Limit Theorems for Model Boltzmann Equations with Several Conserved Quantities

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1. Introduction. In this paper we consider a system of evolution equations of the form

$$(1.1) \quad p_t + vp_x = \frac{1}{\epsilon} Qp \quad (t > 0, x \in \mathbf{R}^1, v \in \{v_1, \dots, v_n\}, \epsilon > 0),$$

$$\lim_{\epsilon \downarrow 0} p(t, x, v) = f(x, v).$$

Here $p = p_\epsilon(t, x, v)$ and $Qp(t, x, v_i) = \sum_{j=1}^N q_{ij}p(t, x, v_j)$; $Q = (q_{ij})$ is a symmetric, negative semi-definite $N \times N$ matrix which has a d -dimensional null space $\mathfrak{X}$ ($1 \leq d < N$); the numbers $\{v_i\}$ are real and distinct and $q_{ii} \neq 0$. The following pair of singular perturbation theorems is proved in Section 3.

Theorem 1. For each $(t, x, v) \in (0, \infty) \times \mathbf{R}^1 \times \{v_1, \dots, v_N\}$,

$$(1.2) \quad \lim_{\epsilon \downarrow 0} p_\epsilon(t, x, v) = \sum_{j=1}^d \{\langle f, e_0^{(j)} \rangle (x + \alpha_1^{(j)} t)\} e_0^{(j)},$$

where $\{e_0^{(j)}\}_{1 \leq j \leq d}$ is a special orthonormal basis for $\mathfrak{X}$ and $\{\alpha_1^{(j)}\}_{1 \leq j \leq d}$ are real numbers given by the formulae $\alpha_1^{(j)} = -\langle e_0^{(j)}, ve_0^{(j)} \rangle$, $1 \leq j \leq d$. The notation $\langle \cdot, \cdot \rangle$ denotes the standard inner product in $\mathbf{R}^{N-1}$.

Theorem 2. For each $\alpha_1 \in \{\alpha_1^{(1)}, \dots, \alpha_1^{(d)}\}$, $(t, x, v) \in (0, \infty) \times \mathbf{R}^1 \times \{v_1, \dots, v_N\}$,

$$(1.3) \quad \lim_{\epsilon \downarrow 0} p_\epsilon(t/\epsilon, x - \alpha_1 t/\epsilon, v)$$

$$= \sum_{\{j: \alpha_1^{(j)} = \alpha_1\}} e_0^{(j)} \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{(2\pi)^{1/2}} \langle f, e_0^{(j)} \rangle (x + y(2\alpha_2^{(j)} t)^{1/2}) dy,$$

where $\alpha_2^{(j)}$ are positive real numbers given by formula (2.10).

In case the off-diagonal elements of Q are positive, the above results reduce to (respectively) the weak law of large numbers and the central limit theorem

for the occupation times of a finite-state Markov chain [2, 8]; in these cases $d = 1$. In general, equation (1.1) may be thought of as the linearization of a non-linear model for the Boltzmann equation [9], corresponding to N different possible velocities together with d conservation laws. A typical example of such a non-linear equation is the system

$$\begin{aligned}\frac{\partial p_1}{\partial t} + \frac{\partial p_1}{\partial x} &= -(p_1 p_3 - p_2^2), \\ \frac{\partial p_2}{\partial t} &= 2(p_1 p_3 - p_2^2), \\ \frac{\partial p_3}{\partial t} - \frac{\partial p_3}{\partial x} &= -(p_1 p_3 - p_2^2),\end{aligned}$$

for which $d = 2$ (velocity and particle number are conserved). Theorem 2 can be re-stated as a singular perturbation theorem for the system

$$(1.4) \quad p_t + \frac{1}{\epsilon} v p_x = \frac{1}{\epsilon^2} Qp.$$

Indeed, if we set $p^\epsilon(t, x, v) = p_\epsilon(t/\epsilon, x, v)$, a simple calculation shows that p^ϵ satisfies (1.4). Theorem 2 then states that after a spatial translation, p^ϵ converges to the stated limit.

The equations we consider are discrete velocity analogues of known linear models with infinitely many possible velocities. An example of such an equation occurs in the Kac-Krook model [4], which we may take in the form

$$\frac{\partial p}{\partial t} + v \frac{\partial p}{\partial x} = \int_{-\infty}^{\infty} p(t, x, v') \left[1 + vv' + \frac{(v^2 - 1)(v'^2 - 1)}{2} \right] \frac{e^{-v'^2/2}}{(2\pi)^{1/2}} dv' - p(t, x, v),$$

corresponding to conservation laws for particle number, velocity, and energy ($d = 3$). In the corresponding matrix models, Theorem 1 describes a pair of "wave propagation modes," with wave velocity given by

$$\bar{v} = \left(\frac{\sum_{i=1}^N v_i^4}{\sum_{i=1}^N v_i^2} \right)^{1/2},$$

together with a pure "heat conduction mode." Theorem 2 then gives a refinement of Theorem 1, corresponding to each of these three modes. It is curious to note that in the case that $d = 2$ and velocity and particle number are conserved, the limiting wave velocity is a different function of (v_i) , namely

$$\bar{v} = \left(\frac{\sum_{i=1}^N v_i^2}{N} \right)^{1/2}.$$

It is noteworthy that $\bar{v}$ and $\bar{v}$ depend on no further detailed knowledge of (q_{ij}) .

From the viewpoint of known singular perturbation theorems, the novel feature of the above results is the different normalizations in Theorem 2. In a sense, Theorem 1 gives the preliminary information on the possible "limiting velocities" and prepares the way for the statement of Theorem 2. To the best of our knowledge, the possibility of several different normalizations is new. In the recent semigroup treatment of random evolutions by T. G. Kurtz [6], it is assumed for the counterpart of Theorem 2 that $\alpha_1^{(j)} = 0$, $1 \leq j \leq d$. As we shall see, this excludes many cases of interest in applications.

From the viewpoint of the kinetic theory of gases, Theorem 1 is closely related to the Euler equations of hydrodynamics. In fact, we show by a direct proof in Section 5 that the limit in Theorem 1 is the solution of the Euler equations associated with the model Boltzmann equation (1.1). The corresponding situation for Theorem 2 is more complicated. The limit in Theorem 2 does not always satisfy the corresponding Navier-Stokes equations. They will be satisfied if, for instance, the null space of Q contains only even powers of the (v_i) . More generally, *the limit in Theorem 2 will satisfy the Navier-Stokes equations whenever the Euler equations are trivial* ($\alpha_1^{(j)} \equiv 0$).

Our results also bear some formal relation to the Chapman-Enskog-Hilbert expansion traditionally associated with the Boltzmann equation. Indeed, in the Chapman-Enskog-Hilbert expansion, equation (1.1) is the starting point for an expansion $p = \sum p_n \epsilon^n$ of the solution. This has been carried out in detail by one of us [1] in case Q is the generator of an irreducible Markov chain. There it is shown that for such an expansion to be convergent the initial data must be very smooth as a function of x . On the other hand, our results, which form a sort of limiting case of the Chapman-Enskog-Hilbert scheme, are valid without any smoothness conditions on f . We suspect that for an adequate discussion of the transition from the Boltzmann equation to the equations of kinetic theory, the methods of this paper may be more appropriate than the use of the full Chapman-Enskog-Hilbert development.

It is interesting to note that if the $\{\alpha_1^{(j)}\}$ are distinct, the limit in Theorem 1 depends only on (v_i) and the null space of Q , but not upon further details of Q . This is in close agreement with certain recent observations in the physical literature [3] concerning the independence of limiting wave velocity and absorption parameters on the details of the intermolecular force law which enters in the Boltzmann equation. The limit in Theorem 2 does depend on some additional information concerning the range of Q , expressed through the positive quantities $\alpha_2^{(j)}$.

In Section 2 we collect the necessary facts concerning the eigenvalues and eigenvectors of $Q - \lambda v$, where λ is a complex parameter and $v = \text{diag } (v_i)$. The symmetry of Q and v guarantees analytic expansions in powers of λ (Section 2A). The negativity properties of Q further guarantee that the eigenvalues stay in the left half of the complex plane when λ is purely imaginary (Section 2B). Then in Section 3 we give the proof of our two limit theorems. Section 4 discusses in detail four classes of examples.

In Section 5 we introduce a basis for the null space of Q in which the system of Euler equations completely decouples. This “hydrodynamical basis” is then used to express the Navier–Stokes equations in a form which facilitates comparison with Theorem 2.

2. Behavior of the eigenvalues.

2A. Degenerate perturbation theory. In this sub-section we collect the formulas which will be used in the sequel. Let Q, v be arbitrary real symmetric $N \times N$ matrices and consider the one-parameter family $Q_\lambda = Q - \lambda v$, where λ is real. We consider the eigenvalue problem

$$(2.1) \quad (Q - \lambda v)e = \alpha e \quad (e \in \mathbf{R}^N, \quad \alpha \in \mathbf{R}^1).$$

Since Q_λ is symmetric, the theorem of Kato (see Lancaster [7, p. 252]) guarantees expansions

$$(2.2) \quad \alpha(\lambda) = \sum_{n=0}^{\infty} \alpha_n \lambda^n, \quad |\lambda| < \lambda_0,$$

$$(2.3) \quad e(\lambda) = \sum_{n=0}^{\infty} e_n \lambda^n, \quad |\lambda| < \lambda_0,$$

convergent in some interval $-\lambda_0 < \lambda < \lambda_0$. Substitution of (2.2)–(2.3) into (2.1) yields the relations

$$(2.4) \quad Qe_0 = \alpha_0 e_0,$$

$$(2.5) \quad Qe_n - ve_{n-1} = \sum_{j=1}^n \alpha_j e_{n-j}, \quad (n \geq 1).$$

We now consider the case in which zero is a d -fold eigenvalue of Q , where $1 \leq d < N$. Let $\{f_0^{(1)}, \dots, f_0^{(d)}\}$ be any linearly independent set of null-vectors for Q . By (2.4), we must have $e_0 = \sum_{i=1}^d c_i f_0^{(i)}$. Substituting into (2.5) for $n = 1$ and taking inner products with $f_0^{(1)}, \dots, f_0^{(d)}$, we have

$$(2.6) \quad \sum_{j=1}^d c_j \langle f_0^{(k)}, v f_0^{(j)} \rangle = -\alpha_1 \sum_{j=1}^d c_j \langle f_0^{(k)}, f_0^{(j)} \rangle \quad (1 \leq k \leq d).$$

This system of d simultaneous linear equations has a non-trivial solution if and only if α_1 is a root of the equation

$$(2.7) \quad \det \{ \langle f_0^{(k)}, v f_0^{(j)} \rangle + \alpha_1 \langle f_0^{(k)}, f_0^{(j)} \rangle \} = 0.$$

In particular, α_1 depends only on v and $\{f_0^{(1)}, \dots, f_0^{(d)}\}$ and not upon the further details of Q .

If the $\{f_0^{(j)}\}$ are orthonormal, equation (2.7) is a standard eigenvalue problem for a symmetric matrix. Hence we can choose the eigenvectors to be orthogonal and of unit length. Otherwise, we first solve equation (2.7) for $\alpha_1 = \alpha_1^{(1)}, \dots, \alpha_1^{(d)}$. For each choice of $\alpha_1^{(k)}$ we can form the corresponding (c_i) and thus determine $e_0 = e_0^{(k)}$.

To compute higher terms in the expansions (2.2)–(2.3), we write equation (2.5) for $n = 1$:

$$(2.8) \quad Qe_1^{(k)} - ve_0^{(k)} = \alpha_1^{(k)} e_0^{(k)} \quad (1 \leq k \leq d).$$

Equation (2.8) can be solved (non-uniquely) for $e_1^{(k)}$ since, by (2.6), $(\alpha_1^{(k)} + v)e_0^{(k)}$ is perpendicular to $f_0^{(1)}, \dots, f_0^{(d)}$. To determine $e_1^{(k)}$ uniquely we must first consider (2.5) for $n = 2$:

$$(2.9) \quad Qe_2^{(k)} - ve_1^{(k)} = \alpha_2^{(k)} e_0^{(k)} + \alpha_1^{(k)} e_1^{(k)}.$$

Taking the inner product with $\{e_0^{(1)}, \dots, e_0^{(d)}\}$ gives d equations which specify $g_1^{(k)}$, the projection of $e_1^{(k)}$ on $\mathfrak{H}$. We write $e_1^{(k)} = Q^{-1}(v + \alpha_1^{(k)})e_0^{(k)} + g_1^{(k)}$, where $g_1^{(k)} \perp \mathfrak{H}$ and $Q^{-1}(v + \alpha_1^{(k)})e_0^{(k)} \perp \mathfrak{H}$.² Taking the inner product of (2.9) with $e_0^{(k)}$, we arrive at the formula

$$(2.10) \quad \alpha_2^{(k)} = -\langle Q^{-1}(v + \alpha_1^{(k)})e_0^{(k)}, (v + \alpha_1^{(k)})e_0^{(k)} \rangle.$$

The positivity of (2.10) will be proved later, under the further condition that Q is negative semi-definite. We summarize the above as

Proposition 2A.1. *Let Q be a real symmetric matrix with a d -fold eigenvalue at zero and let v be an arbitrary symmetric matrix. Then for sufficiently small $\lambda \neq 0$, there exist d solutions of the problem (2.1). These solutions can be expressed in the form (2.2)–(2.3) where α_1 is first determined from the auxiliary eigenvalue equation (2.7). The higher coefficients are determined successively from equation (2.5).*

There is no *a priori* guarantee that the $\{\alpha_1^{(k)}\}$ are all distinct. In fact, if v maps $\mathfrak{H}$ into $\mathfrak{H}^\perp$, we will have $\alpha_1^{(k)} \equiv 0$. In such a case, $e_0^{(k)}$ cannot be determined uniquely from equation (2.6). Nevertheless, the general perturbation theory guarantees the existence of $\{e_0^{(k)}\}$. In many cases they can be uniquely determined from the higher approximations (2.5) for $n > 1$. In these degenerate cases, we have the following fact which will be used in Section 5.

Proposition 2A.2. *If $\alpha_1^{(k)} = \alpha_1^{(l)}$ for $k \neq l$, then*

$$\langle e_0^{(l)}, (v + \alpha_1^{(l)})Q^{-1}(v + \alpha_1^{(k)})e_0^{(k)} \rangle = 0.$$

Proof. From (2.8) we have $e_1^{(k)} = Q^{-1}(v + \alpha_1^{(k)})e_0^{(k)} + \sum_{j=1}^d d_{kj}e_0^{(j)}$ for some constants (d_{kj}) . Hence equation (2.9) becomes

$$Qe_2^{(k)} = (v + \alpha_1^{(k)})\left\{Q^{-1}(v + \alpha_1^{(k)})e_0^{(k)} + \sum_{j=1}^d d_{kj}e_0^{(j)}\right\} + \alpha_2^{(k)}e_0^{(k)}.$$

Taking the inner product with $e_0^{(l)}$, we have

$$0 = \langle e_0^{(l)}, (v + \alpha_1^{(l)})Q^{-1}(v + \alpha_1^{(k)})e_0^{(k)} \rangle + \sum_{j=1}^d d_{kj}\langle e_0^{(l)}, (v + \alpha_1^{(k)})e_0^{(j)} \rangle + \alpha_2^{(k)}\delta_{kl}.$$

Taking (2.8) into account, this becomes

$$0 = \alpha_2^{(kl)} + d_{kl}(\alpha_1^{(k)} - \alpha_1^{(l)}) + \alpha_2^{(k)} \delta_{kl},$$

where we have set $\alpha_2^{(kl)} = \langle (v + \alpha_1^{(l)})e_0^{(l)}, Q^{-1}(v + \alpha_1^{(k)})e_0^{(k)} \rangle$. The conclusion is now immediate.

If λ is complex, the eigenvalues α of the problem (2.1) are solutions of the algebraic equation $\det(Q - \lambda v - \alpha) = 0$. Puiseux's theorem guarantees expansions $\alpha(\lambda) = \sum_{n=0}^{\infty} \alpha_n' \lambda^{nk/r}$, $e(\lambda) = \sum_{n=0}^{\infty} e_n' \lambda^{nk/r}$ convergent in some disc $|\lambda| < \delta$, where k and r are certain integers satisfying $1 \leq r \leq d$ and $0 \leq k \leq r - 1$. But on the real λ -axis, (2.2)–(2.3) are valid. Hence, by the uniqueness of power series, we have the following

Corollary 2A.3. *The power series expansions (2.2)–(2.3) are valid for λ complex, $|\lambda| < \lambda_0$.*

For later purposes it will be useful to introduce the following concept:

Definition 2A.4. A set of vectors $\{e^{(1)}, e^{(2)}, \dots, e^{(d)}\}$ in $\mathfrak{X}$ will be called a *hydrodynamical basis* for (Q, v) if

- i) $\langle e^{(i)}, e^{(k)} \rangle = \delta_{ik}$ ($1 \leq j, k \leq d$),
- ii) $\langle e^{(i)}, ve^{(k)} \rangle = -\alpha_1^{(i)} \delta_{ik}$.

In other words, a hydrodynamical basis is an orthonormal basis for $\mathfrak{X}$ which diagonalizes PvP , where P is the orthogonal projection onto $\mathfrak{X}$.

Proposition 2A.5. *Let (Q, v) be arbitrary symmetric matrices such that Q has a d -dimensional null space ($d \geq 1$). Then there exists a hydrodynamical basis for (Q, v) .*

Proof. Let $\Gamma = \{\lambda_0, \dots, \lambda_p\}$ denote the exceptional points of the algebraic function $\alpha(\lambda)$ defined by $\det(Q - \lambda v - \alpha) = 0$. Let $e^{(k)}(\lambda)$ be the N distinct eigenvalues of $Q - \lambda v$ for $\lambda \notin \Gamma$. For λ real, $\lambda \notin \Gamma$, the $e^{(k)}(\lambda)$ are the eigenvectors of a real symmetric matrix and thus are perpendicular: $\langle e^{(k)}(\lambda), e^{(i)}(\lambda) \rangle = 0$ for $j \neq k$. Letting $\lambda \rightarrow 0$ and noting the expansion (2.3), we have $\langle e_0^{(k)}, e_0^{(i)} \rangle = 0$ for $j \neq k$, in particular for $1 \leq j \neq k \leq d$. Now write equation (2.5) for $n = 1$, set $\alpha_1 = \alpha_1^{(k)}$ ($1 \leq k \leq d$), and take the inner product with $e_0^{(i)}$. We obtain $\langle e_0^{(i)}, Qe_1^{(k)} - ve_0^{(k)} \rangle = \langle e_0^{(i)}, \alpha_1^{(k)} e_0^{(k)} \rangle$. Since $Qe_0^{(i)} = 0$, we arrive at the formula

$$\langle ve_0^{(k)}, e_0^{(i)} \rangle = -\alpha_1^{(k)} \langle e_0^{(k)}, e_0^{(i)} \rangle \quad (1 \leq j, k \leq d),$$

which was to be proved.

2B. Global behavior of the eigenvalues. We now investigate the behavior of $\alpha(\lambda)$ with more specific assumptions on (Q, v) .

Let $Q = (q_{ij})$ be a real $N \times N$ matrix which is

- a) *symmetric* ($q_{ij} = q_{ji}$ for $1 \leq i, j \leq N$),
- b) *negative semi-definite* ($\sum_{1 \leq i, j \leq N} q_{ij} \xi_i \xi_j \leq 0$ for $(\xi_1, \dots, \xi_N) \in \mathbf{R}^N$),
- c) *of minimal dimension* ($q_{ii} \neq 0$ for $1 \leq i \leq N$).

Let $(v_1, \dots, v_N)$ be arbitrary distinct real numbers and let $\alpha^{(1)}(\lambda), \dots,$

$\alpha^{(N)}(\lambda)$ denote the eigenvalues of the matrix $Q - \lambda v = (q_{ik} - \lambda v_i \delta_{ik})$. We shall enumerate the $\alpha^{(j)}(\lambda)$ so that $\alpha^{(j)}(0) = 0$ for $1 \leq j \leq d$.

By conditions a)-b), the eigenvalues of Q lie on the negative real axis. Condition c), together with a)-b), is equivalent to the statement that Q have no rows or columns consisting entirely of zero.³ We now investigate the behavior of $\alpha^{(j)}(\lambda)$ for λ purely imaginary.

Lemma 2B.1. $\operatorname{Re} \alpha^{(j)}(i\gamma) < 0$ for $j = 1, \dots, N$ and γ real $\neq 0$.

Proof. Let $u = (u_i)$ be an eigenvector of $Q - i\gamma v$ with eigenvalue $\alpha = \alpha(i\gamma)$. Then by assumption we have

$$(2.11) \quad \begin{aligned} (Q - i\gamma v)u &= \alpha u, \\ \langle Qu, \bar{u} \rangle - i\gamma \langle vu, \bar{u} \rangle &= \alpha \langle u, \bar{u} \rangle, \end{aligned}$$

where $\bar{u}$ is the vector of complex conjugates. Since Q is symmetric, the first term is real. Thus we must have

$$(2.12) \quad \begin{aligned} \langle Qu, \bar{u} \rangle - (\operatorname{Re} \alpha) \langle u, \bar{u} \rangle &= 0, \\ \gamma \langle vu, \bar{u} \rangle + (\operatorname{Im} \alpha) \langle u, \bar{u} \rangle &= 0. \end{aligned}$$

Since Q is negative semi-definite, we have $\operatorname{Re} \alpha \leq 0$. If $\operatorname{Re} \alpha = 0$, then by a)-b) we must also have $Qu = 0$. Substituting into (2.11), we have $-i\gamma v_i u_i = iku_i$ for $1 \leq j \leq N$, where $\alpha = ik$. This system of equations has a non-trivial solution only if (u_i) has exactly one non-zero entry. But since $Qu = 0$, this contradicts c).

Lemma 2B.2. For $\gamma \rightarrow \pm\infty$, we have $\alpha^{(i)}(i\gamma) = iv_k \gamma + q_{kk} + O(\gamma^{-1})$, for some $1 \leq k \leq N$.

Proof. From the characteristic polynomial $A(\alpha, i\gamma) = \det(Q - i\gamma v - \alpha) = 0$, we obtain for $\gamma \neq 0$

$$(2.13) \quad \begin{aligned} 0 &= \det(Q/\gamma + iv - \alpha/\gamma) \\ &= \det(\epsilon Q + iv - \beta). \end{aligned}$$

Hence $\beta = \alpha/\gamma$ is an analytic function of $\epsilon = 1/\gamma$. When $\epsilon = 0$, the above equation has the distinct roots $\beta = iv_1, \dots, iv_N$. Hence by Puiseux's theorem, there exists an expansion $\beta(\epsilon) = \sum_{n=0}^{\infty} \beta_n \epsilon^n$ with $\beta_0 = iv_j$ for some j . If we substitute this expansion into (2.13), it follows that $\beta_1 = q_{jj}$. This completes the proof.

We now have before us the complete global behavior of $\{\alpha^{(j)}(i\gamma)\}$ for γ real $\neq 0$. These roots stay in the left half of the complex α plane. For $d+1 \leq j \leq N$ they remain at least a fixed positive distance from the imaginary axis for all γ real. The same is true for the first d eigenvalues provided γ is real and bounded away from zero.

Finally, we prove the positivity of (2.10) and related expressions.

Proposition 2B.3. *Let y be a non-zero vector perpendicular to $\mathfrak{N}$. Then $\langle Q^{-1}y, y \rangle < 0$.*

Proof. Let $w = Q^{-1}y$. The above inner product then equals $\langle w, Qw \rangle$, which is non-positive by the negative semi-definiteness of Q . But if it is zero, then by the semi-definiteness of Q , we must have $Qw = 0$, which contradicts $y \neq 0$.

3. Singular perturbation theorems. In this section we give the proofs of the two theorems stated in the Introduction. We consider the system of evolution equations

$$(3.1) \quad \frac{\partial p_\epsilon}{\partial t}(t, x, v) + v \frac{\partial p_\epsilon}{\partial x}(t, x, v) = \frac{1}{\epsilon} Q p_\epsilon(t, x, v)$$

$$(t > 0, x \in \mathbf{R}^1, v \in \{v_1, \dots, v_N\}, \epsilon > 0),$$

$$\lim_{\epsilon \downarrow 0} p_\epsilon(t, x, v) = f(x, v),$$

where $Q = (q_{ij})$ is a symmetric, negative semi-definite matrix of minimal dimension and $v = (v_i)$ are distinct real numbers.

Proof of Theorem 1. Let us assume that as a function of x , $f \in C_c^\infty$, the class of infinitely differentiable functions with compact support. In this case the Fourier transform $\hat{f} = \int_{-\infty}^{\infty} e^{-i\gamma x} f dx$ exists and the equation (3.1) can be solved explicitly. Indeed, the Fourier transform of (3.1) is

$$(3.2) \quad \frac{\partial \hat{p}_\epsilon}{\partial t}(t, \gamma, v) + i\gamma v \hat{p}_\epsilon(t, \gamma, v) = \frac{1}{\epsilon} Q \hat{p}_\epsilon(t, \gamma, v),$$

whose solution is

$$\hat{p}_\epsilon(t, \gamma, v) = \exp [t(Q - i\epsilon\gamma v)/\epsilon] \hat{f}(\gamma, v).$$

In this equation $\hat{p}_\epsilon = \int e^{i\gamma x} p_\epsilon dx$ and “exp” denotes the matrix exponential operating on $\hat{f}(\gamma, \cdot)$, the column vector whose j^{th} component is $\hat{f}(\gamma, v_j)$. To perform the proof, we claim that

$$(3.3A) \quad |\hat{p}_\epsilon(t, \gamma, v)| \leq N \|\hat{f}(\gamma, \cdot)\| \equiv N \langle \hat{f}(\gamma), \hat{f}(\gamma) \rangle^{1/2}$$

and

$$(3.3B) \quad \lim_{\epsilon \downarrow 0} \hat{p}_\epsilon(t, \gamma, v) = \sum_{i=1}^d \langle \hat{f}, e_0^{(i)} \rangle e^{it\alpha_1^{(i)}\gamma} e_0^{(i)}.$$

Theorem 1 then follows by the dominated convergence theorem. To prove (3.3A) we decompose the matrix exponential in the form

$$(3.4) \quad \exp t(Q - i\gamma v)$$

$$= \sum_{j=1}^N \{Z_{j1}(\gamma) + tZ_{j2}(\gamma) + \dots + t^{N-1}Z_{j,N-1}(\gamma)\} \exp(t\alpha^{(j)}(i\gamma)),$$

where $Z_{i1}, \dots, Z_{i, N-1}$ are the component matrices of $Q - i\gamma v$ (see Lancaster [7, p. 173 ff.]). If $Q - i\gamma v$ is diagonalizable, then $Z_{i2} = 0 = \dots = Z_{i, N-1} = 0$ and Z_{i1} is just the orthogonal projection onto the eigenspace corresponding to $\alpha^{(i)}(i\gamma)$. In this case clearly $\|\exp t(Q - i\gamma v)\| \leq N$, so that (3.3A) holds. But the set of γ for which $Q - i\gamma v$ is not diagonalizable are all singular points of the algebraic function $\alpha(i\gamma)$ and hence are at most finite in number. Thus if γ_0 is such a point, we can find $\gamma_n \rightarrow \gamma_0$ such that $\|\exp t(Q - i\gamma_n v)\| \leq N$. Hence $\|\exp t(Q - i\gamma v)\| \leq N$ for all γ real, which is (3.3A).

To prove (3.3B) we again use the representation (3.4) putting in the ϵ 's. Fix γ . By lemmas (2B.1) and (2B.2) we can find $c > 0$ such that

$$\begin{aligned} \exp [t(Q - i\epsilon\gamma v)/\epsilon] \\ = \sum_{j=1}^d \left\{ Z_{j1}(\epsilon\gamma) + \dots + \left(\frac{t}{\epsilon}\right)^{N-1} Z_{j, N-1}(\epsilon\gamma) \right\} \exp (t\alpha^{(j)}(i\epsilon\gamma)/\epsilon) + O(e^{-c t/\epsilon}) \end{aligned}$$

for $\epsilon \downarrow 0$. Now since Q has d linearly independent eigenvectors $\{e_0^{(1)}, \dots, e_0^{(d)}\}$, we must have $Z_{i2}(0) = 0 = \dots = Z_{i, d}(0)$. But the expansions (2.3) show that $\det \{e^{(1)}(\lambda), \dots, e^{(d)}(\lambda)\}$ depends continuously on λ for $|\lambda| < \lambda_0$. Hence for λ sufficiently small, $\{e^{(1)}(\lambda), \dots, e^{(d)}(\lambda)\}$ will be linearly independent, and thus $Z_{i2}(\epsilon\gamma) = 0 = \dots = Z_{i, N-1}(\epsilon\gamma) = 0$ for ϵ sufficiently small. But $\lim_{\epsilon \downarrow 0} Z_{i1}(\epsilon\gamma) = e_0^{(i)} \otimes e_0^{(i)}$, the orthogonal projection onto $e^{(i)}(0)$. Hence

$$\begin{aligned} \lim_{\epsilon \downarrow 0} \hat{p}_\epsilon(t, \gamma, v) &= \sum_{j=1}^d e_0^{(j)} \otimes e_0^{(j)} \lim_{\epsilon \downarrow 0} \exp (t\alpha^{(j)}(i\epsilon\gamma)/\epsilon) \hat{f} \\ &= \sum_{j=1}^d \exp (it\alpha_1^{(j)}\gamma) \langle \hat{f}, e_0^{(j)} \rangle e_0^{(j)}, \end{aligned}$$

which gives (3.3B). This completes the proof of Theorem 1.

Proof of Theorem 2. Without loss of generality, we assume that $\alpha_1 = \alpha_1^{(1)}$. By Fourier inversion we have

$$p_\epsilon(t, x, v) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\gamma x} \hat{p}_\epsilon(t, \gamma, v) d\gamma,$$

and thus

$$(3.5) \quad p_\epsilon(t/\epsilon, x - \alpha_1 t/\epsilon, v) = (2\pi)^{-1} \int_{-\infty}^{\infty} e^{i\gamma x} e^{-i\alpha_1 t\gamma/\epsilon} \hat{p}_\epsilon(t/\epsilon, \gamma, v) d\gamma.$$

Let us again use the decomposition (3.4). Reasoning as before, we note that $p_\epsilon(t, \gamma, v) = \exp [t(Q - i\epsilon\gamma v)/\epsilon] \hat{f}(\gamma, v)$ and that

$$\begin{aligned} (3.6) \quad e^{-i\alpha_1 t\gamma} \exp t(Q - i\gamma v) \\ = \sum_{j=1}^d \{Z_{j1}(\gamma) + \dots + t^{N-1} Z_{j, N-1}(\gamma)\} \exp t[\alpha^{(j)}(i\gamma) - i\alpha_1\gamma] \\ + O(e^{-c t}) \quad (t \uparrow \infty). \end{aligned}$$

Hence, as $\epsilon \downarrow 0$,

$$(3.7) \quad p_\epsilon(t/\epsilon, x - \alpha_1 t/\epsilon, v) \\ = (2\pi)^{-1} \sum_{j=1}^d \int_{-\infty}^{\infty} e^{i\gamma x} \left\{ Z_{j1}(\epsilon\gamma) + \cdots + \left(\frac{t}{\epsilon}\right)^{N-1} Z_{j, N-1}(\epsilon\gamma) \right\} \\ \cdot \exp [t(\alpha^{(j)}(\epsilon\gamma) - i\alpha_1\epsilon\gamma)/\epsilon^2] \hat{f}(\gamma) d\gamma + O(e^{-c t/\epsilon^2}).$$

The sum (3.7) consists of two types of terms. The terms with $\alpha_1^{(j)} \neq \alpha_1$ are (for any fixed x) of the form

$$\int_{-\infty}^{\infty} \exp \frac{i\gamma t(\alpha_1^{(j)} - \alpha_1)}{\epsilon} \tilde{Z}_\epsilon(\gamma) d\gamma,$$

where $\tilde{Z}_\epsilon(\gamma)$ converges in L^1 when $\epsilon \downarrow 0$. By a trivial adaptation of the Riemann–Lebesgue lemma, such a term converges to zero when $\epsilon \downarrow 0$.

For the terms with $\alpha_1^{(j)} = \alpha_1$, it is clear that the corresponding part of the integrand in (3.7) converges to $e^{-i\gamma x} e_0^{(j)} \otimes e_0^{(j)} \exp(-t\alpha_2^{(j)}\gamma^2) \hat{f}(\gamma)$. Hence by the dominated convergence theorem, this part of the integrand converges to the asserted right hand side of Theorem 2. This completes the proof of Theorem 2.

By a slightly refined but totally straightforward analysis of the above reasoning, we may prove error estimates of the form

$$\left| p_\epsilon(t, x, v) - \sum_{j=1}^d \langle f, e_0^{(j)} \rangle (x + \alpha_1^{(j)} t, v) e_0^{(j)} \right| \leq c_1 \epsilon \int |f| dx, \\ \left| p^\epsilon(t, x - \alpha_1 t, v) - \sum_{\{j: \alpha_1^{(j)} = \alpha_1\}} e_0^{(j)} \int_{-\infty}^{\infty} \frac{e^{-v^2/2}}{(2\pi)^{1/2}} \right. \\ \left. \cdot \langle f, e_0^{(j)} \rangle (x + y(2\alpha_2^{(j)} t)^{1/2}, v) dy \right| \leq c_1 \epsilon \int |f|,$$

where c_1 is a constant, independent of (x, v) . The proof of results of this type can be found in [2].

It is worthwhile to note that in both theorems, the convergence is uniform in (x, v) for a fixed t . This fact follows easily from a re-examination of the proof. Finally, although the limit theorems were proved under the condition that the initial data $f \in C_c^\infty$ as a function of x , it is not difficult to extend this to a wider class, e.g., to f such that $\hat{f} \in L^1$.

4. Examples. In most physical applications we impose additional symmetry conditions. We shall assume, in addition to the conditions of 2B, that Q is *bisymmetric* ($q_{ij} = q_{N-j+1, N-i+1}$, $1 \leq i, j \leq N$) and that $v_i + v_{N-i} = 0$ for $1 \leq i \leq N$. Under these further restrictions, $\det(Q - i\gamma v - \alpha)$ is a polynomial in α and γ^2 . On the other hand, Kato's theorem implies that all the eigenvalues of $Q - i\gamma v$ are analytic at $\gamma = 0$. Thus, in particular, the eigenvalues $\alpha^{(j)}(i\gamma)$ with $(d\alpha^{(j)}/d\gamma)(0)$ imaginary must come in pairs. We will see below that when

applied to the first d eigenvalues, this corresponds to travelling waves in opposite directions.

The following examples correspond to conservation of different combinations of particle number, velocity, and energy. The notation $\mathfrak{N} = \{x, y, \dots\}$ means that the null space of Q is generated by the linearly independent vectors $\{x, y, \dots\}$.

Case I. $\mathfrak{N} = \{1\}$.

Proposition 4.1. $\alpha^{(1)}(\lambda) = \alpha_2^{(1)}\lambda^2 + O(\lambda^3)$ (as $\lambda \rightarrow 0$), $e_0^{(1)} = 1/N^{1/2}$, where $\alpha_2^{(1)} = -N^{-1}\langle v, Q^{-1}v \rangle$.

Proof. We use the formulae (2.4)–(2.5) and their consequences. Since $\alpha_0^{(1)} = 0$, $e_0^{(1)} = e_0 \in \mathfrak{N}$, and hence e_0 must be a multiple of 1, by assumption. The normalization determines the constant up to $\pm$. Formula (2.5) for $n = 1$ reads $Qe_1 - ve_0 = \alpha_1 e_0$. Taking the inner product with e_0 and using symmetry gives $\alpha_1^{(1)} = -\langle ve_0, e_0 \rangle$. Recalling that $e_0 = 1/N^{1/2}$ and that $\sum_{i=1}^N v_i = 0$ (from the special assumptions of this section), we see that $\alpha_1^{(1)} = 0$. To compute $\alpha_2^{(1)}$, we use formula (2.10) of Section 2A. Thus, $\alpha_2^{(1)} = -\langle Q^{-1}ve_0, ve_0 \rangle = -(1/N)\langle Q^{-1}v1, v1 \rangle$ which was to be proved.

Remark 1. In the special case $q_{ij} = -\delta_{ij} + 1/N$ ($1 \leq i, j < N$), i.e., where $(I + Q)$ is the orthogonal projection onto the nullspace of Q , the solution of $Qw = v$ is given by $w_i = v_i$. Hence in this case $N\alpha_2 = -\langle v, v \rangle$. In general, α_2 depends on the behavior of Q on $\mathfrak{N}^\perp$ and not just on $\{v_1, \dots, v_N\}$.

Remark 2. The above proposition depends only on the fact that $\sum_{i=1}^N v_i = 0$. If we only assume the general conditions of Section 2, then the above formulas must be slightly modified: $\alpha^{(1)}(\lambda) = \bar{v}_0\lambda + \alpha_2\lambda^2$, $e_0 = 1/N^{1/2}$, where $\bar{v}_0 = (\sum_{i=1}^N v_i)/N$ and $\alpha_2 = -N^{-1}\langle (v - \bar{v}_0), Q^{-1}(v - \bar{v}_0) \rangle$. The reader is invited to verify this simple modification.

Case II. $\mathfrak{N} = \{1, v\}$.

In this case we have $\sum_{i=1}^N q_{ij} = 0$, $\sum_{i=1}^N q_{ij}v_j = 0$ for $1 \leq i \leq N$. The behavior of the eigenvalues and eigenfunctions is given by the following

Proposition 4.2. For $j = 1, 2$,

$$\alpha^{(j)}(\lambda) = (-1)^j \bar{v}\lambda + \frac{1}{2}\sigma^2\lambda^2 + O(\lambda^3) \quad (\text{as } \lambda \rightarrow 0),$$

$$e_0^{(j)} = \frac{v - (-1)^j \bar{v}}{\bar{v}(2N)^{1/2}},$$

where

$$\bar{v} = \left(\sum_{i=1}^N v_i^2 / N \right)^{1/2} \quad \text{and} \quad \sigma^2 = -(1/N\bar{v}^2)\langle Q^{-1}(v^2 - \bar{v}^2), (v^2 - \bar{v}^2) \rangle.$$

Proof. Again we use the formulas (2.4)–(2.5). $Qe_0 = 0$ implies that $e_0 = c_1 + c_2v$ for constants c_1, c_2 . The auxiliary eigenvalue problem (2.6) now reads

$$(4.1) \quad \begin{aligned} c_1 \langle 1, v1 \rangle + c_2 \langle v, v1 \rangle &= -\alpha_1 c_1 \langle 1, 1 \rangle, \\ c_1 \langle v, v1 \rangle + c_2 \langle v, v^2 \rangle &= -\alpha_1 c_2 \langle v, v \rangle. \end{aligned}$$

The condition $v_i + v_{N-i} = 0$ implies that $\langle 1, v1 \rangle = 0 = \langle v, v^2 \rangle$. The further identifications $\langle 1, 1 \rangle = N$, $\langle v, v \rangle = N\bar{v}^2$ simplifies (4.1) to $N\bar{v}^2 c_2 + N\alpha_1 c_1 = 0$, $N\bar{v}^2 c_1 + N\bar{v}^2 \alpha_1 c_2 = 0$. The determinant of these equations (equation (2.7)) is $N^2 \alpha_1^2 \bar{v}^2 - N^2 \bar{v}^4$, of which the two roots are $\alpha_1 = \pm \bar{v}$. The eigenvectors are now determined from (4.1) by simple elimination: The root $\alpha_1 = +\bar{v}$ corresponds to the unnormalized eigenvector $e_0 = v - \bar{v}$ whereas the root $\alpha_1 = -\bar{v}$ corresponds to the unnormalized eigenvector $e_0 = v + \bar{v}$. To determine the constant σ^2 , we again use formula (2.10): The choice $\alpha_1 = +\bar{v}$ gives the value $\alpha_2 = -\langle Q^{-1}(v + \bar{v})(v - \bar{v}), (v + \bar{v})(v - \bar{v}) \rangle / 2N\bar{v}^2$, whereas the choice $\alpha_1 = -\bar{v}$ gives the value $\alpha_2 = -\langle Q^{-1}(v - \bar{v})(v + \bar{v}), (v - \bar{v})(v + \bar{v}) \rangle / 2N\bar{v}^2$. These are clearly equal and in agreement with the form of σ^2 as stated above.

Remark 1. The above analysis depends only on the conditions $\sum_{i=1}^N v_i = 0$, $\sum_{i=1}^N v_i^3 = 0$, which hold under the assumptions of this section. If $\sum_{i=1}^N v_i = N\bar{v}_0 \neq 0$, $\sum (v_i - \bar{v}_0)^3 = 0$, then the above formulae must be modified to $\bar{v} = (\sum_{i=1}^N (v_i - \bar{v}_0)^2 / N)^{1/2}$, $\sigma^2 = -(1/N\bar{v}^2) \langle Q^{-1}((v - \bar{v}_0)^2 - \bar{v}^2), ((v - \bar{v}_0)^2 - \bar{v}^2) \rangle$.

Remark 2. This is the simplest non-trivial example in which it is clear that α_1 depends only on $\mathfrak{N}$ and v whereas α_2 depends on the finer structure of Q . The next example contains this explicit feature together with an additional feature.

Case III. $\mathfrak{N} = \{1, v, v^2\}$.

In this case we have $\sum_{i=1}^N q_{ij} = 0$, $\sum_{i=1}^N q_{ij} v_i = 0$, $\sum_{i=1}^N q_{ij} v_i^2 = 0$. The results below correspond to the physically natural situation of two “wave propagation” modes and one pure “heat conduction” mode.

Proposition 4.3.

$$\begin{aligned} \alpha^{(j)}(\lambda) &= (-1)^j \bar{v} \lambda + \frac{1}{2} \sigma_1^2 \lambda^2 + O(\lambda^3) & (j = 1, 2) \quad (as \lambda \rightarrow 0), \\ \alpha^{(3)}(\lambda) &= \frac{1}{2} \sigma_2^2 \lambda^2 + O(\lambda^3) & (as \lambda \rightarrow 0), \\ e_0^{(j)} &= (v^2 - (-1)^j v \bar{v}) / \bar{v} \bar{v} (2N)^{\frac{1}{2}}, & (j = 1, 2), \\ e_0^{(3)} &= (v^2 - \bar{v}^2) / \bar{v}^2 (N(\bar{v}^2 - \bar{v}^2))^{1/2}, \end{aligned}$$

where

$$\begin{aligned} \bar{v} &= \left(\sum_{i=1}^N v_i^4 / \sum_{i=1}^N v_i^2 \right)^{1/2}, \quad \frac{1}{2} \sigma_1^2 = -\frac{1}{N\bar{v}^2(\bar{v}^2 - \bar{v}^2)} \langle Q^{-1}(v^3 - v\bar{v}^2), (v^3 - v\bar{v}^2) \rangle, \\ \frac{1}{2} \sigma_2^2 &= -\frac{1}{2N\bar{v}^2\bar{v}^2} \langle Q^{-1}(v^3 - v\bar{v}^2), (v^3 - v\bar{v}^2) \rangle. \end{aligned}$$

Proof. As before, we follow the reasoning of Section 2A. In this case the determinantal equation (2.7) reads

$$\begin{vmatrix} N\alpha_1 & N\bar{v}^2 & \alpha_1 N\bar{v}^2 \\ N\bar{v}^2 & \alpha_1 N\bar{v}^2 & N\bar{v}^2 \bar{v}^2 \\ \alpha_1 N\bar{v}^2 & N\bar{v}^2 \bar{v}^2 & \alpha_1 N\bar{v}^2 \bar{v}^2 \end{vmatrix} = 0,$$

whose solutions are $\alpha_1^{(1)} = +\bar{v}$, $\alpha_1^{(2)} = -\bar{v}$, $\alpha_1^{(3)} = 0$. When substituted into (2.6), the third choice gives $e_0^{(3)} = K(v^2 - \bar{v}^2)$, where K is a normalization constant. For the first two roots, a short calculation gives $e_0^{(1)} = K'(v^2 - v\bar{v})$, $e_0^{(2)} = K'(v^2 + v\bar{v})$, where K' is a normalization constant. To find the second term in the expansion of $\alpha(\lambda)$, we again use (2.10). As in Case II, the values of α_2 are the same for $\alpha_1 = \pm\bar{v}$. The relation $\bar{v} < \bar{v}$ follows by applying Schwarz's inequality in the form $(\sum v_i^2)^2 < (\sum v_i^4)(\sum 1)$.

Case IV. $\mathfrak{N} = \{1, v^2\}$.

In this case we have $\sum_{i=1}^N q_{ii} = 0$, $\sum_{i=1}^N q_{ii}v_i^2 = 0$, and necessarily $N > 3$.⁴ This corresponds to the physically unrealistic situation of conservation of particle number and total energy, but no conservation of momentum. This is reflected in the calculations below, which give two heat conduction modes and no wave propagation modes. As we shall see, this case also has some mathematical peculiarities.

Proposition 4.4. For $j = 1, 2$, $\alpha^{(j)}(\lambda) = \frac{1}{2} \sigma_j^2 \lambda^2 + O(\lambda^3)$ (as $\lambda \rightarrow 0$), where $\sigma_j^2 > 0$. In particular, $\alpha^{(i)} = 0$.

Proof. In this case the determinant (2.7) is

$$\begin{vmatrix} N\alpha_1 & N\alpha_1\bar{v}^2 \\ N\alpha_1\bar{v}^2 & N\alpha_1\bar{v}^2\bar{v}^2 \end{vmatrix} = 0$$

or $0 = N^2\alpha_1^2\bar{v}^2(\bar{v}^2 - \bar{v}^2)$. Since $\bar{v} < \bar{v}$, this can only be satisfied if $\alpha_1 = 0$. If we now go back to the equation (2.5) for $n = 1$, we find that it is trivial, and thus we cannot determine e_0 at the first stage of the perturbation scheme. Although somewhat surprising, this is entirely consistent with the results of Section 2. In any case equation (2.4) requires that $e_0 = c_1 + c_2v^2$ for constants c_1, c_2 . Equation (2.5) for $n = 1$ gives $Qe_1 = ve_0$ and hence $e_1 = c_1(Q^{-1}v) + c_2(Q^{-1}v^3) + f_1 + f_2v^2$, where f_1, f_2 are constants and Q^{-1} denotes that value perpendicular to $\mathfrak{N}$. Equation (2.5) for $n = 2$ now gives

$$Qe_2 = v(c_1(Q^{-1}v) + c_2(Q^{-1}v^3) + f_1 + f_2v^2) + \alpha_2(c_1 + c_2v^2).$$

If we take the inner product with $(1, v^2)$, we arrive at two simultaneous linear equations for c_1 and c_2 :

$$(4.2) \quad \begin{aligned} c_1\langle v, Q^{-1}v \rangle + c_2\langle v, Q^{-1}v^3 \rangle + \alpha_2(c_1N + c_2N\bar{v}^2) &= 0, \\ c_1\langle v^3, Q^{-1}v \rangle + c_2\langle v^3, Q^{-1}v^3 \rangle + \alpha_2(c_1\bar{v}v^2 + c_2N\bar{v}^2\bar{v}^2) &= 0. \end{aligned}$$

The possible values of α_2 and hence e_0 are determined by setting the determinant of (4.2) equal to zero; i.e., we must solve

$$(4.3) \quad \begin{vmatrix} N\alpha_2 + \langle v, Q^{-1}v \rangle & N\alpha_2\bar{v}^2 + \langle v, Q^{-1}v^3 \rangle \\ N\alpha_2\bar{v}^2 + \langle v^3, Q^{-1}v \rangle & N\bar{v}^2\bar{v}^2\alpha_2 + \langle v^3, Q^{-1}v^3 \rangle \end{vmatrix} = 0.$$

The strict positivity of σ_j^2 , $j = 1, 2$, follows from the strict positivity of $\alpha_2^{(j)}$, which is a consequence of (2.10) and Proposition 2B.3. It is clear that in this case, e_0 depends on the details of Q outside of $\mathfrak{H}$, in contrast with the previous cases. Since the calculations become difficult, let us restrict ourselves in what follows to the case in which Q^{-1} is minus the identity on $\mathfrak{H}^\perp$. In other words, $Q = -I + R$, where R is the orthogonal projection onto $\mathfrak{H}$. A short calculation shows that the determinant equation (4.3) becomes

$$(4.4) \quad \bar{v}^2(\alpha_2 - \bar{v}^2)(\alpha_2 - \bar{v}^2) = \bar{v}^2(\alpha_2 - \bar{v}^2)^2,$$

where

$$\bar{v} = \left(\sum_{i=1}^N v_i^6 / \sum_{i=1}^N v_i^4 \right)^{1/2}.$$

Proposition 4.5. *Equation (4.4) has distinct roots.*

Proof. If we write (4.4) out as a quadratic equation in the form $A\alpha_2^2 + B\alpha_2 + C = 0$, we find that

$$\begin{aligned} A &= \bar{v}^2 - \bar{v}^2, \\ B &= \bar{v}^2\bar{v}^2 - \bar{v}^2\bar{v}^2, \\ C &= \bar{v}^2\bar{v}^2\bar{v}^2 - \bar{v}^2\bar{v}^4. \end{aligned}$$

Then

$$\begin{aligned} D &= B^2 - 4AC \\ &= (\bar{v}^2\bar{v}^2 - \bar{v}^2\bar{v}^2)^2 - 4(\bar{v}^2\bar{v}^2)(\bar{v}^2 - \bar{v}^2)(\bar{v}^2 - \bar{v}^2) \\ &= b^2(c - a)^2 - 4ab(b - a)(c - b), \end{aligned}$$

where we have set

$$\begin{aligned} a &= \bar{v}^2, \\ b &= \bar{v}^2, \\ c &= \bar{v}^2. \end{aligned}$$

By Schwarz's inequality, we have $0 < a < b < c$. Thus

$$\begin{aligned} D &= b^2\{(c - b) + (b - a)\}^2 - 4ab(b - a)(c - b) \\ &= b^2\{(c - b)^2 + (b - a)^2\} - 2(c - b)(b - a)ab + (c - b)(b - a)(2b^2 - 2ab). \end{aligned}$$

The third term of this expression is clearly positive. As for the first two, the fact that $b > a$ means that the first two terms are greater than $ab\{(c - b)^2 +$

$(b - a)^2 - 2(c - b)(b - a)\} = ab(c - a)^2$, which is positive. Hence $D > 0$, and thus the roots are distinct.

5. The Chapman–Enskog–Hilbert expansion and the equations of hydrodynamics. So far in this paper our approach to the model Boltzmann equations has been purely analytic. We now turn to more physical features of these equations. First, we briefly discuss the Chapman–Enskog–Hilbert expansion for special solutions of

$$(5.1) \quad p_t + vp_x = Qp, \quad \lim_{t \downarrow 0} p(t, x, v) = f(x, v).$$

We make a connection between the eigenvalues $\alpha_j(i\gamma)$, $1 \leq j \leq d$, of $Q - i\gamma v$ and the so-called hydrodynamical operator, which governs the evolution of a certain d -dimensional⁵ piece (hydrodynamical moments) of these special solutions. Because of this connection, the Chapman–Enskog–Hilbert expansion can be used to calculate the coefficients in the expansion in powers of γ of the first d eigenvalues. We illustrate the procedure for two of the cases of the previous section.

The other purpose of this section is to study the relationship between the two limit theorems of Section 1 and the Euler and Navier–Stokes hydrodynamical equations connected with (1.1). In their usual formulation, the latter are written as systems of d coupled partial differential equations involving the d hydrodynamical moments. Instead, we write the equations in terms of certain “derived hydrodynamical moments,” using not the given basis in $\mathfrak{X}$ but the “hydrodynamical basis” $\{e_0^{(k)}\}$. A remarkable simplification occurs in the Euler equations. They decouple into d separate wave equations (with wave velocities $\alpha_1^{(k)}$), one equation for each derived moment. One of the main results of this section is that *the hydrodynamical moments of the limit in Theorem 1 satisfy the Euler equations*. This is easy to verify using our way of writing the latter. The situation with the Navier–Stokes equations is not so transparent since in general the equations for the derived moments do not decouple. However, in special cases, which include the situation when v maps $\mathfrak{X}$ into $\mathfrak{X}^\perp$ (e.g., when $\mathfrak{X}$ is spanned entirely by either even powers of v or odd powers of v), the Navier–Stokes equations decouple into d separate heat equations (with variances $\alpha_2^{(k)}$), one equation for each derived moment. But the derived moments of the limit in Theorem 2 satisfy exactly the same equations. We thus conclude that *in these special cases the hydrodynamical moments of the limit in Theorem 2 satisfy the Navier–Stokes equations*.

Let $\{e_0^{(k)}\}_{1 \leq k \leq d}$ be the basis of $\mathfrak{X}$ defined in Section 2A. We recall the formulae

$$(5.2) \quad \begin{aligned} \langle e_0^{(i)}, e_0^{(k)} \rangle &= \delta_{ik}, \\ \langle e_0^{(i)}, ve_0^{(k)} \rangle &= -\alpha_1^{(k)} \delta_{ik}. \end{aligned}$$

Briefly, $\{e_0^{(k)}\}$ is a basis of $\mathfrak{X}$ which diagonalizes PvP , where P is the orthogonal projection onto $\mathfrak{X}$. If p is a solution of the model equation (5.1), then the *derived*

hydrodynamical moments of p are defined by

$$n_k(t, x) = \langle p, e_0^{(k)} \rangle, \quad (1 \leq k \leq d).$$

We shall speak in terms of the derived moments rather than the hydrodynamical moments as they are usually defined. The two modes of description are entirely equivalent.

The basic assumption behind the Chapman–Enskog–Hilbert expansion is that for large times, p should have a universal functional form, the individual p depending in its details only upon its own hydrodynamical moments. The subclass of special solutions which depend upon their moments in the proposed manner is picked out by expanding p as a formal power series in ϵ :

$$p = p_0 + \epsilon p_1 + \epsilon^2 p_2 + \cdots,$$

which is then required to solve the scaled equation

$$(5.3) \quad \frac{\partial p}{\partial t} + v \frac{\partial p}{\partial x} = \frac{1}{\epsilon} Qp, \quad \lim_{\epsilon \rightarrow 0} p = f.$$

Here, f is a formal power series in ϵ with smooth coefficients. Later, ϵ is set equal to 1 and convergence to an actual solution of (5.1) is hoped for. The term *Hilbert solution* will be used to denote such a formal power series solution evaluated at $\epsilon = 1$. We emphasize that ϵ plays a different role here than in the rest of the paper. In the Chapman–Enskog–Hilbert theory, ϵ is a fixed, real parameter inserted into (5.1) in order to pick out special solutions of that equation while earlier ϵ was allowed to go to zero. Substituting the power series for p into (5.3), we obtain the equations

$$(5.4) \quad \begin{aligned} Qp_0 &= 0, \\ Qp_n &= p_{n-1} + vp'_{n-1}, \quad n \geq 1. \end{aligned}$$

A procedure for solving these equations (which also shows that for a Hilbert solution the Chapman–Enskog assumption is valid) is outlined in [1].⁷ The successive (approximate) hydrodynamical equations (Euler, Navier–Stokes, Burnett, *etc.*) for the moments of the approximations $p_0, p_0 + p_1, p_0 + p_1 + p_2, \text{etc.}$, follow as solubility conditions in the sequence of equations (5.4). Concerning the moments of the full power series for p , one can show that they evolve by an equation of the form

$$(5.5) \quad \frac{\partial}{\partial t} \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix} = H_\epsilon \begin{pmatrix} n_1 \\ \vdots \\ n_d \end{pmatrix},$$

where $H_\epsilon = H_\epsilon(\partial/\partial x)$ (*the hydrodynamical operator*) is a special $d \times d$ matrix each entry of which is an infinite order scalar differential operator. H_ϵ can be explicitly calculated. When evaluated at $\epsilon = 1$, H_ϵ will be written as H .

In [1], a number of facts about Hilbert solutions and their relationship to general solutions of (5.1) are proved for the case $d = 1$ and $q_{ij} > 0$ for $i \neq j$. They all carry over with little change to the present situation and so will not be restated.

For later use we state a connection between the eigenvalues $\alpha^{(i)}(i\gamma)$, $1 \leq j \leq d$, of $Q - i\gamma v$ and the hydrodynamical operator.

Theorem 5.1. *For $|\gamma|$ sufficiently small, the eigenvalues of the $d \times d$ matrix $H_\epsilon(i\gamma)$ are $\alpha^{(i)}(i\gamma)$, $1 \leq j \leq d$.*

Proof. We seek formal power series solutions of (5.1) which have the form of free waves:

$$p = \hat{p}(\gamma, v) e^{\alpha t + i\gamma x}.$$

Here, $p|_{t=0} = \hat{p} e^{i\gamma x}$ is given as a formal power series in ϵ . Substituting into (5.3) shows that α must be an eigenvalue of $\epsilon^{-1}(Q - i\gamma v)$. For p to be a formal power series in ϵ , α must stay finite as $\epsilon \rightarrow 0$, which implies $\alpha = \epsilon^{-1} \alpha^{(i)}(i\gamma \epsilon)$ for some $1 \leq j \leq d$ provided $|\gamma|$ is sufficiently small. Since the evolution of the moments of such a p is determined by $\epsilon^{-1} \alpha_j(i\gamma \epsilon)$, it follows from (5.5) that the latter is an eigenvalue of $H_\epsilon(i\gamma)$. All d eigenvalues of this $d \times d$ matrix are thereby accounted for.

In practice, each of the scalar differential operators $(H(\partial/\partial x))_{jk}$, $1 \leq j, k \leq d$, is calculated term by term as the successive hydrodynamical equations are found. Each new hydrodynamical equation brings in the next order derivative: Euler gives H to first order in $\partial/\partial x$, Navier-Stokes to second order, etc. It turns out that the successive terms in the expansions of the first d eigenvalues can be determined from the corresponding approximations to the hydrodynamical operator. We illustrate this later.

We now turn to the hydrodynamical equations and discuss their connection with the two limit theorems. Our way of deriving the hydrodynamical equations differs from that discussed above, but the two are formally equivalent.⁸ Let us write an arbitrary solution of the Boltzmann equation (5.1) in the form

$$(5.6) \quad p = p_0 + p_\perp,$$

where $p_0 = \sum_{k=1}^d n_k e_0^{(k)}$ and $p_\perp$ is perpendicular to $\mathfrak{H}$. Taking the inner product of both sides of (5.1) with the $e_0^{(k)}$ and using (5.2), we have the so-called *conservation equations*

$$(5.7) \quad \dot{n}_k - \alpha_1^{(k)} n'_k + \langle e_0^{(k)}, v p'_\perp \rangle = 0.$$

The successive hydrodynamical equations are derived by setting $p_\perp = 0$, $p_\perp = p_1$, $p_\perp = p_1 + p_2$, etc., where the p_i are obtained from the Chapman-Enskog-Hilbert expansion. More precisely, each p_n solves a modified version of (5.4); this version is obtained by eliminating the time-derivative $\dot{p}_{n-1}$ on the right-hand side of (5.4) by substitution from the $(n-1)^{\text{th}}$ hydrodynamical equation.

Setting $p_{\perp} = 0$ in (5.7) gives the Euler equations

$$(5.8) \quad \dot{n}_k = \alpha_1^{(k)} n'_k.$$

Thus, in the Eulerian approximation, each of the derived hydrodynamical moments n_k evolves according to a single partial differential equation. It is of interest to note that the moments of the limit in Theorem 1 satisfy the Euler equations.

Theorem 5.2. *Let $\bar{p} = \sum_{i=1}^d \langle f, e_0^{(k)} \rangle (x + \alpha_1^{(k)} t) e_0^{(k)}$ be the right-hand side of (1.2). Then the derived moments $\bar{n}_k$ of $\bar{p}$ satisfy the Euler equations (5.8).*

Proof. This is immediate since

$$\bar{n}_k = \langle f, e_0^{(k)} \rangle (x + \alpha_1^{(k)} t),$$

which is a solution of (5.8).

To obtain the Navier–Stokes equations, we set

$$p_{\perp} = p_1 = Q^{-1}(p_0 + v p'_0),$$

where $p_0 = \sum_{k=1}^d n_k e_0^{(k)}$, the n_k solve the Euler equations (5.8), and p_0 is eliminated by substituting from the latter. Putting $p_{\perp}$ into (5.7) and simplifying yields the equations

$$(5.9) \quad \dot{n}_k - \alpha_1^{(k)} n'_k + \sum_{l=1}^d \langle v e_0^{(k)}, Q^{-1}(v + \alpha_1^{(l)} e_0^{(l)}) n'_l \rangle = 0.$$

These are equivalent to

$$(5.10) \quad \dot{n}_k - \alpha_1^{(k)} n'_k + \sum_{l=1}^d \alpha_2^{(kl)} n''_l = 0,$$

where

$$(5.11) \quad \alpha_2^{(kl)} = \langle (v + \alpha_1^{(k)} e_0^{(k)}), Q^{-1}(v + \alpha_1^{(l)} e_0^{(l)}) \rangle$$

and where we have used the fact that $(v + \alpha_1^{(l)} e_0^{(l)})$ is perpendicular to $\mathfrak{X}$. Hence, in the second approximation, the moments n_k evolve in general according to a coupled system of partial differential equations. In special cases, however, the Navier–Stokes equations decouple into separate heat equations for each derived moment, and the connection with the limit in Theorem 2 is immediate.

Theorem 5.3. *Say v maps $\mathfrak{X}$ into $\mathfrak{X}^{\perp}$. Then the Navier–Stokes equations decouple into*

$$(5.12) \quad \dot{n}_k = \alpha_2^{(k)} n''_k.$$

Let $\bar{p}$ denote the right-hand side of (1.3). Then the derived moments $\bar{n}_k$ of $\bar{p}$ satisfy the Navier–Stokes equations (5.12).

Proof. If v maps $\mathfrak{X}$ to $\mathfrak{X}^{\perp}$, then by (5.2), $\alpha_1^{(k)} = 0$, $1 \leq k \leq d$. Also, according to Proposition 2A.2, $\alpha_2^{(kl)} = -\alpha_2^{(k)} \delta_{kl}$, which gives (5.12). Since

$$\bar{n}_k = \int_{-\infty}^{\infty} \frac{e^{-y^2/2}}{(2\pi)^{1/2}} \langle f, e_0^{(k)} \rangle (x + y(2\alpha_2^{(k)}t)^{1/2}) dy$$

is a solution of (5.12), we are through.

The two cases of Section 4 not covered by this theorem are when $\mathfrak{X} = \{1, v\}$ and $\mathfrak{X} = \{1, v, v^2\}$. In the first case, the Navier–Stokes equations read

$$(5.13) \quad \begin{aligned} n_1 &= -\bar{v}n_1' + \alpha n_1'' + \alpha n_2'', \\ n_2 &= \bar{v}n_2' + \alpha n_1'' + \alpha n_2'', \end{aligned}$$

where $\alpha = -\langle v^2 - \bar{v}^2, Q^{-1}(v^2 - \bar{v}^2) \rangle / N\bar{v}^2$. In the $\{1, v, v^2\}$ case, the Navier–Stokes equations are

$$(5.14) \quad \begin{aligned} n_1 &= -\bar{v}n_1' + \alpha n_1'' + \alpha n_2'' + \tilde{\alpha} n_3'', \\ n_2 &= \bar{v}n_1' + \alpha n_1'' + \alpha n_2'' + \tilde{\alpha} n_3'', \\ n_3 &= \tilde{\alpha} n_1'' + \tilde{\alpha} n_2'' + \tilde{\tilde{\alpha}} n_3'', \end{aligned}$$

where $\alpha = -\beta/2N\bar{v}^2\bar{v}^2$, $\tilde{\alpha} = -\beta/N\bar{v}^2(2\bar{v}^2(\bar{v}^2 - \bar{v}^2))^{1/2}$, $\tilde{\tilde{\alpha}} = -\beta/N\bar{v}^2(\bar{v}^2 - \bar{v}^2)$, and $\beta = \langle v^3 - v\bar{v}^2, Q^{-1}(v^3 - v\bar{v}^2) \rangle$. For neither of these two cases can we make a connection with the limit in Theorem 2.

We close this section by amplifying, in the case $d = 2$, the remark after Theorem 5.1 concerning the calculation of successive terms in the expansions of the first d eigenvalues by means of the corresponding approximations to the hydrodynamical operator. For larger d , the discussion is similar. We assume here that Q and v satisfy the symmetry conditions stated at the beginning of Section 4.

Theorem 5.1 guarantees that for $|\gamma|$ sufficiently small a non-singular matrix $A = A(\gamma)$ exists such that

$$AH(i\gamma)A^T = \begin{bmatrix} \alpha^{(1)}(i\gamma) & 0 \\ 0 & \alpha^{(2)}(i\gamma) \end{bmatrix}.$$

The invariance of trace and determinant implies

$$(5.15) \quad \begin{aligned} \alpha^{(1)} + \alpha^{(2)} &= H_{11} + H_{22}, \\ \alpha^{(1)}\alpha^{(2)} &= H_{11}H_{22} - H_{12}H_{21}, \end{aligned}$$

where $H_{ik} = (H(i\gamma))_{ik}$ and $\alpha^{(i)} = \alpha^{(i)}(i\gamma)$, $j = 1, 2$, are the eigenvalues. Thus, if we substitute into (5.15) the expansions

$$\begin{aligned} H_{ik} &= \sum_{l \geq 1} h_l^{(ik)}(i\gamma)^l, \\ \alpha^{(k)} &= \sum_{l \geq 1} \alpha_l^{(k)}(i\gamma)^l, \end{aligned}$$

and collect coefficients of powers of γ , we see that the $\alpha_l^{(k)}$ satisfy the equations

$$(5.16a) \quad \alpha_i^{(1)} + \alpha_i^{(2)} = h_i^{(11)} + h_i^{(22)},$$

$$(5.16b) \quad \sum_{n=1}^l \alpha_{l-n+1}^{(1)} \alpha_n^{(2)} = \sum_{n=1}^l (h_{l+n-1}^{(11)} h_n^{(22)} - h_{l+n-1}^{(12)} h_n^{(21)}), \quad l \geq 1.$$

Let us rewrite (5.16) for $l = 1$, then for $l \geq 2$.

We have

$$(5.17) \quad \begin{aligned} \alpha_1^{(1)} + \alpha_1^{(2)} &= h_1^{(11)} + h_1^{(22)}, \\ \alpha_1^{(1)} \alpha_1^{(2)} &= h_1^{(11)} h_1^{(22)}, \end{aligned}$$

$$(5.18) \quad \alpha_l^{(1)} + \alpha_l^{(2)} = h_l^{(11)} + h_l^{(22)},$$

$$\alpha_l^{(2)} \alpha_l^{(1)} + \alpha_l^{(1)} \alpha_l^{(2)} = \sum_{n=1}^l (h_{l+1-n}^{(11)} h_n^{(22)} - h_{l+1-n}^{(12)} h_n^{(21)}) - \sum_{n=2}^{l-1} \alpha_{l+1-n}^{(1)} \alpha_n^{(2)},$$

$$l \geq 2.$$

In (5.17) we have used the fact that $h_1^{(12)} = h_1^{(21)} = 0$ (see (5.8)). In (5.18) the empty sum equals zero. Equations (5.17) have the unique solution $\alpha_1^{(k)} = h_1^{(kk)}$, so that these coefficients are known at the Euler level. We solve equations (5.18) iteratively, considering them as linear equations in the unknowns $\alpha_i^{(1)}$, $\alpha_i^{(2)}$. Let us assume we know the $\alpha_m^{(k)}$, $1 \leq m \leq l-1$. Provided the determinant $\alpha_1^{(1)} - \alpha_1^{(2)}$ of the left-hand side of (5.18) is non-zero, then these equations uniquely determine the $\alpha_l^{(k)}$ in terms of the $h_m^{(ik)}$, $1 \leq m \leq l$. In other words, the l^{th} approximate hydrodynamical equations will determine the $\alpha_i^{(k)}$ uniquely. As an example of when this is so, consider $N = \{1, v\}$. Then $h_1^{(11)} = \bar{v} = -h_1^{(22)}$ (see (5.13)) and $\alpha_1^{(k)} = h_1^{(kk)}$, in agreement with the calculation in Section 4. The $\alpha_2^{(k)}$ satisfy

$$(5.19) \quad \begin{aligned} \alpha_2^{(1)} + \alpha_2^{(2)} &= h_2^{(11)} + h_2^{(22)} = -2\langle v^2 - \bar{v}^2, Q^{-1}(v^2 - \bar{v}^2) \rangle / N\bar{v}^2, \\ \bar{v}(\alpha_2^{(1)} - \alpha_2^{(2)}) &= h_2^{(11)} h_2^{(22)} - h_2^{(12)} h_2^{(21)} = 0, \end{aligned}$$

where we have used (5.13) for the $h_2^{(ik)}$. Equations (5.19) have the solution $\alpha_2^{(1)} = \alpha_2^{(2)} = -\langle v^2 - \bar{v}^2, Q^{-1}(v^2 - \bar{v}^2) \rangle / N\bar{v}^2$, in agreement with Section 4.

The question remains as to the non-vanishing of $\alpha_1^{(1)} - \alpha_1^{(2)}$. Because of the normalization of the velocities, it turns out that $\alpha_1^{(1)} = -\alpha_1^{(2)}$ always. If $\alpha_1^{(1)} = 0$, then we write (5.16) for $l = 2$, then for $l \geq 3$. We have

$$(5.20) \quad \begin{aligned} \alpha_2^{(1)} + \alpha_2^{(2)} &= h_2^{(11)} + h_2^{(22)}, \\ \alpha_2^{(1)} \alpha_2^{(2)} &= h_2^{(11)} h_2^{(22)}, \\ \alpha_l^{(1)} + \alpha_l^{(2)} &= h_l^{(11)} + h_l^{(22)}, \end{aligned}$$

$$\alpha_2^{(2)} \alpha_l^{(1)} + \alpha_2^{(1)} \alpha_l^{(2)} = \left(\sum_{n=2}^l (h_{l+2-n}^{(11)} h_n^{(22)} - h_{l+2-n}^{(12)} h_n^{(21)}) \right) - \sum_{n=3}^{l-1} \alpha_{l+2-n}^{(1)} \alpha_n^{(2)},$$

$$l \geq 3.$$

The second equation in (5.20) comes from (5.16b) for $l = 3$. Equation (5.16b) for $l = 2$ is vacuous. In deriving (5.20), we used the fact that $\alpha_1^{(k)} = h_1^{(ik)} =$

$h_2^{(12)} = h_2^{(21)} = 0$. Equations (5.20) have the unique solution $\alpha_2^{(k)} = h_2^{(k)} = \langle v e_0^{(k)}, Q^{-1} v e_0^{(k)} \rangle$ (see (5.10) and (5.11)). Concerning (5.21), the second of these comes from (5.16b) for $l + 1$. We also used here the fact that $\alpha_1^{(k)} = h_1^{(k)} = 0$. Let us solve (5.21) iteratively, considering them as linear equations in the unknowns $\alpha_i^{(k)}$. Provided the determinant $\alpha_2^{(1)} - \alpha_2^{(2)}$ of the left-hand side of (5.19) is non-zero, then these equations uniquely determine the $\alpha_i^{(k)}$ in terms of the $h_m^{(ik)}$, $2 \leq m \leq l$. In other words, as above, the l^{th} approximate hydrodynamical equations will determine the $\alpha_i^{(k)}$. Although we know no general proof concerning this condition on (5.21), it is true for each case we have considered. See, for example, Proposition 4.5, which shows $\alpha_2^{(1)} \neq \alpha_2^{(2)}$ when $N = \{1, v^2\}$ and Q is minus the identity on $N^\perp$.

FOOTNOTES

1. Thus $\langle f, e_0^{(i)} \rangle (x + \alpha_1^{(i)} t) = \sum_{i=1}^N f(x + \alpha_1^{(i)} t, v_i) e_0^{(i)}(v_i)$.
2. The map Q^{-1} denotes the inverse of Q as an operator from $\mathfrak{R}^\perp$ to $\mathfrak{R}^\perp$.
3. This follows from the inequality $q_{ii} q_{jj} \leq q_{ij}^2$.
4. For $N \leq 3$, the conditions of Case IV together with $v_i + v_{N-i} = 0$ contradict the earlier assumption of $q_{ii} \neq 0$.
5. Recall that d is the dimension of $\mathfrak{R}$.
6. In (5.4), $\cdot$ indicates $\partial/\partial t$ and $'$ indicates $\partial/\partial x$.
7. Also, see Grad [5], pp. 254 ff.
8. *Ibid.*, p. 261.
9. With reference to the remark after Theorem 5.1, we see that to first order $(H)_{jk} \sim \alpha_1^{(k)} \delta_{jk} \partial/\partial x$. Thus, the first term in the expansions of the hydrodynamical eigenvalues is $\alpha_1^{(k)} i_\gamma$, which we know.

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