{H \circ \theta_t \mid \mathcal{F}(L^t)\} = k(t - L(t), X_{L(t)}, \bar{H})$$

on $\{0 < L(t) < t\}$.

We now turn to the proof of Theorem 1.13. For this result we assume that condition (1.6) holds. If $\Phi(x, y)$ is a bounded Borel function of $E \times E$, define for $t > 0$ and $x \in E$

$$(4.6) \quad K(t, x, \Phi) = \frac{r(t, x, \Phi)}{r(t, x, 1)}$$

where the ratio is taken to be zero if the denominator vanishes. Here r is the kernel defined in (3.22). Clearly $K(t, x, \Phi)$ is a kernel. Let Y be a bounded previsible process. Then for a fixed $t > 0$ and a bounded Borel function Φ on $E \times E$, $s \rightarrow K(t - s, X_{s-}, \Phi)1_{\{s < t\}}$ is previsible. Therefore using (3.21) twice we obtain

$$\begin{aligned} E^x \{ Y_{L(t)} K(t - L(t), X_{L(t)-}, \Phi); 0 < L(t) < t \} \\ &= E^x \int_0^{t-} Y_s K(t - s, X_s, \Phi) r(t - s, X_s, 1) dA_s \\ &= E^x \int_0^{t-} Y_s r(t - s, X_s, \Phi) dA_s \\ &= E^x \{ Y_{L(t)} \Phi(X_{L(t)}, X_t); 0 < L(t) < t \}. \end{aligned}$$

Again by (4.1) this is equivalent to

$$(4.7) \quad E^x \{ \Phi(X_{L(t)}, X_t) \mid \mathfrak{F}(L^t-) \} = K(t - L(t), X_{L(t)-}, \Phi)$$

on $\{0 < L(t) < t\}$, proving (1.13). In particular if f and h are bounded Borel functions on E

$$(4.8) \quad \text{(i)} \quad E^x \{ f(X_{L(t)}) \mid \mathfrak{F}(L^t-) \} = K(t - L(t), X_{L(t)-}, f \otimes 1),$$

$$\text{(ii)} \quad E^x \{ h(X_t) \mid \mathfrak{F}(L^t-) \} = K(t - L(t), X_{L(t)-}, 1 \otimes h),$$

on $\{0 < L(t) < t\}$. Example (4.13) shows that if (1.6) does not hold, then there may not exist a kernel K satisfying (4.7) or even only (4.8 i).

Clearly, if we condition the formula (4.4) with respect to $\mathfrak{F}(L^t-)$ and compare the result with (4.8) we will obtain a relationship between the kernels k and K . However, we shall develop this relationship from the definitions of k and K . We are assuming here that (1.6) holds so that (4.7) and (4.8) are valid. Recall that $\gamma_t(x, \cdot) = \eta_t(x, \cdot) + q_t(x, \cdot)$ and that $\eta_t(x, \cdot) = 0$ if $x \notin E - F^r$ while $q_t(x, \cdot) = 0$ if $x \in F^r$. Let φ and $\bar{n}$ be as in (1.7) and (3.23). Recall that $\bar{n}(x, \cdot)$ is carried by $E - F^r$. Next define

$$(4.9) \quad N(x, dy) = [1 - \varphi(x)]\epsilon_x(dy) + \bar{n}(x, dy)$$

where ϵ_x is unit mass at x . If f and h are bounded Borel functions, then using the preceding remarks and (3.24)

$$\begin{aligned} (4.10) \quad \int N(x, dy) f(y) \gamma_t(y, h) &= [1 - \varphi(x)] f(x) \gamma_t(x, h) \\ &\quad + \int \bar{n}(x, dy) f(y) q_t(y, h) = r(t, x, f \otimes h). \end{aligned}$$

Comparing this with (4.2) and (4.6) we obtain the following formula, in which $k(t, \cdot, h)f$ denotes the function $y \rightarrow k(t, y, h)f(y)$

$$(4.11) \quad K(t, x, f \otimes h) = K(t, x, k(t, \cdot, h)f \otimes 1).$$

The formulas (4.10) and (4.11) express analytically the relationship between conditioning relative to $\mathfrak{F}(L^t)$ and $\mathfrak{F}(L^t-)$.

Finally we turn to the probabilistic meaning of the kernel N defined in (4.9). To this end it is convenient to introduce a bit more notation. Let $g_t(x) = \gamma_t(x, 1)$. Then $r(t, x, 1) = \int N(x, dy)\gamma_t(y, 1) = Ng_t(x)$. Let $\Gamma = \{(t, x) : g_t(x) = 0\}$ and $\Gamma^* = \{(t, x) : Ng_t(x) = 0\}$. Then it is immediate from (3.17) and (3.21) that for all x and $t > 0$

$$\begin{aligned} P^x[(t - L(t), X_{L(t)}) \in \Gamma; 0 < L(t) < t] &= 0, \\ P^x[(t - L(t), X_{L(t)-}) \in \Gamma^*; 0 < L(t) < t] &= 0, \end{aligned}$$

and so the ratio $Ng_{t-L(t)}(X_{L(t)-})/g_{t-L(t)}(X_{L(t)})$ is well defined and strictly positive almost surely on $\{0 < L(t) < t\}$. This random variable depends only on the set F and on t and we denote it by β_t . Now if f is a bounded Borel function, it follows readily from (4.7) and (4.10) that

$$(4.12) \quad E^x\{\beta_t f(X_{L(t)}) \mid \mathfrak{F}(L^t-)\} = N(X_{L(t)-}, f)$$

on $\{0 < L(t) < t\}$. Thus, aside from the normalizing factor β_t , the kernel $N(x, dy)$ gives "the distribution of $X_{L(t)}$ under the condition $X_{L(t)-} = x$ ". If the definition of N in (4.9) is viewed in this light, then the first term on the right side of (4.9) corresponds to $X_{L(t)} = X_{L(t)-}$ while the second corresponds to $X_{L(t)} \neq X_{L(t)-}$. Formula (4.10) should also be interpreted in this light. It is also instructive to write out explicitly (4.4), (4.7), and (4.12) when F is a finite finely perfect set.

(4.13) Example. We close this section with an example for which (1.6) does not hold and for which there is no kernel K satisfying (4.8 i). The state space E is $[-1, 0] \cup [1, \infty)$. If the process starts from an $x \geq 1$ it translates to the right at unit speed, if it starts from an $x < 0$ it translates to the right at unit speed until reaching 0, and 0 is a holding point with parameter one from which the process jumps to 1. The set $F = [-1, 0) \cup \{1\}$ is finely closed but is neither admissible nor projective. Let $h = 1$ so that $\Lambda_t = t$. Let A_t be the natural one balayage of Λ on F , so that $u_A^1 = P_F^1 U^1 1 = P_F^1 1$. One easily computes $P_F^1 1(x)$, and then checks that

$$(4.14) \quad A_t = \int_0^t 1_{[-1, 0]}(X_s) ds + \frac{1}{2} 1_{\{t \geq T_0 > 0\}}$$

is the unique natural additive functional such that $u_A^1 = P_F^1 1$. Here $T_0 = T_{\{0\}}$. Since both the continuous and discontinuous part of A charge $\{0\}$, it is evident that (1.6) does not hold. Suppose there is a kernel $\bar{K}(t, x, f) = f$ a function of one variable—such that

$$E^x\{f(X_{L(t)}) \mid \mathfrak{F}(L^t-)\} = \bar{K}(t - L(t), X_{L(t)-}, f)$$

on $\{0 < L^t < t\}$. Let $x = -1$. Then $X_{L(t)-} = 0$ on $\{0 < L^t < t\}$. Also $\{L^t = 1\} \in \mathfrak{F}(L^t-)$ and $X_{L(t)} = 0$ on $\{L^t = 1 < t\} = \{1 < t < 1 + S\}$ where S is the holding time at 0. Therefore

$$0 = P^{-1}[X_{L(t)} = 1; L(t) = 1 < t] = \bar{K}(t - 1, 0, \{1\})P^{-1}[L(t) = 1 < t]$$

and since S has an exponential distribution

$$P^{-1}[L(t) = 1 < t] = P^{-1}[S > t - 1] > 0$$

for all $t > 1$. Thus $\bar{K}(t, 0, \{1\}) = 0$ for all $t > 0$. But then if $t > 1$

$$0 \neq P^{-1}\{X_{L(t)} = 1; 0 < L^t < t\} = E^{-1}\{\bar{K}(t - L(t), 0, \{1\}); 0 < L^t < t\} = 0$$

and so no such kernel can exist.

However, a version of Theorem 1.13 does hold without assuming (1.6) provided we are willing to use *two* kernels. See also [8]. The starting point is (3.20). Since B and C (in (3.20)) are continuous we have $B = b * (B + C)$ and $C = (1 - b) * (B + C)$ where b is a function between zero and one. Also D is a purely discontinuous NAF and so by Azema's Radon-Nikodym theorem for NAF's there exists a *previsible* process β_t such that $\beta_t dA_t = dD_t$ and $(1 - \beta_t) dA_t = d(C_t + B_t)$. Actually, in the present situation, it is not hard to see that β may be taken to be the indicator of $\{(t, \omega) : L(t) = t, X_t \notin F^r, X_t = X_{t-}\}$. In particular, $(\beta_t)^2 = \beta_t$ and $\beta_t(1 - \beta_t) = 0$. Define $r^1(t, x, f \otimes h) = [1 - b(x)]f(x)\gamma_t(x, h) + b(x)\rho_t(x, f, h)$ and $r^2(t, x, f \otimes h) = f(x)\gamma_t(x, h)$ where the notation is that of (3.20). Finally define $K^i(t, x, \Phi) = r^i(t, x, \Phi)/r^i(t, x, 1)$, $i = 1, 2$. Then arguing exactly as in the proof of (4.7) one obtains

$$(4.15) \quad E^x\{\Phi(X_{L(t)}, X_t) \mid \mathfrak{F}(L^t-)\} = [1 - \beta(L_t)]K^1(t - L(t), X_{L(t)-}, \Phi) \\ + \beta(L_t)K^2(t - L(t), X_{L(t)-}, \Phi).$$

If $\beta(t)$ is of the form $\psi(X_{t-})$ the two kernels can be combined into a single kernel and (4.15) reduces to (4.7). Note that (1.7) is equivalent to $\beta_t = \psi(X_{t-})$ and that $\varphi = (1 - \psi)b$ in this case.

5. The post L process. So far in this paper we have studied $L(t)$ for a fixed t . Also of great importance is $L = \sup \{s : X_s \in F\}$, the "last exit time" from F . Clearly $L = \lim_{t \rightarrow \infty} L(t)$, and so by letting $t \rightarrow \infty$ in our results we obtain information about L . We shall be particularly interested in the "post L process" $X^L = (X_{L+t})_{t \geq 0}$. For simplicity we assume $\zeta = \infty$ and that F is *transient* in the sense that $L < \infty$ almost surely. Under these assumptions $L(t) = L$ for sufficiently large t . These assumptions will be in force throughout this section. As before $T = T_F$. Of course, F is finely closed and nearly Borel.

Fix now a bounded nonnegative random variable H and $0 < \tau < t$. Let $\bar{H}_t(x) = E^x\{H; T > t - \tau\}$ and let $\bar{H}(x) = \lim_{t \rightarrow \infty} \bar{H}_t(x) = E^x\{H; T = \infty\}$. It is evident that $\{L(t) < \tau\} = \{L(\tau) < \tau; \tau + T \circ \theta_\tau > t\}$ and that $L(t) = L(\tau)$ on this set. Since $L(\tau) \leq \tau$ and $L(\tau)$ is $\mathfrak{F}_\tau$ measurable, it follows that $\mathfrak{F}(L(\tau)) \subset \mathfrak{F}_\tau$. (See the corollary of Lemma 4.5 in [8].) As a result if Z is a bounded nonnegative well measurable process one obtains, using (1.3),

$$\begin{aligned}
E^x \{ Z_{L(t)} H \circ \theta_\tau ; 0 < L(t) < \tau \} \\
&= E^x \{ Z_{L(\tau)} H \circ \theta_\tau ; 0 < L(\tau) < \tau ; T \circ \theta_\tau > t - \tau \} \\
&= E^x \{ Z_{L(\tau)} \bar{H}_t(X_\tau) ; 0 < L(\tau) < \tau \} \\
&= E^x \int_0^{\tau-} Z_s \gamma_{\tau-s}(X_s, \bar{H}_t) d\tilde{A}_s.
\end{aligned}$$

Letting $t \rightarrow \infty$ in this we find

$$(5.1) \quad E^x \{ Z_L H \circ \theta_\tau ; 0 < L < \tau \} = E^x \int_0^{\tau-} Z_s \gamma_{\tau-s}(X_s, \bar{H}) d\tilde{A}_s,$$

where as defined above

$$(5.2) \quad \bar{H}(x) = E^x \{ H ; T = \infty \}.$$

Note that if $H \equiv 1$, then $\bar{H}_t = Q_{t-\tau}1$ and $\bar{H}(x) = P^x[T = \infty]$. Define

$$(5.3) \quad \psi(x) = \lim_{t \rightarrow \infty} Q_t 1(x) = P^x[T = \infty] = P^x[L = 0],$$

and observe that $Q_t \psi = \psi$ for all $t \geq 0$. Also define

$$(5.4) \quad \gamma(x) = \lim_{t \rightarrow \infty} \gamma_t(x, 1),$$

the limit existing since the total mass of an entrance law decreases at t increases, and note that

$$\begin{aligned}
(5.5) \quad \gamma_t(x, \psi) &= \lim_{s \rightarrow \infty} \gamma_t(x, Q_s 1) \\
&= \lim_{s \rightarrow \infty} \gamma_{t+s}(x, 1) = \gamma(x)
\end{aligned}$$

for all $t > 0$. Now, as in Section 4, one obtains from (5.1) and (5.5)

$$(5.6) \quad E^x \{ H \circ \theta_\tau \mid \mathfrak{F}(L) \} = \Gamma(\tau - L, X_L, \bar{H})$$

on $\{0 < L < \tau\}$, where the kernel Γ is given by

$$\begin{aligned}
(5.7) \quad \Gamma(t, x, f) &= \gamma_t(x, f)/\gamma(x) & \gamma(x) \neq 0, \\
&= 0 & \gamma(x) = 0.
\end{aligned}$$

If $H \notin b\mathfrak{F}$, then $|\bar{H}(x)| \leq M\psi(x)$ where $M = \sup |H|$. As a result $|\Gamma(t, x, \bar{H})| \leq M$ for all $t > 0$ and x . In particular, $H \rightarrow \Gamma(t, x, \bar{H})$ is a *bounded* kernel. Also observe that if in (5.1) we set $H = 1$ and let $\tau \rightarrow \infty$ we obtain

$$(5.8) \quad E^x \{ Z_L ; L > 0 \} = E^x \int_0^\infty Z_s \gamma(X_s) d\tilde{A}_s.$$

Letting $Z_s = 1_{\{\gamma=0\}}(X_s)$ in (5.8) we find

$$P^x[X_L \in \{\gamma = 0\}; L > 0] = 0,$$

that is, $\gamma(X_L) > 0$ almost surely on $\{L > 0\}$. The formulas (5.6) and (5.7) should be viewed in the light of this fact.

We now transform (5.6) into a form that is more useful for studying the post L process.

(5.9) Proposition. *Let $H \in b\mathfrak{F}^0$ where $\mathfrak{F}^0 = \sigma(X_s; s \geq 0)$. Then if $t > 0$ one has on $\{L > 0\}$*

$$E^x\{H \circ \theta_{t+L} \mid \mathfrak{F}(L)\} = \Gamma(t, X_L, \bar{H}).$$

Proof. Fix $\Lambda_1 \in \mathfrak{F}(L)$ and let $\Lambda = \Lambda_1 \cap \{L > 0\}$. Then we must show that

$$(5.10) \quad E^x\{H \circ \theta_{t+L}; \Lambda\} = E^x\{\Gamma(t, X_L, \bar{H}); \Lambda\}.$$

Since both sides of (5.10) are measures in H , it suffices to prove (5.10) when

$$(5.11) \quad H = \prod_{i=1}^N f_i(X_{t_i})$$

where each f_i is a bounded continuous function and $0 \leq t_1 < t_2 < \dots < t_N$. Define $L^n = k2^{-n}$ if $(k-1)2^{-n} < L \leq k2^{-n}$, $k \geq 1$, $n \geq 1$ and $L^n = 0$ if $L = 0$. Then $L^n \downarrow L$ as $n \rightarrow \infty$ and for H of the form (5.10), $H \circ \theta_{t+L^n} \rightarrow H \circ \theta_{t+L}$ as $n \rightarrow \infty$. Consequently if $\tau(t, n, k) = t + k2^{-n}$

$$(5.12) \quad \begin{aligned} E^x\{H \circ \theta_{t+L}; \Lambda\} &= \lim_n \sum_k E^x\{H \circ \theta_{\tau(t, n, k)}; k - 1/2^n < L \leq k/2^n; \Lambda\} \\ &= \lim_n \sum_k E^x\{\Gamma(\tau(t, n, k) - L, X_L, \bar{H}); k - 1/2^n < L \leq k/2^n; \Lambda\} \\ &= \lim_n E^x\{\Gamma(t + L^n - L, X_L, \bar{H}); \Lambda\} \end{aligned}$$

where we have used the fact that $L < \tau(t, n, k)$ on $\{k - 1/2^n < L \leq k/2^n\}$ because $t > 0$. But if (ρ_t) is any entrance law for (Q_t) , then $\rho_{t+s}(\bar{H}) = \rho_t(Q_s \bar{H})$, while

$$\begin{aligned} Q_s \bar{H}(x) &= E^x\{E^{X(s)}[H; T = \infty]; s < T\} \\ &= E^x\{H \circ \theta_s; T \circ \theta_s = \infty; s < T\} \\ &= E^x\{H \circ \theta_s; T = \infty\} \rightarrow \bar{H}(x) \end{aligned}$$

as $s \downarrow 0$ since $H \circ \theta_s \rightarrow H$ for any H of the form (5.11). Consequently $\lim_{s \downarrow 0} \rho_{t+s}(\bar{H}) = \rho_t(\bar{H})$. But $(\Gamma(t, x, \cdot))$ is an entrance law for (Q_t) and so (5.10) follows from (5.12), completing the proof of (5.9).

Proposition 5.9 has some interesting consequences.

(5.13) Corollary. *If $H \in b\mathfrak{F}^0$, then for $t > 0$*

$$E^x\{H \circ \theta_{t+L}\} = E^x\{\Gamma(t, X_L, \bar{H})\}.$$

Proof. In view of Proposition 5.9 it suffices to show that

$$(5.14) \quad E^x\{H \circ \theta_{t+L}; L = 0\} = E^x\{\Gamma(t, X_L, \bar{H}); L = 0\}.$$

But $\{L = 0\} = \{T = \infty\}$ and so the left side of (5.14) becomes

$$E^x\{H \circ \theta_t; T = \infty\} = E^x\{H \circ \theta_t; T \circ \theta_t = \infty; t < T\} = Q_t \bar{H}(x).$$

On the other hand the right side of (5.14) is just $\Gamma(t, x, \bar{H})\psi(x)$. From (5.4) and the definition of $\gamma_t(x, \cdot)$ (see below (3.15) and (3.11)) it follows that $\gamma(x) > 0$ if $\psi(x) > 0$. Thus from (5.7) if $\psi(x) > 0$

$$\Gamma(t, x, \bar{H})\psi(x) = \psi(x)\gamma_t(x, \bar{H})/\gamma(x).$$

Moreover if $\psi(x) > 0$ then $x \notin F^r$ and so $\gamma_t(x, \bar{H}) = Q_t \bar{H}(x)/V^1 1(x)$. Finally for such an x , $\gamma(x) = \lim_{t \rightarrow \infty} q_t(x, 1) = \psi(x)/V^1 1(x)$, and combining these facts we see that

$$(5.15) \quad \Gamma(t, x, \bar{H})\psi(x) = Q_t \bar{H}(x)$$

provided $\psi(x) > 0$. But (5.15) is clearly valid if $\psi(x) = 0$, and consequently (5.14) is established completing the proof of (5.13).

If we take $H = f(X_0)$ with $f \in b\mathcal{E}$, then (5.13) becomes

$$(5.16) \quad E^x\{f(X_{t+L})\} = E^x\{\Gamma(t, X_L, f\psi)\}, t > 0,$$

since $\bar{H} = f\psi$ in this case. In particular taking $f = 1_{\{\psi > 0\}}$ we see that $\psi(X_{t+L}) > 0$ almost surely for each fixed $t > 0$. (Note ψ is Borel measurable since $1 - \psi = P_F 1$.) Define for $t > 0$

$$(5.17) \quad \Gamma^*(t, x, f) = E^x\{\Gamma(t, X_L, f\psi)\}$$

and

$$(5.18) \quad \begin{aligned} Q_t^L f(x) &= Q_t(f\psi)(x)/\psi(x) & \text{if } \psi(x) > 0, \\ &= 0 & \text{if } \psi(x) = 0. \end{aligned}$$

It is easy to check, making use of the invariance of ψ under (Q_t) , that $(Q_t^L)_{t \geq 0}$ is a semigroup of subprobability measures with $Q_t^L(x, \cdot)$ carried by $\{\psi > 0\}$ for each x and $Q_t^L(x, 1) = 1$ if $\psi(x) > 0$. If we apply (5.13) with $H = f(X_0)g(X_s)$ for $f, g \in b\mathcal{E}$, then $\bar{H} = fQ_s(\psi g)$ and so for $t > 0$

$$(5.19) \quad \begin{aligned} E^x\{f(X_{t+L})g(X_{t+s+L})\} &= E^x\{\Gamma(t, X_L, fQ_s(\psi g))\} \\ &= \int \Gamma^*(t, x, dy)f(y)Q_s^L g(y). \end{aligned}$$

Now it is known (see [15]) that $X^L = (X_t^L)_{t \geq 0}$ is a strong Markov process and (5.19) states that (Q_t^L) is its transition function and that $\Gamma^*(t, x, \cdot)$ is its entrance law under the measure P^x . The fact that (Q_t^L) is the transition function of X^L was established in [15]. It is clear from (5.15) that $Q_t^L(x, \cdot) = \Gamma(t, x, \cdot)$ when $\psi(x) > 0$. The fact that $\Gamma^*(t, x, \cdot)$ is an entrance law for (Q_t^L) is evident if we set $f = 1$ in (5.19) and use (5.16). Of course, one could establish the fact that

X^L is Markov (relative to the minimal fields $\sigma(X_s^L; s \leq t)$) by applying (5.13) when H is of the form (5.11). In [15] it is shown that X^L is (strong) Markov relative to fields that contain $\mathfrak{F}(L + t)$.

The starting point for the preceding discussion was Theorem 1.3. It is evident that there is an analogous development starting from Theorem 1.8. We content ourselves with stating the basic results and leave the details to the interested reader. In the statements that follow it is assumed that (1.6) holds, that $\zeta = \infty$, and that F is transient.

First of all, if Y is a nonnegative previsible process, H a nonnegative random variable, and $f \in \mathcal{E}_+$, then

$$(5.20) \quad E^x\{Y_L f(X_L) H \circ \theta_\tau; 0 < L < \tau\} = E^x \int_0^{\tau-} Y_s r(\tau - s, X_s, f \otimes \bar{H}) dA_s.$$

Let $r(x) = \lim_{t \rightarrow \infty} r(t, x, 1)$ and define

$$(5.21) \quad \begin{aligned} R(t, x, \Phi) &= r(t, x, \Phi)/r(x) & \text{if } r(x) > 0, \\ &= 0 & \text{if } r(x) = 0. \end{aligned}$$

Then

$$(5.22) \quad E^x\{f(X_L) H \circ \theta_\tau \mid \mathfrak{F}(L-)\} = R(\tau - L, X_{L-}, f \otimes \bar{H})$$

on $\{0 < L < \tau\}$, and if $t > 0$

$$(5.23) \quad E^x\{f(X_L) H \circ \theta_{t+L} \mid \mathfrak{F}(L-)\} = R(t, X_{L-}, f \otimes \bar{H})$$

on $\{L > 0\}$. We now assume that the set F is *admissible*. By the discussion following (1.7), $B + C = I_{F^c} * A$ in this case, and consequently we may assume that the function φ in (1.7) such that $B = \varphi * A$ vanishes off F^c . But then if $x \notin F^c$ it is evident from (3.22) that

$$r(t, x, f \otimes g) = f(x) \gamma_t(x, g).$$

Combining this with (5.23) one readily finds

$$(5.24) \quad E^x\{f(X_L) H \circ \theta_{t+L}\} = E^x\{R(t, X_{L-}, f \otimes \bar{H})\}$$

provided $t > 0$ where $X_{L-} = X_0$ on $\{L = 0\}$. If we define

$$R^*(t, x, f \otimes g) = E^x\{R(t, X_{L-}, f \otimes \psi g)\},$$

then (5.16), (5.17) and (5.24) imply that

$$R^*(t, x, 1 \otimes f) = \Gamma^*(t, x, f).$$

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Added in Proof. It has been pointed out to us that some of the results of this paper are related to the results in the following two references:

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