A Maximum Principle and Gradient Bounds for Linear Elliptic Equations

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1. Introduction. It is well known that if u is a harmonic function, then $|\text{grad } u|^2$ is subharmonic and therefore satisfies the maximum principle. This reasoning can be extended to solutions of second order elliptic equations with constant coefficients, but not to equations with variable coefficients.

The purpose of this paper is to obtain a simple extension to equations with variable coefficients of the above property of Laplace's equation.

Let u be a solution of the uniformly elliptic equation

$$(1.1) \quad Lu - cu \equiv a_{ij}u_{,ij} + b_i u_{,i} - cu = 0, \quad c \geq 0$$

in which $a_{ij}(x)$, $b_i(x)$, $c(x)$ are C^1 functions of $x = (x_1, x_2, \dots, x_N)$ in a domain D in $\mathbf{R}^N$. Here and throughout we use the convention that a repeated subscript indicates summation from 1 to N with respect to that index. Also, we write $u_{,i}$ for $\partial u / \partial x_i$, $u_{,ij}$ for $\partial^2 u / \partial x_i \partial x_j$, and so forth.

In Section 2, we show that there is a constant γ_0 which depends only on the coefficients in (1.1) and their derivatives such that for $\gamma \geq \gamma_0$ the quantity

$$|\text{grad } u|^2 + \gamma u^2$$

satisfies the maximum principle in D .

In Section 3, we show that if there is a nonnegative function ψ which satisfies the inequality $L\psi - c\psi \leq -1$ in D , then a similar and somewhat refined gradient estimate can be obtained for solutions of the inhomogeneous equation $Lu - cu = f$. This refined estimate is especially useful near boundary points where ψ vanishes. We also show how to obtain a modified result when c may be negative.

If the values of u on the boundary ∂D of D are prescribed C^2 functions and if ∂D is sufficiently smooth, one can bound the boundary values of $|\text{grad } u|$ by means of barrier functions. See, *e.g.*, [3, 8]. Our maximum principle can then be used to bound $|\text{grad } u|$ in D .

Bounds for the gradients of solutions of classes of quasilinear equations were obtained in two dimensions by S. Bernstein [1], and in more dimensions by O. Ladyzhenskaya and N. Uraltseva [5, 6] and by J. Serrin [9, 10].

Since these bounds are sufficiently general to work for large classes of non-linear equations, they are of necessity complicated, even for linear equations. The bounds obtained in the linear case with the results of the present paper are simpler.

In Section 4 we extend the results of Section 2 to show that for any fixed function $\alpha(x)$ there is a constant γ_0 such that

$$(1.2) \quad \alpha^2 |\text{grad } u|^2 + \gamma u^2$$

satisfies a maximum principle for $\gamma \geq \gamma_0$. If α is chosen so that it vanishes on ∂D , then we obtain interior estimates for $|\text{grad } u|$ in terms of the boundary values of u and quantities which depend only on the domain (expressed in terms of α and γ). Under certain conditions the quantity γ_0 may be taken as zero in which case we obtain a generalized maximum principle for $\alpha |\text{grad } u|$.

In Section 5 we indicate how the methods of Sections 2 and 3 can be extended to yield bounds for higher derivatives, provided the coefficients of the equation are sufficiently smooth.

The methods given here for elliptic operators are easily modified for second order parabolic operators. In this connection, similar bounds for the derivatives of solutions of parabolic equations have been given by Il'in, Kalashnikov and Oleinik [4] who greatly extended an idea of Bernstein [2]. Since these estimates depend on multiplication of the solution by a suitable exponential in the time variable, they cannot be transferred directly to estimates for solutions of elliptic equations.

In Section 6, we consider elliptic systems of the form

$$(1.3) \quad a_{ij}u_{,i}^\mu + b_i^{\mu\nu}u_{,i}^\nu - c^{\mu\nu}u^\nu = 0, \quad \mu = 1, 2, \dots, M.$$

Since the coupling occurs only in the first-derivative and zero-th derivative terms, we call such a system *semi-coupled*. A maximum principle for the quantity $\sum_{k=1}^M (u^\mu)^2$ for such systems was proved by Miranda [7] provided the coefficients satisfy a certain inequality. Here we show that for solutions of (1.3), the quantity

$$(1.4) \quad u_{,i}^\mu u_{,i}^\mu + \gamma u^\mu u^\mu$$

satisfies a maximum principle if the condition of Miranda is strengthened slightly and γ is sufficiently large. (Repeated superscripts are summed between 1 and M while repeated subscripts are summed between 1 and N .)

2. Second order elliptic equation. We consider the differential equation

$$(2.1) \quad a_{ij}u_{,i} + b_i u_{,i} - cu = 0$$

in a domain D in $\mathbf{R}^N$. We suppose that (a_{ij}) is a uniformly positive definite matrix and that all coefficients are C^1 functions of $x_1, x_2, \dots, x_N$.

We shall say that a function $w(x)$ satisfies the maximum principle in D if either w is constant or w does not attain a maximum value at any interior point of D .

Theorem 1. *Let u be a C^3 solution in D of (2.1). Suppose that a_{ij} , b_i , and c are all C^1 functions in D and that $c = c(x) \geq 0$. Then there is a positive constant γ whose size depends only on the coefficients in (2.1) such that the function*

$$|\text{grad } u|^2 + \gamma u^2$$

satisfies the maximum principle in D .

Proof. We define the function

$$w = |\text{grad } u|^2 + \gamma u^2.$$

A computation yields

$$\begin{aligned} w_{,i} &= 2u_{,k}u_{,ik} + 2\gamma uu_{,i}, \\ w_{,ij} &= 2u_{,k}u_{,ijk} + 2u_{,ik}u_{,jk} + 2\gamma uu_{,ij} + 2\gamma u_{,i}u_{,j}. \end{aligned}$$

Therefore

$$\begin{aligned} (2.2) \quad a_{ij}w_{,ij} + b_iw_{,i} &= 2a_{ij}u_{,k}u_{,ijk} + 2a_{ij}u_{,ik}u_{,jk} + 2\gamma a_{ij}uu_{,ij} \\ &\quad + 2\gamma a_{ij}u_{,i}u_{,j} + 2b_iu_{,k}u_{,ik} + 2\gamma b_iuu_{,i}. \end{aligned}$$

From (2.1) we find

$$(2.3) \quad a_{ij}u_{,ijk} + a_{ij,k}u_{,ij} + b_iu_{,ik} + b_{i,k}u_{,i} - cu_{,k} - c_ku = 0.$$

Multiplying (2.3) by $u_{,k}$ and substituting the result and (2.1) in (2.2), we obtain

$$\begin{aligned} (2.4) \quad a_{ij}w_{,ij} + (b_i - \gamma^{-1}c_{,i})w_{,i} &= 2a_{ij}u_{,ik}u_{,ijk} - 2a_{ij,k}u_{,ik}u_{,ij} - 2b_{i,k}u_{,k}u_{,i} \\ &\quad - 2\gamma^{-1}c_{,i}u_{,k}u_{,ik} + 2\gamma a_{ij}u_{,i}u_{,j} \\ &\quad + 2cu_{,k}u_{,k} + 2\gamma cu^2. \end{aligned}$$

Denoting by (A_{ij}) the matrix which is the inverse of the positive definite matrix (a_{ij}) , we see that if (s_{pk}) is an arbitrary $N \times N$ matrix, then

$$a_{ij}(u_{,ik} + \frac{1}{2}A_{ip}s_{pk})(u_{,jk} + \frac{1}{2}A_{jq}s_{qk}) \geq 0.$$

Therefore the inequality

$$(2.5) \quad s_{jk}u_{,jk} \geq -a_{ij}u_{,ik}u_{,jk} - \frac{1}{4}A_{pq}s_{pk}s_{qk},$$

is valid for an arbitrary matrix (s_{jk}) . We choose $s_{ij} = -a_{ij,k}u_{,k} - \gamma^{-1}c_{,i}u_{,j}$ in (2.5) and substitute the result in (2.4) to obtain

$$\begin{aligned} (2.6) \quad a_{ij}w_{,ij} + (b_i - \gamma^{-1}c_{,i})w_{,i} &\geq 2\gamma a_{ij}u_{,i}u_{,j} \\ &\quad - \frac{1}{2}A_{pq}(a_{pk,l}u_{,l} + \gamma^{-1}c_{,p}u_{,k})(a_{qk,m}u_{,m} + \gamma^{-1}c_{,q}u_{,k}) \\ &\quad - 2b_{i,j}u_{,i}u_{,j} + 2cu_{,k}u_{,k} + 2\gamma cu^2. \end{aligned}$$

Since (a_{ij}) is uniformly positive definite, the right side of (2.6) is positive definite for a sufficiently large value of γ . This value of γ depends only on the coefficients in (2.1) and their first derivatives. Thus w satisfies an elliptic differential inequality, and its maximum cannot occur at an interior point of D unless w is identically constant.

As a consequence of Theorem 1, we see that if

$$(2.7) \quad |\text{grad } u| \leq M, |u| \leq m \quad \text{on } \partial D,$$

then

$$|\text{grad } u|^2 + \gamma u^2 \leq M^2 + \gamma m^2 \quad \text{in } D.$$

Furthermore, if all the coefficients of (2.1) are constant, then we may choose $\gamma = 0$ and Theorem 1 states that $|\text{grad } u|$ satisfies the maximum principle. If only c is constant, then $|\text{grad } u|$ still satisfies the maximum principle provided that the right side of (2.6), with the terms involving γ deleted, is a positive semi-definite quadratic form in $\text{grad } u$. In this case we need to assume that (2.1) is elliptic in D but not that it is uniformly elliptic. Moreover, c may be a negative constant.

If c is not constant but is never zero, we can divide the equation (2.1) by $|c|$ and apply the preceding analysis. With the appropriate choice for the matrix (s_{ij}) in (2.5) we get the following result.

Theorem 2. *Let u be a C^3 solution of the (not necessarily uniformly) elliptic equation (2.1). If $c(x)$ never vanishes and if the matrix*

$$2c\delta_{ij} - b_{i,i} - b_{j,i} + c^{-1}(b_{ic,i} + b_{jc,i}) \\ - \frac{1}{2}A_{mn}(a_{mk,i} - c^{-1}a_{mkc,i})(a_{nk,i} - c^{-1}a_{nkc,i})$$

is positive semi-definite in D , then $|\text{grad } u|$ satisfies the maximum principle in D . If $c \equiv 0$, then the same result is true if the matrix

$$-b_{i,i} - b_{j,i} - \frac{1}{2}A_{mn}a_{mk,i}a_{nk,i}$$

is positive semi-definite in D .

3. Inhomogeneous equations. The gradient estimate given in Theorem 1 can be adjusted to yield estimates for the first derivatives of solutions of inhomogeneous second order elliptic equations. Let u be a solution of the uniformly elliptic equation

$$(3.1) \quad a_{ij}u_{,ij} + b_i u_{,i} - cu = f,$$

where the coefficients and f are C^1 functions and $c \geq 0$ throughout D . Let $\psi = \psi(x)$ be any nonnegative function which satisfies in D the differential inequality

$$(3.2) \quad a_{ij}\psi_{,ij} + b_i\psi_{,i} - c\psi \leq -1.$$

Theorem 3. Let u be a C^3 solution in D of the uniformly elliptic equation (3.1) with $c \geq 0$, and suppose that $\psi \geq 0$ satisfies the differential inequality (3.2) in D . Then there is a positive constant γ depending only on the coefficients in (3.1), on ψ , and on their first derivatives such that the function

$$(3.3) \quad w = |\text{grad } u|^2 + \gamma u^2 - (u^2 + \beta)\psi$$

with

$$\beta = \max_{x \in D} [|\text{grad } f|^2 + 3\gamma^2 f^2]$$

satisfies the maximum principle in D .

Proof. Proceeding as in the proof of Theorem 1, we have

$$\begin{aligned} a_{ij}w_{,ij} + b_iw_{,i} - cw &= 2a_{ij}u_{,ik}u_{,jk} - 2a_{ij,k}u_{,k}u_{,ij} - 2b_{i,j}u_{,i}u_{,j} \\ &+ 2c_{,k}uu_{,k} + 2f_{,k}u_{,k} + 2(\gamma - \psi)a_{ij}u_{,i}u_{,j} + 2(\gamma - \psi)fu \\ &- (u^2 + \beta)(a_{ij}\psi_{,ij} + b_i\psi_{,i} - c\psi) + cu_{,k}u_{,k} + (\gamma - \psi)cu^2 - 4a_{ij}uu_{,i}\psi_{,j}. \end{aligned}$$

We employ inequality (2.5) as well as the elementary inequalities

$$\begin{aligned} 2uc_{,k}u_{,k} &\geq -\frac{1}{3}u^2 - 3(c_{,k}u_{,k})^2, \\ 2f_{,k}u_{,k} &\geq -|\text{grad } f|^2 - |\text{grad } u|^2, \\ 2(\gamma - \psi)fu &\geq -\frac{1}{3}u^2 - 3(\gamma - \psi)^2f^2, \\ -4a_{ij}uu_{,i}\psi_{,j} &\geq -\frac{1}{3}u^2 - 12(a_{ij}u_{,i}\psi_{,j})^2. \end{aligned}$$

Taking (3.2) into account we find

$$\begin{aligned} (3.4) \quad a_{ij}w_{,ij} + b_iw_{,i} - cw &\geq [2(\gamma - \psi)a_{ii} - \frac{1}{2}A_{mn}a_{mk,i}a_{nk,i} - 2b_{i,j} + (c - 1)\delta_{ii} \\ &- 3c_{,i}c_{,i} - 12a_{im}a_{in}\psi_{,m}\psi_{,n}]u_{,i}u_{,i} + (\gamma - \psi)cu^2 \\ &+ [\beta - |\text{grad } f|^2 - 3(\gamma - \psi)^2f^2]. \end{aligned}$$

We choose γ so large that the above quadratic form in $u_{,i}$ is nonnegative definite, and we also increase γ , if necessary, so that $\gamma \geq \psi$ throughout D . We then select β so that

$$\beta = \max_{x \in D} [|\text{grad } f|^2 + 3\gamma^2 f^2].$$

Then $a_{ij}w_{,ij} + b_iw_{,i} - cw \geq 0$ in D , and hence w satisfies the maximum principle.

Remarks. (i) If D is contained in a slab, say $0 \leq x_1 \leq \xi$, then it is easy to construct a nonnegative function ψ which satisfies (3.2). We set

$$\psi = A(1 - e^{-\alpha x_1})$$

and find that

$$(3.5) \quad a_{ij}\psi_{,ij} + b_i\psi_{,i} - c\psi = A[(-a_{11}\alpha^2 + b_1\alpha)e^{-\alpha x_1} - c(1 - e^{-\alpha x_1})].$$

Since the equation (3.1) is uniformly elliptic, a_{11} has a positive lower bound. Also, we have $c \geq 0$. Therefore we may choose α and then A so that the right side of (3.5) is always less than -1 .

(ii) If c has a positive lower bound in D , we may simply choose $\psi = \max_{x \in D} [1/c]$.

(iii) The hypothesis $c \geq 0$ in Theorem 3 may be removed if we can find a function φ which has a positive lower bound in D and which satisfies in D the differential inequality

$$(3.6) \quad a_{ii}\varphi_{,ii} + b_{i\varphi,i} - c\varphi \leq 0.$$

Then the function $v = u/\varphi$ satisfies the uniformly elliptic differential equation

$$a_{ii}v_{,ii} + \bar{b}_{i\varphi,i} - \bar{c}v = f/\varphi,$$

where

$$\bar{c} = (a_{ii}\varphi_{,ii} + b_{i\varphi,i} - c\varphi)/\varphi \leq 0.$$

Hence we may apply Theorem 3 to v (or Theorem 1 if $f = 0$) to obtain a bound for $|\text{grad } u|$.

If $c \geq 0$ in D and

$$|u| \leq m, \quad |\text{grad } u| \leq M \quad \text{on } \partial D,$$

then Theorem 3 yields the bound

$$(3.7) \quad |\text{grad } u|^2 \leq M^2 + \gamma m^2 + (m^2 + \beta)\psi.$$

If γ is positive, then the bound in (3.7) does not yield an optimum estimate for points near the boundary of D , even if ψ vanishes on ∂D . A desirable alternative to (3.7) is one such that the estimate for $|\text{grad } u|^2$ tends to M^2 as the point $x \in D$ tends to the boundary of D . The following theorem shows that such a bound is attainable when an auxiliary function ψ can be found which vanishes on ∂D .

Theorem 4. *Let u be a C^3 solution of the uniformly elliptic equation (3.1) with $c \geq 0$. Let ψ be a nonnegative solution of the differential inequality (3.2), and let γ and β be chosen as in Theorem 3. Then there exists a nonnegative constant η depending only on the coefficients in (3.1) and their first derivatives with the property that the function*

$$|\text{grad } u|^2 - \eta[M^2 + \gamma m^2 + (m^2 + \beta) \max_{x \in D} \psi + \max_{x \in D} |\text{grad } f|^2] \psi$$

satisfies the maximum principle in D .

Proof. We define

$$v = |\text{grad } u|^2.$$

As in the derivation of (2.6), we find that

$$a_{ij}v_{,ij} + b_{i,j}v_{,i} - 2cv \geq [-\frac{1}{2}A_{mn}a_{mk,i}a_{nk,j} - b_{i,i} - b_{i,i}]u_{,i}u_{,i} + 2c_{,k}uu_{,k} + 2f_{,k}u_{,k}.$$

The right side of the above inequality is a quadratic form in $\text{grad } u$, u , and $\text{grad } f$. Hence there exists a nonnegative constant η such that

$$a_{ij}v_{,ij} + b_{i,j}v_{,i} - 2cv \geq -\eta (|\text{grad } u|^2 + \gamma u^2 + |\text{grad } f|^2).$$

We now define

$$y = v - \eta[M^2 + \gamma m^2 + (m^2 + \beta) \max_{x \in D} \psi + \max_{x \in D} |\text{grad } f|^2] \psi.$$

We see from Theorem 3 and (3.2) that y satisfies the inequality

$$a_{ij}y_{,ij} + b_{i,j}y_{,i} - 2cy \geq 0.$$

Therefore y satisfies the maximum principle in D .

It follows that if $|u| \leq m$ and $|\text{grad } u| \leq M$ on ∂D , then

$$(3.8) \quad |\text{grad } u|^2 \leq M^2 + \eta[M^2 + \gamma m^2 + (m^2 + \beta) \max_{x \in D} \psi + \max_{x \in D} |\text{grad } f|^2] \psi.$$

The bound (3.8) is of particular interest when ψ vanishes on all or part of the boundary of D . In the vicinity of such points the estimate for $|\text{grad } u|^2$ is near M^2 , as is desirable.

4. Interior gradient estimates. As in Section 2, we consider the uniformly elliptic differential equation

$$(4.1) \quad a_{ij}u_{,ij} + b_{i,j}u_{,i} - cu = 0$$

in a domain D in $\mathbb{R}^N$. We introduce the function

$$(4.2) \quad w(x) = [\alpha(x)]^2 |\text{grad } u|^2 + \gamma u^2,$$

and we investigate conditions under which w satisfies a maximum principle in D . The following result is a straightforward extension of Theorem 1.

Theorem 5. *Let u be a C^3 solution in D of (4.1), and suppose that $c(x) \geq 0$. Let $\alpha(x)$ be a C^2 function in D . Then there is a positive constant γ depending only on α and on the coefficients in (4.1) such that w satisfies the maximum principle in D .*

Proof. We have

$$(4.3) \quad \begin{aligned} a_{ij}w_{,ij} + (b_i - \gamma^{-1}\alpha^2 c_{,i})w_{,i} &= 2\alpha^2 a_{ij}u_{,ik}u_{,jk} + 8\alpha\alpha_{,i}a_{ij}u_{,k}u_{,jk} \\ &\quad - 2\alpha^2 a_{ij,k}u_{,k}u_{,ij} - 2\alpha^4 \gamma^{-1} c_{,i}u_{,k}u_{,jk} \\ &\quad + [(2\alpha\alpha_{,pq} + 2\alpha_{,p}\alpha_{,q})a_{pq}\delta_{ij} + 2\gamma a_{ij} \\ &\quad + 2\alpha\alpha_{,p}b_p\delta_{ij} - 2\alpha^2 b_{i,j} + 2\alpha^2 c\delta_{ij} \\ &\quad - 2\gamma^{-1}\alpha^3 \alpha_{,p}c_{,p}\delta_{ij}]u_{,i}u_{,j} + 2\gamma cu^2. \end{aligned}$$

We employ inequality (2.5) choosing

$$s_{ij} = 4\alpha^{-1}\alpha_{,k}a_{ik}u_{,i} - a_{ij,k}u_{,k} - \alpha^2\gamma^{-1}c_{,i}u_{,i}.$$

In this way (4.3) yields the inequality

$$(4.4) \quad \begin{aligned} a_{ij}w_{,ij} + (b_i - \gamma^{-1}\alpha^2c_{,i})w_{,i} &\geq 2[\gamma a_{ij} + \{a_{pq}(\alpha\alpha_{,pq} - 3\alpha_{,p}\alpha_{,q}) + \alpha b_p\alpha_{,p} \\ &\quad + \alpha^2c + 2\gamma^{-1}\alpha^3c_{,p}\alpha_{,p} - \tfrac{1}{4}\gamma^{-2}\alpha^6A_{pq}c_{,p}c_{,q}\}\delta_{ij} \\ &\quad - \tfrac{1}{4}\alpha^2A_{pq}a_{pk,i}a_{qk,i} - \alpha^2b_{i,i} \\ &\quad + (2\alpha\alpha_{,q} - \tfrac{1}{2}\gamma^{-1}\alpha^4A_{pq}c_{,p})a_{qi,i}]u_{,i}u_{,i} + 2\gamma cu^2. \end{aligned}$$

Since (a_{ij}) is uniformly positive definite, the right side of (4.4) is nonnegative for sufficiently large γ . Therefore

$$a_{ij}w_{,ij} + (b_i - \gamma^{-1}\alpha^2c_{,i})w_{,i} \geq 0 \quad \text{in } D.$$

Hence w satisfies the maximum principle in D .

If $|u| \leq m$, $|\text{grad } u| \leq M$ for $x \in \partial D$, Theorem 4 yields the inequality

$$[\alpha(x)]^2 |\text{grad } u|^2 \leq \alpha_1^2 M^2 + \gamma m^2$$

for $x \in D$, where $\alpha_1 = \max_{x \in \partial D} |\alpha(x)|$. With various choices of α , a variety of gradient estimates may be obtained. For example, let $P \in D$ and choose $\alpha(P) = 1$, $\alpha(x) = 0$ on ∂D . Then we find

$$|\text{grad } u(P)| \leq \gamma^{1/2}m,$$

where, of course, γ depends on the selection of α as well as on the coefficients of the operator in (4.1). The quantity γ depends (through the selection of α) on the distance of P from ∂D . In general, $\gamma \rightarrow \infty$ as $P \rightarrow \partial D$.

5. Higher derivatives. Second derivatives of solutions of the equation

$$(5.1) \quad a_{ij}u_{,ij} + b_i u_{,i} = 0$$

may be estimated by the same method as that used in Section 2. We introduce the function

$$w(x) = u_{,ik}u_{,ik} + \gamma_1 u_{,k}u_{,k} + \gamma_2 u^2$$

and we compute the quantity $a_{ij}w_{,ij} + b_i w_{,i}$. We differentiate (5.1) twice and use inequalities similar to those in Section 2 to show that we may choose γ_1 and γ_2 so large (their sizes depending only on the coefficients in (6.1) and the first two derivatives of the coefficients), that the inequality

$$a_{ij}w_{,ij} + b_i w_{,i} \geq 0$$

holds throughout D . Hence w satisfies the maximum principle. More generally, if the coefficients in (5.1) have the appropriate order of smoothness, then higher derivatives may be estimated by means of the maximum principle.

Theorem 6. Let $u \in C^{k+2}(D)$ be a solution of (5.1) in D . Then there exist positive constants $\gamma_{i,k}$, $i = 1, 2, \dots, k$ with $\gamma_{k,k} = 1$ which depend only on the coefficients and the first k partial derivatives of the coefficients in (5.1) such that

$$w(x) = \sum_{|\alpha| \leq k} \gamma_{|\alpha|,k} |D^\alpha u|^2$$

satisfies the maximum principle in D .

By the methods of Section 3 we can show that if u is a solution of (3.1), if the coefficients and f are in C^k , and if ψ is a nonnegative function satisfying (3.2), then we can construct an expression of the form

$$w = \sum_{|\alpha| \leq k} \gamma_{|\alpha|,k} |D^\alpha u|^2 - (u^2 + \beta)\psi$$

which satisfies the maximum principle in D . Here β is the maximum of a quadratic form in f and its partial derivatives of all orders up to k .

6. Elliptic systems. Let $\mathbf{u} = \{u^1, u^2, \dots, u^M\}$ be a solution of the system of M equations

$$(6.1) \quad Lu \equiv a_{ij} u_{,i}^{\mu} + b_i^{\mu\nu} u_{,i}^{\nu} - c^{\mu\nu} u^{\nu} = 0, \quad \mu = 1, 2, \dots, M.$$

Here repeated Latin subscripts are summed from 1 to N while repeated Greek superscripts are summed from 1 to M . This system has two special properties: First, it is *semi-coupled* which means that the μ -th equation contains second derivatives of u^μ only, and second, the coefficients of the second derivatives are the same for all equations.

We shall assume that there is a vector field $\mathbf{e} = \{e_1, e_2, \dots, e_N\}$ and a positive constant σ such that

$$(6.2) \quad a_{ij} \xi_i^{\mu} \xi_j^{\mu} - b_i^{\mu\nu} \xi_i^{\mu} \xi_j^{\nu} + e_i \xi_i^{\mu} \xi_j^{\mu} + c^{\mu\nu} \xi_i^{\mu} \xi_j^{\nu} \geq \sigma (\xi_i^{\mu} \xi_i^{\mu} + \xi_i^{\nu} \xi_i^{\nu})$$

holds for all real matrices (ξ_i^{μ}) and all real vectors $\xi = (\xi^1, \xi^2, \dots, \xi^M)$, uniformly in D . Moreover, we suppose, as before, that the coefficients in (6.1) are C^1 functions in D . Under these hypotheses we establish the following analog of Theorem 1.

Theorem 7. Let $\mathbf{u}$ be a C^3 solution of (6.1) in D , and suppose that the coefficients satisfy (6.2) for some positive constant σ . Then there is a nonnegative number γ_0 which depends only on σ and bounds for the coefficients in (6.1) and their first derivatives such that for $\gamma \geq \gamma_0$, the function

$$(6.3) \quad w(x) \equiv u_{,k}^{\mu} u_{,k}^{\mu} + \gamma u^{\mu} u^{\mu}$$

satisfies the maximum principle in D .

Proof. A computation shows that

$$\begin{aligned}
\frac{1}{2}(a_{ij}w_{,ij} + e_i w_{,i}) &= a_{ij}u_{,ik}^{\mu}u_{,ik}^{\mu} - a_{ij,k}u_{,k}^{\mu}u_{,ij}^{\mu} - b_i^{\mu\nu}u_{,k}^{\mu}u_{,ik}^{\nu} \\
&\quad - b_{i,k}^{\mu\nu}u_{,k}^{\mu}u_{,i}^{\nu} + c^{\mu\nu}u_{,k}^{\mu}u_{,k}^{\nu} + c_{,k}^{\mu\nu}u_{,k}^{\mu}u_{,k}^{\nu} \\
&\quad + e_i u_{,ik}^{\mu}u_{,k}^{\mu} \\
&\quad + \gamma[a_{ij}u_{,ij}^{\mu}u_{,i}^{\mu} - b_i^{\mu\nu}u_{,i}^{\mu}u_{,i}^{\nu} + e_i u_{,i}^{\mu}u_{,i}^{\mu} + c^{\mu\nu}u_{,i}^{\mu}u_{,i}^{\nu}].
\end{aligned}$$

Application of the inequalities (6.2) and (2.5) yields

$$\begin{aligned}
\frac{1}{2}(a_{ij}w_{,ij} + e_i w_{,i}) &\geq \gamma\sigma[u_{,i}^{\mu}u_{,i}^{\mu} + u^{\mu}u^{\mu}] - b_{i,k}^{\mu\nu}u_{,k}^{\mu}u_{,i}^{\nu} + c^{\mu\nu}u_{,k}^{\mu}u_{,k}^{\nu} + c_{,k}^{\mu\nu}u_{,k}^{\mu}u_{,k}^{\nu} \\
&\quad - \frac{1}{4}A_{pq}(a_{pl,k}u_{,k}^{\mu} + b_p^{\mu\nu}u_{,l}^{\nu} - e_p u_{,l}^{\mu})(a_{ql,m}u_{,m}^{\mu} + b_q^{\mu\nu}u_{,l}^{\nu} - e_q u_{,l}^{\mu}).
\end{aligned}$$

The right side of the above inequality is clearly nonnegative definite for γ sufficiently large. Hence w satisfies the maximum principle.

Remarks. (i) For the case of a single equation such as (2.1), the condition (6.2) reduces to the uniform ellipticity condition $a_{ij}\xi_i\xi_j \geq \sigma\xi_i\xi_i$ together with the condition $c \geq \sigma > 0$.

(ii) The condition (6.2) is satisfied for some $\sigma > 0$ if and only if the two matrices (a_{ij}) and

$$(6.4) \quad c^{\mu\nu} - \frac{1}{4}A_{pq}(b_p^{\mu\tau} - e_p\delta^{\mu\tau})(b_q^{\nu\tau} - e_q\delta^{\nu\tau})$$

are uniformly positive definite. Here $\delta^{\mu\tau}$ is the usual Kronecker delta.

(iii) If $b_i^{\mu\nu} = b_i\delta^{\mu\nu}$, then we can choose $e_i = b_i$ to see that (6.2) is satisfied for some $\sigma > 0$ if and only if both (a_{ij}) and $(c^{\mu\nu})$ are uniformly positive definite.

(iv) From the proof of Theorem 7 we see that the e_i may be allowed to depend on the u^{μ} . Making the best such choice, we find that the hypothesis that (6.4) is positive definite may be replaced by the condition

$$(6.5) \quad (c^{\mu\nu} - \frac{1}{4}A_{pq}b_p^{\mu\tau}b_q^{\nu\tau})\xi^{\mu}\xi^{\nu} + \frac{1}{4}(\xi^{\mu}\xi^{\mu})^{-1}A_{pq}b_p^{\alpha\beta}b_q^{\gamma\delta}\xi^{\alpha}\xi^{\beta}\xi^{\gamma}\xi^{\delta} \geq \rho\xi^{\mu}\xi^{\mu}$$

for some constant $\rho > 0$.

(v) By employing the vector field $\mathbf{e}$ in the proof of Theorem 3 we could improve the constant γ needed for the maximum principle. We compute

$$a_{ij}w_{,ij} + e_i w_{,i} - cw$$

and then maximize the lower bound in (3.4) with respect to e_i .

(vi) If we replace w by the function $[\alpha(x)]^2 u_{,k}^{\mu}u_{,k}^{\mu} + \gamma u^{\mu}u^{\mu}$ a theorem analogous to Theorem 5 can be obtained for systems.

(vii) If we replace w as given in (6.3) by the function $u^{\mu}u^{\mu}$ and go through the same proof as Theorem 7, we see that $u^{\mu}u^{\mu}$ satisfies the maximum principle under the two slightly weaker conditions: (a) that (6.1) is elliptic and (b), that either (6.2) holds for some vector field $\mathbf{e}$ with $\sigma = 0$ or (6.5) holds with $\rho = 0$. This result was proved in a similar manner by C. Miranda [7] who first considered this class of systems.

If, instead of (6.1), we consider the inhomogeneous system $\mathbf{Lu} = \mathbf{F}$ where

$\mathbf{F} = (f^1, \dots, f^M)$, then an analog to Theorem 3 can be established provided a nonnegative function φ can be found which satisfies the differential inequality

$$a_{ij}\varphi_{,ij} + e_i\varphi_{,i} \leq -1.$$

Then by the same reasoning as in Section 3, we find that the function

$$w = u_{,k}^{\mu}u_{,k}^{\mu} + \gamma u^{\mu}u^{\mu} - \varphi \max_{x \in D} [f_{,k}^{\mu}f_{,k}^{\mu} + \frac{1}{2}\sigma^{-1}\gamma f^{\mu}f^{\mu}]$$

satisfies the maximum principle.

The results in Section 4 on higher derivatives clearly have extensions for systems of the type considered here.

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