Commuting Observables in a σ -Orthologic

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Communicated by the Editors

1. Introduction. A σ -logic is a σ -orthocomplete orthomodular poset. Interest in this type of logic was motivated by the 1936 treatise of Birkhoff and J. von Neuman [1] in which they suggested that the logic of quantum mechanics, that is, the propositions associated with a quantum physical system, do not form a Boolean Algebra. Hence the conventional theory of probability, as laid out by Kolmogorov [9] was not applicable to quantum physics. Most contemporary practitioners of quantum logic seem to agree that a quantum logic is (at least) an orthomodular poset [4] [5] [11] [12] [13] [14] [15].

Two of the major differences involved in dealing with σ -orthocomplete orthomodular posets rather than Boolean Algebras in probability theory, have to do with the compatibility of elements and the definition of an observable. One characterization of a Boolean Algebra is that it is an orthomodular poset for which each one of its elements is compatible, in the sense of Mackey [10], with each of its other elements. This is a direct consequence of Varadarajan's proof that the center—the set of all elements which are compatible with all other elements—of a σ -complete orthomodular poset is a Boolean sub σ -algebra [15, Proposition 4.1]. Varadarajan also resolves the problem of observables in [15] by defining an observable on a σ -logic in a natural way, which, in a Boolean Algebra, reduces to the conventional notion.

In [2] and [3] Foulis and Randall introduced a new type of logic, called an orthologic, which represents the propositions associated with a general sample space, one in which the execution of more than one physical experiment is allowed. (The orthologic associated with a classical sample space is, indeed, a Boolean Algebra.) Most, if not all, of the logics which have been associated with quantum mechanics are orthologics. The notion of compatibility can be defined in an orthologic precisely as it was defined by Mackey. Furthermore, even though every orthologic need not be an orthomodular poset, the center of an orthologic enjoys precisely the same property as the center of an orthomodular poset, that is, the center is a Boolean Algebra [7]. Hence, any Boolean Algebra can also be characterized as an orthologic in which each of its elements is compatible with each of its other elements. An observable can be defined

in an orthologic precisely in the same manner as Varadarajan defined it in a σ -logic. Hence the two major points of departure in the work of Varadarajan from conventional probability theory are essentially the same as those in an orthologic, however the structure of an orthologic is weaker.

Varadarajan and Ramsay used these concepts in [15] and [13] to prove the following theorem: If A and B are two simultaneously observable observables in a σ -logic, then there exists an observable C and Borel functions f and g , satisfying the conditions that $A = C \circ f^{-1}$ and $B = C \circ g^{-1}$. The purpose of this paper is to show that this theorem is true in a wider class of "logics" than σ -logics, namely, in a large class of orthologics.

2. Definitions. We begin by defining the terms which Varadarajan and Ramsay used in the statement of the theorem. Suppose that $(L, \leq, 0, 1)$ is a partially ordered set satisfying $0 \leq x \leq 1$ for each $x \in L$. If $x, y \in L$, and the supremum (respectively, infimum) of x and y exists in L , we shall write it as $x \vee y$ (respectively $x \wedge y$). Then L is said to be an *orthocomplemented poset* if there exists a map $' : L \rightarrow L : x \mapsto x'$ satisfying the following properties for all $x, y \in L$:

- C1. $(x')' = x$
- C2. $x \wedge x' = 0, x \vee x' = 1$
- C3. $x \leq y$ implies $y' \leq x'$

If $x, y \in L$ we say that x is *orthogonal* to y , denoted $x \perp y$, if and only if $x \leq y'$. An orthocomplemented poset L is called an *orthomodular poset* or a *logic* if it satisfies the following two additional properties:

OMO: If $x \perp y$ then $x \vee y$ exists in L .

OMI: If $x \leq y$ then $y = x \vee (y \wedge x')$.

If L is a logic and $x, y \in L$, we say that x is *compatible* with y , or x *commutes* with y , in symbols $x \mathcal{C} y$ if there exists elements $x_1, y_1, d \in L$, which are pairwise orthogonal, such that $x = x_1 \vee d$ and $y = y_1 \vee d$. If $A, B \subseteq L$ we will write $A \mathcal{C} B$ to mean $a \mathcal{C} b$ for all $a \in A$ and for all $b \in B$. If $A \subseteq L$ and $b \in L$ we will write $b \mathcal{C} A$ to mean $\{b\} \mathcal{C} A$. The *center* of L , denoted $\mathcal{C}(L)$, is the set of all $x \in L$ having the property that $x \mathcal{C} L$.

An orthomodular poset L is called a σ -*orthocomplete orthomodular poset*, or σ -*logic*, if the supremum of every countable family of pairwise orthogonal elements of L exists in L .

We will write $\underline{B}$ for the Borel subsets of the real line, $\mathbb{R}$ for the real line, and $\mathbb{N}$ for the natural numbers. An *observable* [10, p. 66] on a σ -logic L is a function $a : \underline{B} \rightarrow L$ which satisfies the following properties:

- 01. If $E, F \in \underline{B}$ and $E \cap F = \emptyset$, then $a(E) \perp a(F)$.
- 02. If $\{E_i : i \in \mathbb{N}\}$ is a countable family of pairwise disjoint sets in $\underline{B}$ then $a(\bigcup_{i \in \mathbb{N}} E_i) = \bigvee_{i \in \mathbb{N}} a(E_i)$.
- 03. $a(\emptyset) = 0, a(\mathbb{R}) = 1$

If a and b are observables then they are said to be *simultaneously observable* if and only if $a(E) \mathcal{C} b(F)$ for all $E, F \in \underline{B}$.

We turn now to some definitions regarding orthologies. An *orthologic* is a system $(L, \leq, 0, 1, \perp)$ satisfying the following properties:

- L1. $(L, \leq, 0, 1)$ is a partially ordered set such that $0 \leq x \leq 1$ holds for every $x \in L$.
- L2. $\perp$ is a symmetric binary relation defined on L such that, for every $x \in L$, if $x \perp x$, then $x = 0$.
- L3. If $x, y \in L$ with $x \perp y$, then the supremum, $x \vee y$, exists in L .
- L4. If $x, y, z \in L$ with $x \perp y$, $y \perp z$ and $x \perp z$, then $x \perp (y \vee z)$.
- L5. If $x \in L$, then there exists $y \in L$ with $x \perp y$ and $x \vee y = 1$.
- L6. If $x, y \in L$, then $x \leq y$ holds if and only if for every $z \in L$ the condition $z \perp y$ implies the condition $z \perp x$.

The elements $x \in L$ are called *propositions*. If $x, y \in L$ with $x \leq y$, then we say that x *implies* y . If $x, y \in L$ with $x \perp y$, we say that x and y *refute* each other or that x is *orthogonal* to y . If $x, y \in L$ and $x \perp y$ we shall write $x \oplus y$ for the supremum of x and y . We will also use the notation $x \vee y$ and $x \wedge y$ as defined earlier. We define the terms *compatibility* or *commutivity*, *center* and *σ -orthocomplete* for an orthologic, precisely as they were defined for a logic.

If L is an orthologic, then L' is said to be a *sublogic* of L if $L' \subseteq L$ and $(L', \leq, 0, 1, \perp)$ is an orthologic, where $\leq$ and $\perp$ are the restrictions of $\leq$ and $\perp$ from L to L' . Note that if $\{0, 1\} \subseteq L' \subseteq L$ then L' is a sublogic of L if and only if the following conditions hold:

- S1. If $x, y \in L'$ with $x \perp y$ then $x \oplus y \in L'$.
- S2. If $x \in L'$ then there exists $y \in L'$ with $x \oplus y = 1$.
- S3. If $x, y \in L'$ and if for all $z \in L'$ the condition $z \perp y$ implies the condition $z \perp x$ then $x \leq y$.

If $L' \subseteq L$ then L' is said to be a *Boolean sublogic* of L in case L' is a sublogic of L and also a Boolean Algebra under the same operations. As remarked earlier, to verify that a sublogic L' of L is a Boolean sublogic, it is sufficient to show that for each $x, y \in L'$, x commutes with y in L' . A Boolean sublogic of L is called a *block* if it is not properly contained in any other Boolean sublogic of L . If L is a σ -orthocomplete orthologic and L' is a Boolean sublogic of L , then L' is called a *Boolean sub σ -logic* of L if whenever $\{x_i : i \in N\}$ is a countable family of pairwise orthogonal elements of L' then $\bigoplus_{i \in N} x_i \in L'$. Note that this implies that $\bigoplus_{i \in N} x_i$ is the supremum of $\{x_i : i \in N\}$ in L' as well as in L .

If $e \in L$ we write $L[0, e]$ for the set of all $x \in L$ with $0 \leq x \leq e$. The element $e \in L$ is called *regular* if $(L[0, e], \leq, 0, e, \perp)$ is an orthologic, where $\leq$ and $\perp$ are the restrictions of $\leq$ and $\perp$ in L . More precisely, $e \in L$ is regular if and only if the following two conditions hold:

- R1. If $x \in L$ with $x \leq e$ then there exists $y \in L$ with $y \leq e$, satisfying the properties that $x \perp y$ and $x \oplus y = e$.
- R2. If $x, y \in L$ with $x \leq e$ and $y \leq e$, and if for every $z \in L$ with $z \leq e$ the condition $z \perp y$ implies the condition $z \perp x$, then $x \leq y$ holds.

An orthologic is called *regular* if each of its elements, other than 0, is regular.

We define the concepts of *observable* and *simultaneously observable* observables

for a σ -orthologic precisely as they were defined for a σ -logic. A similar argument to that given in [15, Proposition 3.5] shows that if x and y are observables on a σ -orthologic and E and F are Borel sets then $x(E) \subseteq y(F)$ if and only if there exists an observable z and Borel sets G and H with $x(E) = z(G)$ and $y(F) = z(H)$, that is, if and only if the propositions $x(E)$ and $y(F)$ can be observed simultaneously.

We will say that L is a σ -orthologic if it is a regular, σ -orthocomplete orthologic having the property that every block in L is a Boolean sub σ -logic of L . Our objective is to prove the theorem of Varadarajan and Ramsay, which was stated in §1, in the setting of a σ -orthologic.

We comment briefly on the difference between σ -logics and σ -orthologics. It is straightforward to show that every σ -logic (logic) is a σ -orthologic (respectively, orthologic). The converse is false. We will say that an orthologic L is an *orthocomplemented orthologic* if $(L, \leq, 0, 1)$ is an orthocomplemented poset and for all $x, y \in L$, $x \perp y$ if and only if $x \leq y'$. Then an orthologic is a logic if and only if it is an orthocomplemented orthologic and satisfies OMI. We emphasize that an orthologic, which is an orthocomplemented poset and satisfies OMI, via the orthogonality relation induced by the orthocomplementation, may fail to be a logic because this orthogonality relation is not the orthogonality relation on the orthologic. Cf. [§5] On the other hand, if L is an orthologic, and there exists a function $' : L \rightarrow L : x \rightarrow x'$ satisfying $x \perp y$ if and only if $x \leq y'$, then L is an orthocomplemented orthologic, but L need not satisfy the OMI and hence need not be a logic. Cf. [8, p. 18].

If L is an orthocomplemented σ -orthologic, then the regularity condition on L guarantees that L is a σ -logic (respectively, logic) if and only if it is an orthocomplemented orthologic [8]. We showed in [7] that an orthologic L is an orthomodular poset if and only if for each $x \in L$ there exists a unique $y \in L$ with $x \oplus y = 1$.

We will postpone exhibiting examples of σ -orthologics which are not σ -logics until §5. The implications made by the choice of vocabulary provide much of the necessary insight into the motivation for the definition of an orthologic. For further motivation the interested reader is referred to [2] and [3].

3. Properties of σ -orthologics. The proofs presented by Varadarajan and Ramsay rely strongly on the proof in [15] that if A and B are Boolean subalgebras of a σ -logic L , having the property that $A \subseteq B$, then there exists a smallest Boolean sub σ -algebra of L containing $A \cup B$. Our first task is to prove the equivalent of this theorem in a σ -orthologic. That is, we wish to prove the following:

Theorem 1. *If A and B are Boolean sublogics of a σ -orthologic L , having the property that $A \subseteq B$, then there exists a smallest Boolean sub σ -logic of L containing $A \cup B$.*

The proof will follow from a series of lemmas about regular orthologics.

Lemma 1. *Let L be a regular orthologic. If $x, y \in L$ and $x \perp y$ then $x \vee y$ and $x \wedge y$ exist in L . Furthermore, if A is any Boolean sublogic of L containing x and y , then the supremum (infimum) of x and y in A is equal to $x \vee y$ (respectively $x \wedge y$).*

Proof. Write $x = x_1 \oplus d$ and $y = y_1 \oplus d$, where $x_1 \perp y_1$. Then $x \vee y = x_1 \oplus y_1 \oplus d$. Clearly d is a lower bound for x and y . Suppose $r \in L$ is also a lower bound. If $s \in L$ with $s \leq x$ and $s \perp d$ then $s \perp y$ and hence $s \perp r$. Thus $r \leq d$ and $x \wedge y = d$. If $x, y \in A$ then we may choose x_1, y_1 and d to be in A , and hence $x \vee y$ and $x \wedge y$ are the supremum and infimum of x and y in A .

We remark that the existence of $x \vee y$ in the above lemma does not require the regularity condition. However, there exist non-regular orthologies, even orthocomplemented ones, in which $x \wedge y$ need not exist even when $x \perp y$ [8, page 21].

Lemma 2. *Let L be a regular orthologic. Assume that x, t, t_0 are pairwise commuting elements of L with $t \oplus t_0 = 1$. Then $x = (x \wedge t) \oplus (x \wedge t_0)$.*

Proof: Write $x = x_1 \oplus d, t = t_1 \oplus d, x = x_2 \oplus s$ and $t_0 = t_2 \oplus s$, where $x_1 \perp t_1$ and $x_2 \perp t_2$. It follows from Lemma 1 that $(x \wedge t) \oplus (x \wedge t_2) = d \oplus s$. Since $d \oplus s \leq x$ and x is regular, there exists $a \in L$ with $d \oplus s \oplus a = x$. Then $a \perp d$ and $a \perp t_1$ implies that $a \perp t$. Similarly, $a \perp t_0$ and hence $a = 0$. So $d \oplus s = x$.

Lemma 3. *Let L be a regular orthologic, and let A be a Boolean sublogic of L . Assume that $t, t_0 \in L$ having the properties that $\{t, t_0\} \subseteq A$ and $t \oplus t_0 = 1$. Then:*

- (i) *If $x, y \in A$ then $(x \wedge t) \vee (y \wedge t) = (x \vee y) \wedge t$.*
- (ii) *There exists a smallest Boolean sublogic F of L with the property that $A \cup \{t, t_0\} \subseteq F$.*
- (iii) *If B is a Boolean sublogic of L with $t, t_0 \in B$ and if $A \subseteq B$, then $F \subseteq B$.*

Proof: Assume first that $x \perp y$. Then $(x \wedge t) \oplus (y \wedge t)$ exists in L and is clearly less than or equal to $(x \oplus y) \wedge t$. Assume $z \in L$ with $z \leq t$ and $z \perp (x \wedge t) \oplus (y \wedge t)$. Then $z \perp (x \wedge t) \oplus (y \wedge t) \oplus t_0$, so by Lemma 2, $z \perp x \oplus y$. Hence $(x \oplus y) \wedge t = (x \wedge t) \oplus (y \wedge t)$.

We now drop the condition $x \perp y$. Write $x = x_1 \oplus d$ and $y = y_1 \oplus d$ where $x_1 \perp y_1$ and $x_1, y_1, d \in A$. By the argument in the preceding paragraph, $(x \wedge t) = (x_1 \wedge t) \oplus (d \wedge t)$ and $y \wedge t = (y_1 \wedge t) \oplus (d \wedge t)$. Now $(x \wedge t) \vee (y \wedge t)$ and $(x \vee y) \wedge t$ exist in L by Lemma 1.

It is enough to show that $(x \vee y) \wedge t \leq (x \wedge t) \vee (y \wedge t)$, since the other inequality is clear. The proof is similar to that in the first paragraph, with the additional observation that $x \vee y = x_1 \oplus y_1 \oplus d$.

(ii) For each $x \in A$ there exists a unique element in A which is effective as the orthocomplement of x in A . We will denote this element by x' . Let $F = \{(x \wedge t) \oplus (y \wedge t_0) : x, y \in A\}$. By Lemma 2, $A \subseteq F$. Clearly $t, t_0 \in F$.

We will verify first that F is a sublogic of L . The fact that if $f_1, f_2 \in F$ with $f_1 \perp f_2$ then $f_1 \oplus f_2 \in F$, follows easily from part (i). If $f = (x \wedge t) \oplus (y \wedge t_0)$ where $x, y \in A$, then $g = (x' \wedge t) \oplus (y' \wedge t_0)$ is an element of F , is orthogonal to f , and it follows from (i) that $f \oplus g = 1$. Assume that $f_1 = (x_1 \wedge t) \oplus (y_1 \wedge t_0)$ and $f_2 = (x_2 \wedge t) \oplus (y_2 \wedge t_0)$ are elements of F , where $x_1, x_2, y_1, y_2 \in A$. Assume further that for all $f \in F$, $f \perp f_2$ implies $f \perp f_1$. We must show that $f_1 \leq f_2$. Let $f_3 = (x_2' \wedge t) \oplus (y_2' \wedge t_0)$. Then $f_3 \in F$ and $f_3 \perp f_2$ so $f_3 \perp f_1$. Thus $(x_2' \wedge t) \perp (x_1 \wedge t)$ and $(y_2' \wedge t_0) \perp (y_1 \wedge t_0)$. But $x_1 = (x_1 \wedge x_2) \oplus (x_1 \wedge x_2')$ and hence $x_1 \wedge t = [(x_1 \wedge x_2) \oplus (x_1 \wedge x_2')] \wedge t$. Then by (i), $x_1 \wedge t = (x_1 \wedge x_2 \wedge t) \oplus (x_1 \wedge x_2' \wedge t)$. But since $(x_1 \wedge x_2' \wedge t) = 0$, it follows that $x_1 \wedge t \leq x_2 \wedge t$. A similar argument, which uses the fact that $(y_2' \wedge t_0) \perp (y_1 \wedge t_0)$, shows that $y_1 \wedge t_0 \leq y_2 \wedge t_0$. Hence $f_1 \leq f_2$, and F is a sublogic of L .

We have left only to show that F is Boolean; for, if it is, then it must be contained in every Boolean sublogic of L which contains $A \cup \{t, t_0\}$ —in fact, it must be contained in every sublogic of L containing $A \cup \{t, t_0\}$ which is a lattice—and hence it is the smallest such. As remarked earlier, the center of any orthologic is a Boolean Algebra, and hence it is enough to show that $F = \mathcal{C}(F)$. To this end, assume that $f_1, f_2 \in F$, defined as in the preceding paragraph. We must show that $f_1 \mathcal{C} f_2$ in F . Since $x_1 \mathcal{C} x_2$ in A , there exists $a, c, d \in A$ with $x_1 = a \oplus d$, $x_2 = c \oplus d$, and $a \perp c$. By (i), $x_1 \wedge t = (a \wedge t) \oplus (d \wedge t)$ and $x_2 \wedge t = (c \wedge t) \oplus (d \wedge t)$. Similarly, since $y_1 \mathcal{C} y_2$ in A , there exists $b, e, g \in A$, which are pairwise orthogonal, with $y_1 \wedge t_0 = (b \wedge t_0) \oplus (g \wedge t_0)$ and $y_2 \wedge t_0 = (e \wedge t_0) \oplus (g \wedge t_0)$. Write $u = (a \wedge t) \oplus (b \wedge t_0)$, $v = (d \wedge t) \oplus (g \wedge t_0)$ and $w = (c \wedge t) \oplus (e \wedge t_0)$. Then $f_1 = u \oplus v$, $f_2 = w \oplus v$, each of u, v and w is in F , and they are pairwise orthogonal. Hence $f_1 \mathcal{C} f_2$.

(iii) Assume that B is a Boolean sublogic of L with $t, t_0 \in B$ and $A \mathcal{C} B$. Let $f = (x \wedge t) \oplus (y \wedge t_0)$ be an arbitrary element of F , where $x, y \in A$. It is enough to show that $f \mathcal{C} B$. By (ii), $F_x = \{(b \wedge x) \oplus (g \wedge x') : b, g \in B\}$ is the smallest Boolean sublogic of L containing $B \cup \{x, x'\}$. Let $z \in B$ be arbitrary. Let z' denote the unique orthogonal complement of z in B . Then $z, z', x \wedge t \in F_x$ implies $z \mathcal{C} (x \wedge t)$ and $z' \mathcal{C} (x \wedge t)$. So by Lemma 2, $x \wedge t = (x \wedge t \wedge z) \oplus (x \wedge t \wedge z')$. A similar argument shows that $y \wedge t_0 = (y \wedge t_0 \wedge z) \oplus (y \wedge t_0 \wedge z')$. But then $(x \wedge t) \oplus (y \wedge t_0) = [(x \wedge t \wedge z) \oplus (y \wedge t_0 \wedge z)] \oplus [(x \wedge t \wedge z') \oplus (y \wedge t_0 \wedge z')]$. Now there exists $e \in L$ with $z = [(x \wedge t \wedge z) \oplus (y \wedge t_0 \wedge z)] \oplus e$, and since $e \leq z$, it follows that $e \perp [(x \wedge t \wedge z') \oplus (y \wedge t_0 \wedge z')]$. Hence $z \mathcal{C} f$.

Lemma 4. *If A and B are Boolean sublogics of a regular orthologic L having the property that $A \mathcal{C} B$, then there exists a smallest Boolean sublogic of L containing $A \cup B$.*

We will only give here a brief description of the proof, since it follows exactly the same lines as the proof presented by Ramsay [13, Lemma 10]. If $X \subseteq L$, write $[X]$ for the smallest Boolean sublogic of L containing X if it exists. Let $\mathcal{E}$ be a maximal inclusion chain in the set of Boolean sublogics E of B having the

property that $[A \cup E]$ exists and $[A \cup E] \subseteq B$. This chain is non-trivial by the last lemma. Set $T = \bigcup \{[A \cup E] : E \in \mathcal{E}\}$. It is straightforward to verify that $T = [A \cup (\bigcup \mathcal{E})]$, $\bigcup \mathcal{E} \in \mathcal{E}$ and $T \subseteq B$. Then Lemma 3(iii) may be used to show that $B = \bigcup \mathcal{E}$.

We are now ready to prove Theorem 1. Suppose A and B are Boolean sublogics of a σ -orthologic L and $A \subseteq B$. Then by Lemma 4 there exists a smallest Boolean sublogic F of L with $A \cup B \subseteq F$. Extend F to a block, via Zorn's Lemma, and call it F' . Let $\mathcal{G}$ denote the set of all Boolean sub σ -logics G of L with the property that $A \cup B \subseteq G \subseteq F'$. Since $F' \in \mathcal{G}$, $\mathcal{G} \neq \emptyset$. It is straightforward to verify that $\bigcap \mathcal{G}$ is an element of $\mathcal{G}$.

4. Main Theorem. *If a and b are two simultaneously observable observables in a σ -orthologic L , then there exists an observable c and Borel functions f and g , satisfying the conditions that $a = c \circ f^{-1}$ and $b = c \circ g^{-1}$.*

Again we will follow the outline of the proof presented by Ramsay. We therefore refer the interested reader to [13] for the details.

Proof. Write A for the image of a and B for the image of b . It is straightforward to show that A and B are Boolean sub σ -logics of L . $A \subseteq B$ by hypothesis, so by Theorem 1, there exists a smallest Boolean sub σ -logic D of L containing $A \cup B$. Since A and B are both σ -homomorphisms, applying [15, Proposition 3.15] it follows that A and B are countably generated as Boolean σ -algebras. If D_A and D_B are their respective countable generating sets, then $A \cup B = [D_A] \cup [D_B] \subseteq [D_A \cup D_B] \subseteq [A \cup B] = D$, and hence $[A \cup B] = [D_A \cup D_B] = D$, so $D_A \cup D_B$ is a countable generating set for D . Again applying [15, Proposition 3.15] we have that there exists a σ -homomorphism $c : D \rightarrow D$. Then c is an observable. The remainder of the proof is identical with that appearing in [13].

5. Examples. Let G denote any undirected graph with no loops, and let V denote its set of vertices. If $x, y \in V$, write $x \# y$ to mean that x and y are connected by an edge. If $A \subseteq V$, write $A^* = \{y \in V : y \# a \text{ for all } a \in A\}$ and A^{**} for $(A^*)^*$. Let $L = \{A^{**} : A \subseteq V\}$. Then L is an orthologic, where the partial order is containment and $A^{**} \perp B^{**}$ if and only if $A \subseteq B^*$. Every finite orthologic arises in this way. It is orthomodular if and only if whenever E is the set of vertices of a clique of G and $E \subseteq \{x\}^* \cup \{y\}^*$ for $x, y \in V$, then $x \# y$ [2].

Examples of σ -orthologies which are not logics may be found in [8, p. 18(ii) and p. 12]. These examples are finite, but may be extended to arbitrary cardinalities in the obvious ways. The familiar cartesian product and horizontal sum [6] of families of σ -orthologies provide further examples. If at least one of the σ -orthologies in the family is not a logic then neither is the product or sum.

We contend that in the search for the weakest structures which satisfy the theorem of Varadarajan and Ramsay, in order that the physical interpretations make sense, we are surely not interested in anything weaker than σ -orthocomplete

orthologies in which the blocks are sub σ -orthocomplete. The graph below provides some justification for our regularity condition.

The theorem does not hold for the logic affiliated with this graph. The logic enjoys all the properties of a σ -orthologic except regularity.

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Date communicated: DECEMBER 14, 1972