Generalized Hurwitz–Riemann Formulas

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1. Introduction. Let $f: M^n \rightarrow N^n$ be a continuous map, where M^n and N^n are compact, connected, oriented n -manifolds. It will be assumed throughout that f has nonzero degree. A point y in N^n is *pointwise (componentwise) deficient* if $f^{-1}(y)$ has fewer than $|\deg f|$ points (components). Define the *pointwise (componentwise) deficiency* $d_f(y)$ ($D_f(y)$) of a deficient point y to be the difference $|\deg f| - K(y)$, where $K(y)$ is the number of points (components) in $f^{-1}(y)$. Let d_f (D_f) be the set of pointwise (componentwise) deficient points. Then Hopf has shown [2, p. 275, (4)] that for 2-manifolds

$$\sum \{d_f(y) : y \in d_f\} \leq |\deg f| \chi(N^2) - \chi(M^2)$$

where χ is the Euler Characteristic. Hopf's result relied heavily on a discovery by Kneser [3] that for such maps $\chi(M^2) \leq |\deg f| \chi(N^2)$. Sections 2 and 3 will simplify and extend the Kneser–Hopf approach to show the somewhat stronger 1.1.

1.1 Theorem. Let $f: (M^2, \partial M^2) \rightarrow (N^2, \partial N^2)$ be a continuous map such that

- (1) $\deg f \neq 0$;
- (2) M^2 and N^2 are compact, connected, oriented 2-manifolds with boundary.

Then

$$\sum \{D_f(y) : y \in D_f\} + \text{rank im } i^* \leq |\deg f| \chi(N^2) - \chi(M^2)$$

where i^* is the homomorphism in cohomology (or homology) in dimension one induced by the inclusion map $i: f^{-1}(D_f) \rightarrow M^2$.

Any homology or cohomology theory (e.g., Čech) with coefficients in a principal ideal domain may be used. A somewhat stronger version of 1.1 is given in 3.3.

A map $f: X \rightarrow Y$ is *quasimonotone* if for any continuum K in Y with $\text{int } K \neq \emptyset$, $f^{-1}(K)$ has but a finite number of components and each of these maps onto K under f . Both proper open and monotone onto maps between manifolds are quasimonotone [7, p. 151, (8)]. Ziemer [8] has shown that for quasimonotone maps $f: M^2 \rightarrow N^2$ between compact, connected 2-manifolds without boundary (oriented or not) there exists a minimal integer d such that $f^{-1}(y)$ has at most

d components for all y in N^2 . By defining deficiency in terms of this constant d , rather than $|\deg f|$, we prove the following in Section 4 using integral Alexander cohomology.

1.2. Theorem. *Let $f : M^2 \rightarrow N^2$ be a continuous map such that*

- (1) *f is quasimonotone and d is as above, and*
- (2) *M^2 and N^2 are closed, connected 2-manifolds. Then*

$$\sum \{D_f(y) : y \in N^2\} + \sum \{\dim H^1(f^{-1}(y); Z) : y \in N^2\} = d\chi(N^2) - \chi(M^2).$$

This theorem will rely heavily on Ziemer's work [8].

A map $f : X \rightarrow Y$ is light if $\dim f^{-1}(y) = 0$ for all y in Y . For light open maps, 1.2 is a generalization of the following version of the Hurwitz–Riemann Equation.

1.3. Theorem. *Let $f : M^2 \rightarrow N^2$ be a light open map where M^2 and N^2 are closed, connected 2-manifolds. Then*

$$\sum \{d_f(y) : y \in N^2\} = |\deg f| \chi(N^2) - \chi(M^2).$$

For monotone maps [7, p. 70] 1.2 yields

1.4. Corollary. *Let $f : M^2 \rightarrow N^2$ be a continuous map such that*

- (1) *f is monotone and surjective, and*
- (2) *M^2 and N^2 are closed, connected 2-manifolds. Then*

$$\sum \{\dim H^1(f^{-1}(y), Z) : y \in N^2\} = \chi(N^2) - \chi(M^2).$$

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2. By a *true cyclic element* of a space Y will be meant a nondegenerate connected subset which has no cut points and which is maximal with respect to the property of being a connected subset without cut points. For a thorough discussion of cyclic element theory consult [7, Chapter 4]. A *generalized cactoid* is a continuous curve whose every true cyclic element is a 2-manifold and all but a finite number of which are 2-spheres [4, p. 854].

2.1. Remark. Any generalized cactoid C_0 , which is also a finite two dimensional polyhedron, may be characterized as the finite union of closed 2-manifolds X_j ($j = 1, \dots, m$) together with a 1-dimensional polyhedron X_0 such that $X_i \cap X_j$ is a finite set ($i \neq j$; $i, j = 0, 1, \dots, m$).

Since 1.3 is readily proved in the oriented simplicial case and this is the version which will be needed in the development of 1.1, we give its proof here for completeness's sake.

2.2. (Proof of 1.3 in the oriented simplicial case). Suppose that two 2-simplices of M^2 are mapped with opposite orientations. Join these two simplices by a chain of 2-simplices each successive pair of which has an edge in common.

Then it must be that two successive members of this chain are mapped with opposite orientations. Such an occurrence indicates a fold by the map f and this contradicts the openness of the map. Thus every 2-simplex of M^2 is mapped with the same orientation. It follows then that every 2-simplex of N^2 has $|\deg f|$ inverses. The same holds true then for every 1-simplex in N^2 . Noting that the 0-simplices of N^2 fall short of $|\deg f|$ inverses by exactly the deficiency completes the theorem.

The following statement will be used several times in the proofs that follow.

2.3. Lemma. *Let $f : S^2 \rightarrow N^2$ be a continuous map where N^2 is a closed, connected, oriented 2-manifold other than the 2-sphere, S^2 . Then $\deg f = 0$.*

Proof. Let H be the space obtained from a torus by removing a small disk. Define $p : N^2 \rightarrow T$, the torus, with degree one by first collapsing the complement of H in N^2 to a point, and then mapping the identification space obtained identically onto the torus. Let δ_1 and δ_2 be the positive generators of $H^1(T, \mathbb{Z})$. The element $\delta_1 \cdot \delta_2$ (cup product) generates $H^2(T, \mathbb{Z})$. Knowing as we do the degree of p , it follows that $p^*(\delta_1 \cdot \delta_2)$ generates $H^2(N^2, \mathbb{Z})$. Let σ be the generator of $H^2(S^2, \mathbb{Z})$. Then $(\deg f) \sigma = f^*(p^*(\delta_1 \cdot \delta_2)) = (f^*p^*(\delta_1)) \cdot (f^*p^*(\delta_2)) = 0 \cdot 0 = 0$ and so $\deg f = 0$.

The next result is due to Kneser [3]. Robinson has achieved an extension of this result [5] in his thesis. Since Kneser's proof relies heavily on an involved proof in an earlier paper and Robinson's is unpublished we will present details of a new and considerably shorter proof.

2.4. Theorem. (Kneser [3]). *Let $f : M^2 \rightarrow N^2$ be a continuous map such that (1) M^2 and N^2 are closed, connected, oriented 2-manifolds and (2) $\deg f \neq 0$. Then $\chi(M^2) \leq |\deg f| \chi(N^2)$.*

Proof. Observe first that the theorem is certainly true when $N^2 = S^2$. Let $A = \{(M^2, f, N^2)\}$ for which the theorem is false. If $A \neq \emptyset$, then choose $(M^2, f, N^2) \in A$ such that $\chi(M^2)$ is maximal.

Claim 1. *We may assume the triple (M^2, f, N^2) has been chosen with f light and simplicial.*

Proof. Certainly, by virtue of simplicial approximation, f may be assumed simplicial. By taking appropriate subdivisions of M^2 and N^2 we can obtain a monotone light factorization $f = h \circ g$ for which the factor maps are simplicial and the intermediate space C_k is a finite simplicial complex [6]. Further, C_k is obtained from a generalized cactoid C_0 by k 2-point identifications [4]. Roberts and Steenrod have shown [4] that $b_1(M^2) \geq b_1(C_k)$ from which it follows that $b_1(M^2) \geq b_1(C_0)$ where b_1 is the 1-dimensional betti number mod 2. Characterize C_0 as in 2.1. Then h induces light simplicial maps $h_j : X_j \rightarrow N^2$ ($j = 1, \dots, m$). Since g is monotone and simplicial we may define orientations for the orientable manifold components of X such that $\deg f = \sum \deg h_j$. Note that the constraint induced above on the 1-betti number of X leads to $\chi(M^2) \leq \chi(X_i)$

for X_j orientable. If the inequality is always strict then the maximality of $\chi(M^2)$ leads to

$$\chi(M^2) \leq \sum \chi(X_i) \leq \sum |\deg h_i| \chi(N^2) \leq |\deg f| \chi(N^2)$$

where the sums are restricted to those j for which $\deg h_i \neq 0$. In this case $(M^2, f, N^2) \notin A$. Thus it must be that $\chi(M^2) = \chi(X_j)$ for some fixed j . Since $b_1(M^2) \geq b_1(C_0)$ we have then that $X_i = S^2$ for $i \neq j$. Using 2.3 and $\deg f = \sum \deg h_i$ we see $\deg f = \deg h_j$. That is, the triple (X_j, h_j, N^2) satisfies the claim.

Claim 2. We may assume the triple (M^2, f, N^2) has been chosen with f light, open and simplicial.

Proof. Choose (M^2, f, N^2) as in Claim 1. Let $X_0 = M^2$ and $f_0 = f$. The proof is constructive and will proceed inductively on the 1-simplices s_j ($j = 1, \dots, q$) of X_0 . At each step we will define a cellular map $f_i : X_i \rightarrow N^2$ such that

- (1) $\deg f_i = \deg f$ and
- (2) $X_i \cong M^2$ ($j = 1, \dots, q$).

Cellular maps $p_i : X_i \rightarrow X_{i+1}$ will be defined and we will set

$$P_{i+1} = p_i \circ p_{i-1} \cdots p_0.$$

We will proceed inductively.

First step: ($j = 1$). If f_0 does not fold about s_1 set $X_1 = X_0$, $f_1 = f_0$ and $p_0 = \text{identity}$. If f_1 folds about s_1 , then excise the interior of $\sigma_1 \cup \sigma_2$ (union) where σ_1 and σ_2 are the two 2-simplices with face s_1 . Now identify the remaining corresponding 1-simplices bounding σ_1 and σ_2 so as to induce a well-defined map f_1 from f_0 . The space X_1 will be the domain of f_1 and p_0 will be defined naturally.

Inductive step—($1 < j = k < q$). If f_{k-1} does not fold about $P_{k-1}(s_k)$ set $X_k = X_{k-1}$, $f_k = f_{k-1}$ and $p_{k-1} = \text{identity}$. If f_{k-1} folds about $P_{k-1}(s_k)$, then excise the interior of $\sigma_1 \cup \sigma_2$ where σ_1 and σ_2 are the two 2-cells with face $P_{k-1}(s_k)$. Identify the remaining corresponding 1-cells bounding σ_1 and σ_2 so as to induce a well-defined map g from f_{k-1} . Let Y be the domain of g . Now σ_1 and σ_2 may have intersected in either

- a) 3 edges or
- b) 1 or 2 edges or
- c) $P_{k-1}(s_k)$ and a third vertex.

Case (a) implies that X_{k-1} was a sphere and 2.3 eliminates this as a possibility.

In case (b) set $X_k = Y$, $f_k = g$ and let p_{k-1} be the natural map from X_{k-1} to X_k . Now proceed to $j = k + 1$.

In case (c) observe that the space Y could just as well have been obtained by identifying 2 points of some manifold L . The map g induces a map $t : L \rightarrow N^2$. Note that $\chi(L) = \chi(X_{k-1}) + 2$ (count cells). This contradicts our maximality condition if L is connected. Let L_1 and L_2 be the components of L . Note $\chi(L_i) \geq \chi(X_{k-1})$ ($i = 1, 2$). If the inequality is always strict our maximality condition

leads to

$$\chi(M^2) = \chi(X_{k-1}) < \sum_i \chi(L_i) \leq \sum |\deg t_i| \chi(N^2) \leq |\deg f| \chi(N^2)$$

where the sums are restricted to those i for which $\deg t_i \neq 0$. But then $(M^2, f, N^2) \notin A$. Therefore $\chi(X_{k-1}) = \chi(L_i)$ for some i . Call this component X_k and the corresponding induced map f_k . Let p_{k-1} be the natural map. Since the other component of L is a sphere (Euler Characteristic is 2), 2.3 assures us $\deg f_k = \deg f_{k-1}$. This completes case (c).

Proceeding through $j = q$ gives us a triple $(X_q, f_q, N^2) \in A$ with f_q light, open and cellular. A barycentric subdivision of domain and range completes the claim.

Claim 2 and reference to 2.2 completes the theorem.

3. We are now in a position to begin an attack on 1.1. we will start by strengthening Hopf's result [2, p. 275 (4)].

3.1. Theorem. *Let $f : M^2 \rightarrow N^2$ be a continuous map such that (1) M^2 and N^2 are closed, connected, oriented 2-manifolds and (2) $\deg f \neq 0$ then*

$$\sum \{D_f(y) : y \in D_f\} \leq |\deg f| \chi(N^2) - \chi(M^2).$$

Proof. To prove the theorem it will suffice to show the equation holds true when the left hand sum is restricted to any finite subset B of D_f . Let $B = \{y_i\}_{i=1}^m$ be any such set. The components A_{ij} of $f^{-1}(y_i)$ are isolated. Choose disjoint disk neighborhoods $V_i(y_i)$ and $\delta > 0$ such that

- (a) $f(A_{ij} + \delta) \subset V_i$ for all i and j ,
- (b) $\{A_{ij} + \delta\}$ are all disjoint.

For each A_{ij} pick X_{ij} a compact, connected, oriented 2-manifold with boundary such that

- (a) $A_{ij} \subset \text{interior } X_{ij}$,
- (b) $X_{ij} \subset A_{ij} + \delta$.

The boundary of the X_{ij} will consist of disjoint circles S_{ijk}^1 embedded in M^2 . The image of $(M^2 - \bigcup_{ij} \text{int } X_{ij})$ will be compact and will not contain B . Choose disk neighborhoods $U_i(y_i)$ such that $\bigcup_i (\overline{U_i(y_i)}) \cap f(M^2 - \bigcup_{ij} \text{int } X_{ij}) = \emptyset$. Note $\bar{U}_i \subset V_i$. For each S_{ijk}^1 choose a bicollar ϵ such that

- (a) $(S_{ijk}^1 \times [-\epsilon, \epsilon]) \subset A_{ij} + \delta$,
- (b) $S_{ijk}^1 \times \{0\}$ corresponds to S_{ijk}^1 ,
- (c) $(S_{ijk}^1 \times [-\epsilon, 0]) \subset (\text{int } X_{ij} - A_{ij})$,
- (d) $(S_{ijk}^1 \times (0, \epsilon]) \subset [(A_{ij} + \delta) - X_{ij}]$.

Define d_{ijk} to be the degree of $f| (S_{ijk}^1 \times \{\epsilon\}) \rightarrow (V_i - U_i)$. Define $g : M^2 \rightarrow N^2$ by

- (a) $g : [X_{ij} - U_i(S_{ijk}^1 \times (-\epsilon, 0])] \rightarrow \{y_i\}$,
- (b) $g : (S_{ijk}^1 \times \{0\}) \rightarrow \text{boundary } U_i$ with degree d_{ijk} ,
- (c) $g : (S_{ijk}^1 \times [-\epsilon, 0]) \rightarrow U_i$ a homotopy between (a) and (b),
- (d) $g : (S_{ijk}^1 \times (0, \epsilon]) \rightarrow (V_i - U_i)$ a homotopy between $g| (S_{ijk}^1 \times \{0\})$ and $f| (S_{ijk}^1 \times \{\epsilon\})$,

(e) $g = f$ elsewhere.

Then

(a) $\deg f = \deg g$,

(b) $\sum \{D_f(y) : y \in B\} = \sum \{D_g(y) : y \in B\}$,

(c) $\{g^{-1}(U_i)\} = \{X_{ii}\}$,

(d) g maps the boundary of X_{ii} into the boundary of U_i . Remove the U_i from N^2 and the X_{ii} from M^2 . Let $\bar{N}^2$ and $\bar{M}^2$ be the double of the resulting spaces.

Case 1. $\bar{N}^2 \neq S^2$. Let $\bar{P}$ be the components of $\bar{M}^2$ which are not spheres. Redefine the X_{ii} to include any disk components of $M^2 - \cup X_{ii}$ which they might bound. With this redefinition $\bar{M}^2$ becomes $\bar{P}^2$. The map g induces $\bar{g} : \bar{P} \rightarrow \bar{N}^2$ and $\bar{g}_i : Y_i \rightarrow \bar{N}^2$ for each component Y_i of $\bar{P}$. Note that $|\deg g| \leq \sum |\deg \bar{g}_i|$. (This would surely be true if we had not redefined the X_{ii} . However, 2.3 indicates we only eliminated components which were mapped with degree 0.) Now

$$(1) \quad \chi(\bar{N}^2) = 2(\chi(N^2) - m) \quad \text{and}$$

$$(2) \quad \chi(\bar{P}) = 2\chi(M^2 - \cup X_{ii}) = 2(\chi(M^2) - \sum [1 - b_1(X_{ii})]).$$

However, 2.4 shows

$$(3) \quad \chi(\bar{P}) = \sum \chi(Y_i) \leq \sum |\deg \bar{g}_i| \chi(\bar{N}^2) \leq |\deg g| \chi(\bar{N}^2) = |\deg f| \chi(\bar{N}^2).$$

Substituting (1) and (2) into (3) we get

$$2(\chi(M^2) - \sum [1 - b_1(X_{ii})]) \leq |\deg f| 2(\chi(N^2) - m)$$

which implies

$$\sum \{D_f(y) : y \in B\} + \sum b_1(X_{ii}) \leq |\deg f| \chi(N^2) - \chi(M^2).$$

This implies the conclusion of the theorem.

Case 2. $\bar{N}^2 = S^2$ in which case $N^2 = S^2$ and there is only one deficient point. In this case redefine the X_{ii} and $\bar{M}^2$ exactly as in case 1. Notice that $\chi(\bar{P}) \leq 0$ and therefore $(\chi(M^2) - \sum \chi(X_{ii})) \leq 0$. There are two possibilities:

(a) Each X_{ii} is a 2-manifold with boundary. Then $\chi(X_{ii}) = 1 - b_1(X_{ii})$. Substituting into the above and adding $|\deg f|$ to both sides we get $\chi(M^2) + |\deg f| - \sum 1 + \sum b_1(X_{ii}) \leq |\deg f|$. That is, $\sum \{D_f(y) : y \in B\} + \sum b_1(X_{ii}) \leq |\deg f| - \chi(M^2)$. Since $N^2 = S^2$ we may deduce the required conclusion.

(b) Some X_{ii} is a 2-manifold without boundary. In this case this X_{ii} is in fact the only X_{ii} and it is M^2 . Note $\sum \{D_f(y) : y \in B\} = |\deg f| - 1$. Hence $\sum \{D_f(y) : y \in B\} + \sum b_1(X_{ii}) = |\deg f| - 1 + 2 - \chi(M^2) \leq |\deg f| \chi(N^2) - \chi(M^2)$. This completes the theorem.

3.2. Remark. It should be noted that somewhat more than necessary was proved in 3.1. That is, the somewhat stronger result $\sum \{D_f(y) : y \in D_f\} +$

$\sum b_i(X_{ii}) \leq |\deg f| \chi(N^2) - \chi(M^2)$ was shown, where the X_{ii} had been modified to exclude disk components in the complement. Note that the X_{ii} are orientable 2-manifolds and that the free R -modules $H_1(\cup X_{ii}; R) \approx H^1(\cup X_{ii}; R)$ have rank independent of R for any nontrivial principal ideal domain. This result will lead to a proof of 1.1.

3.3. Theorem. *Let $f : (M^2, \partial M^2) \rightarrow (N^2, \partial N^2)$ be a continuous map such that*

(1) $\deg f \neq 0$,

(2) M^2 and N^2 are compact, connected, oriented 2-manifolds with boundary.

Let K be the union of the 2-cell components of $M^2 - f^{-1}(D_f)$ and set $L = K \cup f^{-1}(D_f)$. Then

$$\sum \{D_f(y) : y \in D_f\} + \text{rank } H^1(L; Z) \leq |\deg f| \chi(N^2) - \chi(M^2).$$

Proof. Case 1. $\partial M = \partial N = \emptyset$. Construct a monotone sequence $\{X_n\}$ of manifolds with boundary such that

(1) $f^{-1}(D_f) \subset \text{int } X_n \subset [X_{n-1} \cap (f^{-1}(D_f) + 1/n)]$ and

(2) a component U of $M^2 - X_n$ is a disk if and only if U is contained in some disk component of $M^2 - f^{-1}(D_f)$. (Recall that the monotone union of 2-cells is a 2-cell [1].) Let $\tilde{X}_n$ be the manifold obtained from X_n by adjoining the 2-cell components of $(M^2 - X_n)$ to X_n . (This is just the redefinition of the X_{ii} that took place in 3.1.) Note that the $\tilde{X}_n$ are nested. The statement in 3.2 (proved in 3.1) implies that

$$\sum \{D_f(y) : y \in D_f\} + \text{rank } H^1(\tilde{X}_n; Z) \leq |\deg f| \chi(N^2) - \chi(M^2).$$

This inequality holds for all n and the continuity of Alexander cohomology implies the result.

Case 2. M and N arbitrary. Double the domain and range of f inducing a new map $f \& f : M \& M \rightarrow N \& N$. Note that

(1) $\deg f = \deg f \& f$,

(2) $\sum \{D_{f \& f}(y) : y \in N \& N\} = 2 \sum \{D_f(y) : y \in N\}$,

(3) $\text{rank } H^1(L \& L) = 2(\text{rank } H^1(L))$.

From Case 1 we have

$$\sum \{D_{f \& f}(y) : y \in N \& N\} + \text{rank } H^1(L \& L) \leq |\deg f \& f| \chi(N \& N) - \chi(M \& M).$$

Substituting (1)–(3) into the above we get the conclusion of the theorem.

3.4. Proof of 1.1. Assume first that the manifolds are without boundary. Choose manifolds X_{ii} approximating $f^{-1}(D_f)$ of the type referred to in 3.2. Now consider the following diag.

$$\begin{array}{ccc} H_1(f^{-1}(D_f)) & \xrightarrow{i^*} & H_1(M) \\ \searrow & & \nearrow \\ & H_1(\cup X_{ii}) & \end{array} \quad \begin{array}{ccc} H^1(f^{-1}(D_f)) & \xleftarrow{i^*} & H^1(M) \\ \nwarrow & & \swarrow \\ & H^1(\cup X_{ii}) & \end{array}$$

In both cases we see that $\text{rank im } i^*$ is bounded above by $\sum b_1(X_{ii})$. Our observations in 3.2 then imply the result for this case. When M and N are not closed the result can again (see 3.3) be achieved by doubling the spaces.

4. In this section our result will be sharpened in a particularly nice way for the class of quasimonotone maps. First we will familiarize the reader with a proof of Zeimer's [8] in order to establish notation for the proof of 1.2.

4.1. Lemma (Zeimer). *Let $f : M^2 \rightarrow N^2$ be a quasimonotone map between compact connected 2-manifolds. Then there exists an integer d and a point y_0 in N^2 such that*

- (1) $f^{-1}(y_0)$ has exactly d components,
- (2) $f^{-1}(y)$ has d or fewer components for any point y other than y_0 in N^2 .

Sketch of proof. Let $f = h \circ g$ be the monotone light open factorization of f [7, Chapt. 8, §8], and let $g(M^2) = X$ be the factor space. Then X is obtained by performing k , 2 point identifications on a generalized cactoid C_0 for some integer k [4]. The map h induces $\bar{h} : C_0 \rightarrow N^2$ light and open. The space C_0 consists of a finite union of 2-manifolds X_i ($j = 1, \dots, m$) with $X_i \cap X_j$ finite for $j \neq i$ [8, p. 664-5, (3.4) & (3.5)]. Since the h_i are light and open the conclusion of the lemma is immediate.

4.2. Proof of 1.2. Recall the notation of 4.1. By 1.3 we have $\sum \{D(y) : y \in N^2\} = |\deg h_i| \chi(N_2) - \chi(X_i)$ ($j = 1, \dots, m$). Thus for $\bar{h} : C_0 \rightarrow N^2$ we have $\sum \{D_{\bar{h}}(y) : y \in N^2\} = d\chi(N^2) - \chi(C_0)$. For the map $h : X \rightarrow N^2$ then we restate Zeimer's result [8],

$$(i) \quad \sum \{D_h(y) : y \in N^2\} = d\chi(N^2) - \chi(X).$$

Set $B = \{y \in X : H^1(g^{-1}(y), Z) \neq 0\}$.

Claim. The set B is finite.

Proof. Let B_0 be the finite set of cut points of C_0 . Let B_1 be the k points identified in going from C_0 to X . Let $r = b_1(M^2)$. Then if B is infinite we may pick $r + 1$ distinct points x_i ($i = 0, \dots, r$) in B which are not in B_0 or B_1 . About each $g^{-1}(x_i)$ place a connected 2-manifold A_i such that

- (1) the A_i are distinct,
- (2) $H^1(A_i) \neq 0$,
- (3) $M^2 - \cup A_i$ is connected and
- (4) boundary $A_i = S^1$ a circle.

Item (3) follows from the fact that $M^2 - \cup g^{-1}(x_i)$ is connected (g is monotone) [7, p. 138, (2.2)].

To justify item (4) choose an arbitrarily small disk neighborhood U of x_i . Then $g^{-1}(x_i)$ does not separate $g^{-1}(U)$ [7, p. 138, (2.2)]. Choose any manifold Y containing $g^{-1}(x_i)$ and lying inside $g^{-1}(U)$. If $\text{bnd } Y \neq S^1$, then join the components of the boundary by tubes in $g^{-1}(U - x_i)$. Removing the interiors of the tubes from Y will lead to an A_i satisfying (1) through (4) for sufficiently small U .

Excise the A_i and replace them with disks D_i ($i = 1, \dots, r + 1$). Now $\chi(M^2 - \cup A_i) + \chi(\cup A_i) = \chi(M^2) = 2 - r$. Since $\chi(\cup A_i) \leq 0$ we know

$\chi(M^2 - \cup A_i) \geq 2 - r$. Thus $\chi(M^2 - \cup A_i + \cup D_i) = \chi(M^2 - \cup A_i) + \chi(\cup D_i) \geq 3$ since $\chi(\cup D_i) = r + 1$. This of course is impossible as no closed 2-manifold has so large an Euler Characteristic. This completes the claim.

We can now say $\chi(X) - \chi(B) = \chi(X - B)$ (here the Euler Char. is being computed with cohomology groups using the cohomology sequence with compact supports). Now $\chi(X - B) = \chi(M^2 - g^{-1}(B))$. (Vietoris mapping theorem). Thus $\chi(X) - \chi(B) = \chi(M^2) - \chi(g^{-1}(B))$. That is $\chi(X) = \chi(M^2) + \sum \{H^1(g^{-1}(y)) : y \in B\}$. Substituting the above into (i) completes the proof of 1.2.

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