On Bers' Theorem on Isolated Singularities

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A well-known theorem of Bers ([1]) states that every solution of the minimal surface equation in a punctured disk extends to a solution in the full disk; *i.e.*, all isolated singularities are removable. The surprising feature of this result is that one does not even have to assume the solution bounded, as is the case, for example, for harmonic functions.

If one considers surfaces in $\mathbf{R}^n$, given non-parametrically by

$$x_k = f_k(x_1, x_2), \quad k = 3, \dots, n,$$

and if we denote by f the vector $(f_3, \dots, f_n)$, then the condition that the surface be a minimal surface is that f satisfy the vector equation

$$(1) \quad (1 + |f_y|^2)f_{xx} - 2(f_x \cdot f_y)f_{xy} + (1 + |f_x|^2)f_{yy} = 0,$$

where $x = x_1$, $y = x_2$ (see, for example, [6], p. 17).

As soon as the codimension is at least two, it turns out that equation (1) can have isolated singularities, even with bounded solutions. An example in $\mathbf{R}^4$ ([7], p. 287) is

$$f(x, y) = (r^2 + 4)^{1/2} (\sin \theta, -\cos \theta),$$

where (r, θ) are polar coordinates in the x, y -plane. Note that in this example, *neither* coordinate x_3 nor x_4 tends to a limit at the singular point (in this case, the origin). This fact turns out not to be an accident, but rather is characteristic of non-removable singularities. Namely, the following result holds.

Theorem. *If a vector f satisfies (1) in a punctured disk, and if all but one of the components tends to a limit at the center, then so does the other component, and f extends to a solution in the full disk.*

In the case $n = 3$, this theorem reduces to Bers' original theorem. The proof follows essentially the same lines as Bers' proof, except that Bers makes extensive use of a representation theorem. Since no such theorem is available in higher codimension, we use a different reasoning, which even in three dimensions leads somewhat more directly to the result. (Bers, on the other hand,

develops a much more elaborate theory, with his theorem on removable singularities appearing as a special case.)

We begin with a result about parametric surfaces which may be of some interest in its own right.

Theorem 1. *Let D be a doubly-connected plane domain with boundary components C, C' . Let $x : D \rightarrow \mathbf{R}^n$ define a doubly-connected minimal surface.*

If $n - 1$ of the coordinate functions tend to a limit at one of the boundary components, say C , then the surface extends analytically over $x(C)$ to a generalized minimal surface. There are two possibilities:

Case 1. The extended surface is obtained by adding a single point. This happens $\Leftrightarrow$ all n coordinates tend to a limit at $C \Leftrightarrow$ the component C is conformally a point boundary of $x(D)$.

Case 2. The curve C maps onto a line segment. The extended surface is obtained by reflecting $x(D)$ across this line.

Proof. By the uniformization theorem, we may map D diffeomorphically onto an annular domain

$$A : 0 \leq r_1 < |w| < r_2 \leq \infty,$$

where C corresponds to $|w| = r_1$, such that u, v are isothermal parameters on the surface; $w = u + iv$. Then each coordinate x_k is harmonic in w , and the functions

$$\phi_k(w) = \frac{\partial x_k}{\partial u} - i \frac{\partial x_k}{\partial v}$$

are analytic and satisfy

$$(2) \quad \sum_{k=1}^n \phi_k^2 \equiv 0.$$

(For standard properties of minimal surfaces in $\mathbf{R}^n$, see [6].)

Case 1. $r_1 = 0$. In this case, each of the functions $x_k(w)$ has an isolated singularity at the origin. Since $n - 1$ of them, say $x_1, \dots, x_{n-1}$, tend to a limit at $w = 0$, the singularity is removable. Hence the functions $\phi_1, \dots, \phi_{n-1}$ are analytic in $|w| < r_2$. By (2), so is the function ϕ_n^2 . Thus ϕ_n is bounded and also has a removable singularity at the origin. The same is then true of $x_n = \operatorname{Re} \int \phi_n$. Thus $x(w)$ extends analytically over the origin, defining a generalized minimal surface in $|w| < r_2$.

Case 2. $r_1 > 0$. We may as well assume $r_1 = 1$, and that the functions $x_1, \dots, x_{n-1}$ tend to zero as $|w| \rightarrow 1$. Setting $w = e^{-i\zeta}$, we obtain the composed map $x(\zeta)$ from the infinite strip $S_0 : 0 < \eta < \log r_2$ into $\mathbf{R}^n$, $\zeta = \xi + i\eta$. We may now apply the reflection principle for minimal surfaces in $\mathbf{R}^n$ (see [6], p. 54). Namely, by the reflection principle for harmonic functions, defining $x_k(\bar{\zeta}) = -x_k(\zeta)$ extends x_k to be harmonic in the strip $S : -\log r_2 < \eta < \log r_2$, for $k = 1, \dots, n - 1$. Thus, each of the functions

$$\phi_k(\zeta) = \frac{\partial x_k}{\partial \xi} - i \frac{\partial x_k}{\partial \eta}, \quad k = 1, \dots, n-1,$$

is analytic in S and pure imaginary on $\eta = 0$. By (2), the function ϕ_n^2 , defined in S_0 , extends continuously to the real axis, taking on non-negative real values there. Hence ϕ_n extends to the real axis and is real-valued there. Setting $\phi_n(\bar{\zeta}) = \overline{\phi_n(\zeta)}$ extends ϕ_n to an analytic function in S , and defining

$$x_n(z) = \operatorname{Re} \int_0^z \phi_n(\zeta) d\zeta,$$

we have a harmonic extension of x_n to S , satisfying $x_n(\bar{z}) = x_n(z)$. It follows that the extended functions $x_k(\zeta)$ have the same real periods in S as in S_0 , and hence we get a harmonic map of the annulus $1/r_2 < |w| < r_2$, extending the original map. Equation (2) continues to hold, so that the extended map defines a generalized minimal surface. On the circle $|w| = 1$, the functions $x_1, \dots, x_{n-1}$ all vanish, while x_n has a finite maximum and minimum. The image of this circle is therefore a line segment along the x_n -axis, and the way the functions x_k were extended shows that the image of $1/r_2 < |w| < 1$ is the reflection of the original surface in the x_n -axis. Note that the function x_n cannot be constant on $|w| = 1$, since that would mean the entire circle $|w| = 1$ would map into a point, which is impossible for a generalized minimal surface ([6], Lemma 7.2). Thus, if all n coordinate functions tend to a limit at the boundary component C , Case 1 must hold. This shows the equivalence of the various statements in Case 1, and completes the proof of the theorem.

Before going on, we recall the lemma of Lewy (see [1], p. 309): *A one-one harmonic mapping of a plane domain is a diffeomorphism.*

Proposition 1. *If, in Case 1 of Theorem 1, the surface $x(D)$ has a one-one projection onto any plane, then the extended surface is regular.*

Proof. Suppose the point $x(C)$ is the origin, and assume that $x(D)$ projects one-one onto the x_1, x_2 -plane. We then obtain a one-one harmonic map of the punctured disk $0 < |w| < r_2$ into the x_1, x_2 -plane. No point $w_0 \neq 0$ can map into the origin since, if it did, the set $|w - w_0| < |w_0|/2$ would map onto a neighborhood N of the origin by Lewy's lemma, and since $x(w)$ extends continuously to $w = 0$, there would also be points arbitrarily near $w = 0$ mapping into N . Thus the extended map of $|w| < r_2$ into the x_1, x_2 -plane is a one-one harmonic map with the origin going into the origin, and applying Lewy's lemma again shows that the extended map is a diffeomorphism. Hence the extended surface is a regular surface.

Proposition 2. *In Case 2 of Theorem 1, the surface $x(D)$ can not have a one-one projection onto any plane orthogonal to the line segment $x(C)$.*

Proof. Let $x(C)$ lie along the x_n -axis, and suppose that $x(D)$ has a one-one projection on the x_1, x_2 -plane. We then have a generalized minimal surface

defined in $1/r_2 < |w| < r_2$, with x_1 and x_2 equal to zero on $|w| = 1$, and x_n non-constant on $|w| = 1$. Let b, c be points on $|w| = 1$ where x_n takes on its minimum and maximum, respectively. At each of these points, the arcs of $|w| = 1$ on either side of the point must map onto the same part of the x_n -axis. Hence the map $x(w)$ is not one-one in any neighborhood of the points $w = b, w = c$, and hence these must be branch points; *i.e.*, all the ϕ_k must vanish at b and c .

In order to study the map in a neighborhood of the point b , make the preliminary transformation $w = be^{-i\xi}$. Then the map

$$\xi, \eta \rightarrow x_1, x_2$$

is a harmonic map in a disk $|\zeta| < \epsilon$, is one-one in the upper half-disk $\eta > 0$, and maps the diameter $\eta = 0$ onto the origin. It follows that x_1 and x_2 have expansions

$$(3) \quad \begin{aligned} x_1 &= \operatorname{Re} \{i\alpha\zeta^\ell + \cdots\}, \\ x_2 &= \operatorname{Re} \{i\beta\zeta^m + \cdots\}, \end{aligned}$$

where α and β are real, non-zero, and $\ell, m \geq 2$ since b is a branch point. In case $\ell = m$, make the preliminary transformation

$$\begin{aligned} y_1 &= \alpha x_1 + \beta x_2, \\ y_2 &= -\beta x_1 + \alpha x_2. \end{aligned}$$

Then the map $\xi, \eta \rightarrow y_1, y_2$ has the same properties, but y_2 vanishes to a higher order than y_1 . Thus in the expansion (3), we may assume $m > \ell \geq 2$.

We next assert that $m \leq 3$. Namely, if $m \geq 4$, then the set $x_2 = 0$ would include at least three arcs in the upper half-plane $\eta > 0$, emanating from the origin. Each of these must map one-one onto a segment of the x_1 -axis starting at $x_1 = 0$. Thus at least two must map onto the same segment, contradicting our hypothesis.

The only possibility is therefore $\ell = 2, m = 3$. For sufficiently small $\epsilon > 0$, the set of points in the half-disk $|\zeta| < \epsilon, \eta > 0$, where $x_2 = 0$ consists of exactly two arcs joining the origin to points on $|\zeta| = \epsilon$. These two arcs bound a sector in which x_2 is everywhere positive or everywhere negative, say the former. Then this sector maps one-one onto a domain in the x_1, x_2 -plane bounded by an interval of the x_1 -axis containing the origin, and an arc in the upper half-plane joining the end points of the interval. Since the map continues to be a diffeomorphism at each boundary point of the sector other than the origin, we conclude that image in the x_1, x_2 -plane contains in its interior an open set including all points of a *closed* half-disk except for the origin.

Now exactly the same argument in a neighborhood of the point c yields a second such domain in the image (after a possible rotation in the x_1, x_2 -plane). But any two such domains have a non-empty intersection, so that the map could not be globally one-one.

Theorem 2. *Any solution of the minimal surface equation (1) in a punctured disk, such that all but one of the f_k tends to a limit at the center, extends to a solution in the full disk.*

Proof. Let D be the punctured disk, say $0 < x_1^2 + x_2^2 < \delta^2$. Then the solution of (1) defines a minimal surface $x(D)$ in $\mathbb{R}^n$ for which x_1 and x_2 tend to zero at the center, and all the other components x_k except for at most one also tend to a limit. We may therefore apply Theorem 1. Since the surface $x(D)$ obviously has a one-one projection on the x_1, x_2 -plane, it follows from Proposition 2 that Case 2 in Theorem 1 cannot occur. Then by the proof of Proposition 1, the extended surface is given by a parametric map $x(w)$ in a full disk, such that the projection $w \rightarrow x_1, x_2$ is a diffeomorphism. It follows that the extended surface is a solution of (1) in the full disk $x_1^2 + x_2^2 < \delta^2$.

Remarks 1. Theorem 1 remains true if we assume that the map $x : D \rightarrow \mathbb{R}^n$ defines a generalized minimal surface. The proof is unchanged.

2. One may ask whether, in Case 2 of Theorem 1, the surface $x(D)$ could have a one-one projection on *any* plane, not necessarily orthogonal to the line $x(C)$. The answer is, it can, as is shown by the following example, adapted from a construction due to Courant ([2], p. 123).

Let P be a pentagon in the x, y -plane containing two equal sides A_1, A_2 subtending an angle of 120° , two equal sides B_1, B_2 perpendicular to A_1, A_2 , respectively, and a side C . Let A_2 be the interval $[-1, 0]$ on the x -axis, and let B_2 lie along the positive y -axis. Let f be a continuous function on the boundary of P which is zero except for some interval in C where it is positive. Then there exists a solution of the minimal surface equation (1) in the interior of P taking on these boundary values. Further, this solution is strictly positive inside P . We may reflect the solution over A_1 and then again over the image of A_2 under the original reflection. We then reflect this entire surface over the y -axis. The resulting surface is a solution of (1) in a doubly-connected domain D , the union of six pentagons, whose inner boundary component is the interval $[-1, 1]$ on the x -axis. The solution is zero along this interval, so that y and z both tend to zero at this component. This surface thus provides an example of Case 2 in Theorem 1, and, furthermore, it has a one-one projection on the x, y -plane. By reflecting in the x -axis, we get the extended surface which has simple branch points at $x = \pm 1$.

As a final note, we recall that Bers' theorem has been extended by Nitsche ([5]) and de Giorgi-Stampacchia ([3]), who showed that, for solutions of (1) when $n = 3$, not only an isolated point but any compact set whose one-dimensional Hausdorff measure is zero is a removable singularity. The above example shows that there exist solutions having a non-removable singularity along a line segment, even when the solution takes smooth values along that segment (in this case, identically zero). On the other hand, a result of Kinderlehrer ([4]) shows that such a solution must obey a generalized maximum principle.

For example, there is no solution taking the value zero on the line segment, and positive values on the outside boundary.

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