An Extension of the Frobenius Group Commutator Theorem

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$\mathfrak{B}(\mathfrak{H})$ denotes the space of bounded linear operators on a complex Hilbert space $\mathfrak{H}$ and $\mathbf{G}(\mathfrak{H})$ is the group of invertibles in $\mathfrak{B}(\mathfrak{H})$. For $T \in \mathfrak{B}(\mathfrak{H})$ the numerical range of T is given by

$$W(T) = \{(Tx, x) \mid (x, x) = 1\},$$

where $(\cdot, \cdot)$ is the inner product in $\mathfrak{H}$. If $A, B \in \mathbf{G}(\mathfrak{H})$ the multiplicative (or group) commutator $T \in \mathbf{G}(\mathfrak{H})$ is

$$(1) \quad T = ABA^{-1}B^{-1}.$$

Observe that $T = I$ if and only if A commutes with B , that is if and only if $[A, B] = AB - BA = 0$.

Several authors (see [5], [6], [8]) have studied the multiplicative commutator T under the additional hypothesis that

$$(2) \quad [T, A] = TA - AT = 0.$$

The classical result along these lines is the Frobenius theorem [3] which may be stated as

Proposition A. *Let $\dim \mathfrak{H}$ be finite. If (1) and (2) hold with A and B unitary, then $0 \in W(A)$ or $[A, B] = 0$.*

In [6] Putnam gives the following generalization in which the finiteness hypothesis on $\dim \mathfrak{H}$ is removed. We restate it as

Proposition B. *Let (1) and (2) hold with A unitary. Then $0 \in \overline{W(B)}$ or $\sigma(T) = \{1\}$.*

Observe that this proposition implies the Frobenius result even when $\dim \mathfrak{H}$ is not finite since the only unitary T for which $\sigma(T) = \{1\}$ is the identity.

Putnam's proposition is strengthened in [1] where the conclusion $\sigma(T) = \{1\}$ is replaced by $T = I$. This result is what we call

Proposition B'. *Let (1) and (2) hold with A unitary. Then $0 \notin \overline{W(B)}$ or $[A, B] = 0$.*

In a different direction in an attempt to weaken the condition A unitary there appears in [5] the following

Proposition C. *Let $\dim \mathfrak{H}$ be finite. If (1) and (2) hold and if both A and T are normal, then $0 \notin W(B)$ or $[A, B] = 0$.*

The purpose of this paper is to prove a theorem which contains all of these results, namely

Theorem 1. *Let (1) and (2) hold. If A is normal, then $0 \notin \overline{W(B)}$ or $[A, B] = 0$.*

We also find that the verbatim extension of Proposition C to infinite dimensions is valid.

Theorem 2. *Let (1) and (2) hold. If A and T are normal, then $0 \notin W(B)$ or $[A, B] = 0$.*

This is rather surprising for one would expect that in the infinite dimensional case the replacement $0 \notin \overline{W(B)}$ for $0 \notin W(B)$ would be necessary.

§1. The main results. For $T \in \mathcal{B}(\mathfrak{H})$, $\sigma(T)$ will designate the spectrum of T . It is well known that $W(T)$ is convex and that if $\dim \mathfrak{H}$ is finite $W(T) = \overline{W(T)}$. Critical to what follows is the observation which we catalog as

Property P. *If $P = P^* = P^2$, $P \neq 0$, and PTP is viewed as an operator on $P\mathfrak{H}$, then $W(PTP) \subset W(T)$.*

Other useful properties of the numerical range are described in detail in [4].

Theorem 1, the central result of this paper, will be presented as a consequence of Theorem 3, which we are now prepared to prove. As will be seen, the statement of Theorem 3 displays each hypothesis individually while the equivalent Theorem 1 tends to conceal what is assumed in the compact commutator expression.

Theorem 3. *If $S, N, B \in \mathcal{B}(\mathfrak{H})$ and satisfy*

i) N normal,

ii) $SN = NS$,

iii) $SB = BN$,

then $0 \notin \overline{W(B)}$ or $S = N$.

Proof. From iii) $(S - N)B = BN - NB = [B, N]$. Therefore $\|(S - N)B\| \leq 2 \|B\| \inf_{\lambda \in \mathbf{C}} \|N - \lambda I\|$ and if B is invertible $\|S - N\| \leq \|B\| \|B^{-1}\| \text{diam}(\sigma(N))$ where we have used the fact that N is normal. From i) and ii) it follows that S belongs to the commutant of the von Neumann algebra generated by N . Hence if E is the spectral measure of N and M is any Borel set of $\mathbf{C}$, $SE(M) = E(M)S$ so that $E(M)$ reduces S . Let S_M and N_M denote the restrictions of S and N to $E(M)\mathfrak{H}$ and B_M the compression of B to $E(M)\mathfrak{H}$

$(B_M = E(M)B|_{E(M)\mathfrak{S}})$. As elements of $\mathfrak{B}(E(M)\mathfrak{S})$ S_M , N_M , B_M also satisfy conditions i), ii), iii). Now assume $0 \notin \overline{W(B)}$. Then B is invertible and $\|B^{-1}\| \leq \delta^{-1}$, where δ is the distance from the origin to $\overline{W(B)}$. Moreover, since B_M is the compression of B to $E(M)\mathfrak{S}$ Property P implies $W(B_M) \subset W(B)$ if $E(M) \neq 0$. Accordingly $0 \notin \overline{W(B)}$ and $E(M) \neq 0$ imply B_M is invertible in $\mathfrak{B}(E(M)\mathfrak{S})$ and $\|B_M^{-1}\| \leq \delta^{-1}$. Furthermore,

$$(3) \quad \|S_M - N_M\| \leq \|B\| \delta^{-1} \text{diam}(M).$$

Finally, for each $\epsilon > 0$, we can find a finite family of disjoint Borel subsets $\{M_i\}_{i=1}^r$ of $\mathbf{C}$ such that $\text{diam}(M_i) < \epsilon$, $E(M_i) \neq 0$, $i = 1, \dots, r$, and $\sigma(N) \subset \bigcup_{i=1}^r M_i$. We see from (3) and from the construction of the M_i (in particular, that $\sum_{i=1}^r E(M_i) = I$ and that $E(M_i)\mathfrak{S}$ reduces both N and S , $i = 1, \dots, r$) that

$$\|(S - N)x\|^2 = \sum_{i=1}^r \|(S_{M_i} - N_{M_i})E(M_i)x\|^2 \leq (\|B\| \delta^{-1}\epsilon)^2 \|x\|^2.$$

Hence $\|S - N\| \leq \|B\| \delta^{-1}\epsilon$ and therefore $S = N$.

As claimed our main result easily follows from what has been shown above.

Theorem 1. *Let (1) and (2) hold. If A is normal, then $0 \notin \overline{W(B)}$ or $[A, B] = 0$.*

Proof. $[T, A] = 0$ implies $AT^{-1} = T^{-1}A = BAB^{-1}$. Set $S = AT^{-1}$; then $SA = AS$ and $SB = BA$. Hence the conditions of Theorem 2 are satisfied by S, A, B . Then $0 \notin \overline{W(B)}$ implies $S = T^{-1}A = A$ so $T = I$ and $[A, B] = 0$.

If *a priori* knowledge that S is normal is available, Theorem 3 may be improved somewhat. This improvement will enable us to obtain the literal extension of the Marcus and Thompson Proposition C expressed in Theorem 2. We note that Theorem 4 is contained in [10].

Theorem 4. *If $S, N, B \in \mathfrak{B}(\mathfrak{S})$ and satisfy i), ii), iii) of Theorem 3, and moreover if S is normal, then $0 \notin W(B)$ or $S = N$.*

Proof. Since both N and S are normal and commute, then as above $E(M_1)F(M_2) = F(M_2)E(M_1)$ for all Borel subsets M_1, M_2 . Here E and F are the spectral measures of N and S , respectively. Moreover, in view of the Fuglede-Putnam Theorem, it is not hard to see that $F(M)B = BE(M)$ for all Borel sets M . Because E and F commute, either $E = F$ or there exists a Borel set M_0 and $x_0 \neq 0$ such that either $E(M_0)x_0 = x_0$ and $F(M_0)x_0 = 0$ or $E(M_0)x_0 = 0$ and $F(M_0)x_0 = x_0$. For either eventuality

$$0 = (BE(M_0)x_0, x_0) = (F(M_0)Bx_0, x_0) = (Bx_0, x_0).$$

Consequently if $0 \notin W(B)$, then $E = F$ and $N = S$.

Applying precisely those substitutions made in the proof of Theorem 1 we obtain

Theorem 2. *Let (1) and (2) hold. If A and T are normal, then $0 \notin W(B)$ or $[A, B] = 0$.*

It is natural to ask whether the condition $0 \notin W(B)$ in these theorems can be relaxed to $0 \notin \text{co } \sigma(B)$. The finite dimensional example below shows such a weakening invalid. For the example put

$$N = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 & -1 & 0 \\ 3 & 0 & 2 \\ 4 & 0 & 3 \end{bmatrix} \quad \text{and} \quad S = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

In this case $\sigma(B) = \{1\}$, the hypotheses of Theorems 3 or 4 are satisfied, but $N \neq S$.

By a theorem of Williams [9] for any neighborhood V of $\text{co } \sigma(B)$ there exists a similarity R_V such that $W(R_V B R_V^{-1}) \subset V$. Thus for the example above there exists a similarity R such that $0 \notin W(R B R^{-1})$. Then putting $N' = R N R^{-1}$, $B' = R B R^{-1}$ and $S' = R S R^{-1}$, we observe that N' and S' are diagonal, that ii) and iii) of Theorem 3 hold for S' , N' , B' , and that $0 \notin W(B')$. That $N' \neq S'$ shows that the normality condition i) of Theorem 3 can not be replaced by the condition N diagonal.

§2. Remarks. 1. Theorem III of [2] appears to be an improvement of Theorem 1. This theorem, under the same hypotheses as Theorem 1, makes the stronger conclusion $0 \notin \text{co } \sigma(B)$ or $[A, B] = 0$. However, the example at the end of the last section shows this conclusion to be erroneous. In reality what appears to be shown in [2] is that under the hypotheses of Theorem 1 and under the *additional hypothesis* that A commute with $B^n A B^{-n}$, for all $n > 1$, then $0 \notin \text{co } \sigma(B)$ implies $[A, B] = 0$.

2. Theorem 1 may be framed as a theorem for W^* -algebras. If $\mathfrak{A}$ is a W^* -algebra with identity, then we speak of the numerical range of $A \in \mathfrak{A}$ as $V(A) = \{f(A) \mid f \in \mathfrak{A}^*, f(I) = 1 = \|f\|\}$. For $\mathfrak{A} = \mathcal{B}(\mathfrak{H})$ and $T \in \mathfrak{A}$, it is well known that $V(T) = \overline{W(T)}$. Further, all the techniques used in the proof of Theorem 1 carry over to this more general setting yielding a proof of

Theorem 1'. *Let $\mathfrak{A}$ be a W^* -algebra with unit. If $A, B, C \in \mathfrak{A}$ such that $C = A B A^{-1} B^{-1}$, $[A, C] = 0$ and $[A, A^*] = 0$, then $0 \in V(B)$ or $[A, B] = 0$.*

3. We observe that our Theorem 1 may be viewed as a multiplicative analog for the linear commutator theorem of Putnam [7],

Theorem D. *Suppose $[A, [A, B]] = 0$ and that A is normal. Then $[A, B] = 0$.*

If we define $\{A, B\} = A B A^{-1} B^{-1}$, then Theorem 1 has the formulation

Theorem 1. *(alternate statement). Suppose $\{A, \{A, B\}\} = I$, A is normal, and $0 \notin W(B)$. Then $\{A, B\} = I$.*

4. It is also of some interest to note that the proof of Theorem 3 may be simplified when $\mathfrak{H}$ is finite dimensional. Let $\{\lambda_1, \dots, \lambda_k\} = \sigma(N)$ so that $N = \sum_{i=1}^k \lambda_i P_i$: P_i is the orthogonal projection on $\ker(N - \lambda_i I)$. Then ii) implies $S P_i = P_i S$. Therefore $S P_i B P_i = P_i S B P_i = P_i B N P_i = \lambda_i P_i B P_i$. If $0 \notin W(B)$, then $0 \notin W(B_i)$ where $B_i = P_i B|_{P_i \mathfrak{H}}$ so that B_i is invertible. Hence $S P_i = \lambda_i P_i$ and $S = \sum_{i=1}^k \lambda_i P_i = N$.

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