Mollification Does Not Imply Convergence in Area

W. D. PEPE

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1. Introduction. In 1933, Morrey [M1] showed that if $f : I \times I \rightarrow E^n$ for $n \geq 2$ satisfies $f_i \in W_{p_i}^1$ and $p_i^{-1} + p_j^{-1} \leq 1$ for $i \neq j$, then the Lebesgue area of f on $I \times I$ is given by the usual formula where I is the unit interval, $f = (f_1, \dots, f_n)$, and $W_{p_i}^1$ is the Sobolev space of real functions with p_i^{th} —summable partial derivatives (cf. [M2]). In 1970, Goffman, Ziemer [GZ] generalized this result for maps from I^k to E^n under similar hypotheses on the partial derivatives. These conditions on the partial derivatives imply that when the functions are regularized or mollified, the resulting smooth maps have Jacobians which converge in L^1 to the Jacobian of the original function and that the Lebesgue area of the smooth maps converges to the area of the original function. A more general question is: Does mollification of a function always produce a sequence of smooth maps whose Lebesgue area converge to the Lebesgue area of the original function? Both Professors Goffman and Ziemer conjectured that this was false. In this paper, a function, f , will be exhibited for each pair p and q so that $p > 1$, $q > 1$, and $1/p + 1/q > 1$, and whose mollifiers have Lebesgue area tending to infinity and with $f_1 \in W_p^1$ and $f_2 \in W_q^1$ and whose Lebesgue area is zero. Thus, the conditions of Morrey, Goffman, and Ziemer concerning mollification are sharp. However, this does not settle the question of when, in general, the area formula holds, but merely shows that the techniques of Goffman and Ziemer, used to prove the main result of [GZ], cannot be generalized.

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2. The function. Decompose $Q \equiv [0, 2] \times [0, 1]$ into $A_{k,i}$, $k = 1, 2, 3, \dots$, $j = 1, 2, \dots, 2^{k-1}$ and $\{2\} \times [0, 1]$ where:

$$(1) \quad A_{k,i} = \{(x, y) : |x - x_{k,i}| \leq 2^{-k} \text{ and } |y - y_{k,i}| \leq 2^{-k}\};$$

and,

$$(2) \quad x_{k,i} = 2 - 3 \cdot 2^{-k}, \text{ and } y_{k,i} = 2^{-k} + (j - 1) \cdot 2^{-(k+1)}.$$

Define $f_{k,i} : A_{k,i} \rightarrow E^2$ by:

$$(3) \quad f_{k,i}(x, y) = (2^{ka}[1 - 2^k |x - x_{k,i}|] \cdot [1 - 2^k |y - y_{k,i}|], \\ 2^{kb}[1 - 2^k |x - x_{k,i}|]^2 \cdot [1 - 2^k |y - y_{k,i}|]^2).$$

Observe that $f_{k,i} \equiv 0$ on the boundary of $A_{k,i}$ so the following function, f , is well defined on all of E^2 :

$$(4) \quad f(x, y) = \begin{cases} (0, 0), & (x, y) \notin \bigcup_{k=1}^{\infty} \bigcup_{i=1}^{2^{k-1}} A_{k,i}; \\ f_{k,i}(x, y), & (x, y) \in A_{k,i}. \end{cases}$$

Further, if (I) $a < 0$ and $b < 0$, then f is continuous on E^2 as

$$\lim_{k \rightarrow \infty} \sup_{(x,y) \in A_{k,i}} |f_{k,i}(x, y)| = 0.$$

It is also easy to see that the Lebesgue area of f on Q , $\mathcal{L}(f, Q)$, is zero, for the Lebesgue measure of $f(Q)$ is zero.

3. A computational convenience. Consider the transformations:

$$(5) \quad \begin{aligned} T_1(x, y) &= (x, y); \\ T_2(x, y) &= (-x, y); \\ T_3(x, y) &= (x, -y); \text{ and,} \\ T_4(x, y) &= (-x, -y). \end{aligned}$$

Then, with $B_k = [0, 1/2^k] \times [0, 1/2^k]$, $k = 1, 2, 3, \dots$, $A_{k,i}$ is the union of the four non-overlapping cubes $h_{i,i,k}(B_k)$, $i = 1, 2, 3, 4$, where,

$$(6) \quad h_{i,i,k}(x, y) = T_i(x, y) - T_i\left(\frac{1}{2^k}, \frac{1}{2^k}\right) + (x_{k,i}, y_{k,i}).$$

With $g_k : B_k \rightarrow E^2$ given by:

$$(7) \quad g_k(x, y) = (2^{k(2+a)}xy, 2^{k(4+b)}x^2y^2);$$

then

$$(8) \quad f|_{h_{i,i,k}(B_k)} = g_k \circ h_{i,i,k}^{-1}.$$

As $h_{i,i,k}$ is an isometry leaving the axes parallel, and using the symmetry of g_k ;

$$(9) \quad \left| \frac{\partial f_i}{\partial x} \right| = \left| \frac{\partial g_{k,i}}{\partial x} \right| = \left| \frac{\partial g_{k,i}}{\partial y} \right| = \left| \frac{\partial f_i}{\partial y} \right|,$$

for all k , and $i = 1, 2$ where $g_k = (g_{k,1}, g_{k,2})$.

4. The Sobolev class of f_i . Using (1), (3), (4), (6), (8), and (9):

$$(10) \quad \begin{aligned} \iint_Q \left| \frac{\partial f_i}{\partial x} \right|^p dx dy &= \sum_{k=1}^{\infty} \sum_{i=1}^{2^{k-1}} \iint_{A_{k,i}} \left| \frac{\partial f_i}{\partial x} \right|^p dx dy \\ &= \sum_{k=1}^{\infty} 2^{k-1} \cdot 4 \cdot \iint_{B_k} \left| \frac{\partial g_{k,i}}{\partial x} \right| dx dy = \iint_Q \left| \frac{\partial f_i}{\partial y} \right|^p dx dy. \end{aligned}$$

As $\partial g_{k,1}/\partial x = 2^{k(2+a)}y$, using (10):

$$\begin{aligned}
 \iint_Q \left| \frac{\partial f_1}{\partial x} \right|^p dx dy &= \iint_Q \left| \frac{\partial f_1}{\partial y} \right|^p dx dy \\
 &= \sum_{k=1}^{\infty} 2^{k+1} \int_0^{2^{-k}} \int_0^{2^{-k}} 2^{pk(2+a)} y^p dx dy \\
 &= \frac{2}{p+1} \sum_{k=1}^{\infty} 2^{k(pa+p-1)}.
 \end{aligned}
 \tag{11}$$

In a similar fashion:

$$\iint_Q \left| \frac{\partial f_2}{\partial x} \right|^q dx dy = \iint_Q \left| \frac{\partial f_2}{\partial y} \right|^q dx dy = \frac{2^{1+q}}{(1+q)(1+2q)} \sum_{k=1}^{\infty} 2^{k(q+bq-1)}.
 \tag{12}$$

So, from (11) and (12), $f_1 \in W_p^1$ and $f_2 \in W_q^1$ provided:

$$\text{(II) } p < \frac{1}{a+1}, \quad a+1 > 0, \quad q < \frac{1}{p+1}, \quad \text{and } b+1 > 0.$$

5. Mollification. Regularizing or mollifying a function is accomplished by selecting a non-negative function, c , of class C^∞ with support in $I \times I$ so that $\int \int_{I \times I} c(x, y) dx dy = 1$ and forming $c_\epsilon(x, y) = \epsilon^{-2} c(x/\epsilon, y/\epsilon)$ then the mollified function, $f * c_\epsilon$, given by:

$$c_\epsilon * f(x, y) = \int_{-\epsilon}^{\epsilon} \int_{-\epsilon}^{\epsilon} c_\epsilon(s, t) f(x+s, y+t) ds dt;
 \tag{13}$$

is a C^∞ function.

We will assume that c is invariant under T_i as given in (5) and that

$$\int_{-1}^1 \int_{-1}^1 t^2 c(s, t) ds dt = \int_{-1}^1 \int_{-1}^1 s^2 c(s, t) ds dt \equiv M.
 \tag{14}$$

Then mollifying f in $h_{i,j,k}(B_k)$ for $\epsilon < 1/2^{k+2}$ is the same as mollifying g_k and composing with $h_{i,j,k}^{-1}$ for all (x, y) , at least ϵ away from the complement of $h_{i,j,k}(B_k)$. Thus, from (8), and because $h_{i,j,k}$ is an isometry:

$$\mathfrak{L}(c_\epsilon * f, A_{i,k}) \geq 4\mathfrak{L}(c_\epsilon * g_k, B_{k,\epsilon}),
 \tag{15}$$

where

$$B_{k,\epsilon} = [\epsilon, 2^{-k} - \epsilon] \times [\epsilon, 2^{-k} - \epsilon].
 \tag{16}$$

Mollifying g_k in $B_{k,\epsilon}$ gives:

$$c_\epsilon * g_{k,1}(x, y) = \iint 2^{k(2+a)}(x+s)(y+t)c_\epsilon(s, t) ds dt = 2^{k(2+a)}xy; \quad \text{and}
 \tag{17}$$

$$\begin{aligned}
 c_\epsilon * g_{k,2}(x, y) &= \iint 2^{k(4+a)}(x+s)^2(y+t)^2 c_\epsilon(s, t) ds dt \\
 &= 2^{k(4+b)}[x^2y^2 + Mx^2\epsilon^2 + My^2\epsilon^2 + N\epsilon^4];
 \end{aligned}
 \tag{18}$$

where $N = \int_{-1}^1 \int_{-1}^1 s^2 t^2 c(s, t) ds dt$. Computing the Jacobian of $c_\epsilon * g_k$ gives:

$$Jc_\epsilon * g_k(x, y) = 2^{k(6+a+b)}\epsilon^2 \cdot 2M \cdot (y^2 - x^2).
 \tag{19}$$

Finally, as $c_\epsilon * g_k$ is smooth:

$$(20) \quad \begin{aligned} \mathfrak{L}(c_\epsilon * g_k, B_{k,\epsilon}) &= \iint_{B_{k,\epsilon}} |Jc_\epsilon * g_k(x, y)| \, dx \, dy \\ &= 2M\epsilon^2 2^{k(6+a+b)} \left\{ \frac{1}{2} \left(\frac{1}{2^k} - \epsilon \right)^3 \left(\frac{1}{3} \frac{1}{2^k} - \epsilon \right) + \epsilon^3 \left(\frac{1}{3 \cdot 2^k} - \frac{\epsilon}{2} \right) \right\}. \end{aligned}$$

As $\epsilon \leq 1/2^{k+2}$, all terms in (20) are positive. Letting $\epsilon = 1/2^{n+2}$, we have from (1), (15), and the ϵ^3 term of (20):

$$(21) \quad \begin{aligned} \mathfrak{L}(c_\epsilon * f, Q) &\geq \sum_{k=1}^n \sum_{j=1}^{2^{k-1}} \mathfrak{L}(c_\epsilon * f, A_{k,j}) \geq \sum_{k=1}^n 2^{k-1} \cdot 4\mathfrak{L}(c_\epsilon * g_k, B_k) \\ &\geq 4M \sum_{k=1}^n 2^k \frac{2^{k(6+a+b)}}{2^{5n+10}} \left(\frac{1}{3 \cdot 2^k} - \frac{1}{2^{n+3}} \right) \\ &= \frac{M}{2^8} \left\{ \frac{2^{6+a+b}}{3} \cdot \frac{2^{n(1+a+b)}}{2^{6+a+b} - 1} - \frac{2^{-5n}}{8} - \frac{2^{7+a+b}}{8} \frac{2^{n(1+a+b)}}{2^{7+a+b} - 1} - \frac{2^{-7n}}{8} \right\} \\ &= \frac{M}{2^{2-a-b}} \left\{ \frac{1}{3} \cdot \frac{1}{2^{6+a+b} - 1} - \frac{1}{4} \frac{1}{2^{7+a+b} - 1} \right\} 2^{n(1+a+b)} + c_n 2^{-5n}; \end{aligned}$$

where c_n is a bounded sequence. Thus provided (III) $1 + a + b > 0$, the sequence diverges.

6. The conditions on a and b . We have the following conditions on a and b :

(I) $a < 0$ and $b < 0$;

(II) $1 + a > 0$, $p < (1 + a)^{-1}$, $(1 + b) > 0$, and $q < (1 + b)^{-1}$;

and,

(III) $1 + a + b > 0$.

Given any p and $q > 1$ so that $1/p + 1/q > 1$, with c so that

$$(22) \quad 1 < c < 1/p + 1/q, \text{ define:}$$

$$(23) \quad a = \frac{1}{cp} - 1 \quad \text{and} \quad b = \frac{1}{cq} - 1.$$

As c , p , and q are greater than one, (I) holds. If (III) holds, $1 + a > 0$ and $1 + b > 0$. As $1 + a + b = 1 + (1/c)(1/p + 1/q) - 2$, and from (22), $1 < (1/c)(1/p + 1/q)$, so (III) holds. To verify the remainder of (II),

$$(24) \quad p < cp = \frac{1}{1-a} \quad \text{and} \quad q < qc = \frac{1}{1-b}.$$

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University of Kentucky

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