Ergodicity, Uniformity and Unique Ergodicity.

G. HANSEL & J. P. RAOULT

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Introduction. Let X be a compact metric space, $\mathfrak{B}$ its Borel σ -algebra, Q a probability on $\mathfrak{B}$, and U a homeomorphism of X onto itself. The dynamical system $(X, \mathfrak{B}, Q, U)$ is said to be *uniquely ergodic* [1] iff Q is the unique probability on $\mathfrak{B}$ which is preserved by U .

The origin of our work can be found in the following problem, which was set up by W. Parry: May a uniquely ergodic dynamical system have a strictly positive entropy? Many questions and answers arose from this problem. First, F. Hahn and Y. Katznelson [2] have demonstrated that Parry's problem got a positive answer: they have proved that it is possible to exhibit a uniquely ergodic dynamical system of any large entropy. R. I. Jewett [3] has shown that any dynamical system $(W, \mathfrak{A}, P, T)$, where $(W, \mathfrak{A}, P)$ is a Lebesgue probability space, and T an invertible and weakly mixing transformation on W , is "isomorphic" ([4], p. 53) to a uniquely ergodic dynamical system. Hahn and Katznelson's result is an immediate consequence of Jewett's. K. Jacobs [5] has generalized Jewett's work to the case of a weakly mixing measurable flow on a Lebesgue space.

R. I. Jewett had let open the problem of extending his result to the case where T is only supposed to be an ergodic (not necessarily weakly mixing) transformation. This generalization can be deduced from a result obtained here, about the "uniform" subsets of an ergodic system: given an ergodic dynamical system $(W, \mathfrak{A}, P, T)$, a subset $A \in \mathfrak{A}$ is said to be *uniform* iff sequence $(n^{-1} \sum_{i=0}^{n-1} 1_A \circ T^i)_{n \in \mathbb{N}}$ converges to $P(A)$ for the $\mathcal{L}^\infty(W, \mathfrak{A}, P)$ semi-norm (that is to say, uniformly almost everywhere); we demonstrate (Theorem 1) that, if $\mathfrak{A}$ is separable and T is invertible, then there exists a denumerable subalgebra $\mathfrak{A}'$, stable by T (and even such that $T^{-1}(\mathfrak{A}') = \mathfrak{A}'$), dense in $\mathfrak{A}$ (equipped with the pseudo-metric $\delta(A, B) = P(A \triangle B)$) and whose elements are uniform subsets of W .

It follows (Theorem 2) that $(W, \mathfrak{A}, P, T)$ is "conjugate" ([4], p. 66) to a uniquely ergodic system $(X, \mathfrak{B}, Q, U)$, where X is the Stone space associated to $\mathfrak{A}'$ and, therefore, is a polish space (*i.e.*, metrizable, separable and complete). Moreover, if $(W, \mathfrak{A}, P)$ itself is isomorphic to a polish probability space, then $(W, \mathfrak{A}, P, T)$ and $(X, \mathfrak{B}, Q, U)$ are isomorphic (Theorem 3): this is an extension to Jewett's result.

Let us note that W. Krieger [6] too, has obtained this generalization by a quite different method, which, in particular, relies on Shannon-Mac-Millan's theorem.

1. Notations-Definitions.

1.1. Let $(W, \mathcal{A}, P, T)$ be a dynamical system, such that

- (i) $\mathcal{A}$ is a separable σ -algebra (*i.e.*, generated by a sequence $(A_n)_{n \in \mathbb{N}}$),
- (ii) T is an invertible, measure preserving and ergodic mapping of W onto itself.

Moreover, in paragraphs 1 to 8, probability P is supposed to be non-atomic.

We denote by $\mathcal{A}^*$ the set of all subsets $A \in \mathcal{A}$, such that $P(A) > 0$.

1.2 We denote by W^k the set of finite sequences $B = (w_0, w_1, \dots, w_{k-1})$ of length k in W . For every sequence $B = (w_0, w_1, \dots, w_{k-1})$, we denote by B^* the sequence obtained by cancelling the last term of B , that is

$$B^* = (w_0, w_1, \dots, w_{k-2});$$

we denote by $|B|$ the length of B . Let $B \in W^k$ and $B' \in W^{k'}$; we define the *product* $B \times B' \in W^{k+k'}$ by juxtaposition of sequences B and B' : if $B = (w_0, w_1, \dots, w_{k-1})$ and $B' = (w'_0, w'_1, \dots, w'_{k'-1})$, then $B \times B' = (w_0, w_1, \dots, w_{k-1}, w'_0, w'_1, \dots, w'_{k'-1})$; B and B' are said to be the factors of this product. This notion has a trivial generalization to a finite number of factors.

1.3. Let $k \in \mathbb{N}$, and let $B = (w_i)_{0 \leq i < k}$ be a finite or denumerable sequence of elements of W ; B is said to be a *block* iff, for all i such that $0 \leq i < k$, we have $w_i = T^i(w_0)$.

Let $K \in \mathcal{A}^*$; we denote by U_K the *return time* into K , that is to say the mapping $U_K : W \rightarrow \mathbb{N}^*$ defined by $U_K(w) = \inf \{k \in \mathbb{N}^* / T^k(w) \in K\}$. The system being supposed to be ergodic, U_K is almost everywhere finite.

A block $(T^i(w))_{0 \leq i < k}$ is said to be a *K-block* iff $w \in K$ and $k = U_K(w)$; in this case, we put $K(w) = (T^i(w))_{0 \leq i < k}$. Let $K(w)$ be a finite *K-block*; $(K(w))^*$ is said to be a *reduced K-block*. Let $F \subset K$; we put $K(F) = \{T^i(w) / w \in F, 0 \leq i < U_K(w)\}$. K is said to be a *bounded* subset of W iff $\sup_{w \in K} U_K(w) < \infty$.

Let a and b be two positive integers, such that $a \leq b$; K is said to be *(a, b)-bounded* iff, for all $w \in K$, we have $a \leq U_K(w) \leq b$.

1.4. We denote by $\mathcal{A}^k$ the set of finite sequences $\underline{A} = (A_0, A_1, \dots, A_{k-1})$ of length k in $\mathcal{A}$.

Let $\underline{A} = (A_0, A_1, \dots, A_{k-1}) \in \mathcal{A}^k$; let $m \in \{1, 2, \dots, k\}$ and $A \in \mathcal{A}$; we put ${}^m \underline{A} = (A_0, A_1, \dots, A_{m-1})$ and $(\underline{A}, A) = (A_0, A_1, \dots, A_{k-1}, A)$. We denote by $\sigma(\underline{A})$ the algebra of subsets of W generated by $\{T^{-i}(A_i) / (i, j) \in \{0, 1, \dots, k-1\}^2\}$.

1.5. Let $\underline{A} \in \mathcal{A}^k$, $B = (w_0, w_1, \dots, w_{n-1}) \in W^n$, with $n \geq k$; let $A \in \sigma(\underline{A})$ and $m \in \{1, 2, \dots, k\}$; we put $f_B^{A, m} = n^{-1} \sum_{i=0}^{n-m} 1_A(w_i)$; we remark that $0 \leq f_B^{A, 1} - f_B^{A, k} \leq (k-1)/n$.

Sequence B is said to be $(\underline{A}, \epsilon)$ -uniform iff, for all $A \in \sigma(\underline{A})$, we have $|f_B^{A,k} - P(A)| < \epsilon$. Sequence B is said to be $(\underline{A}, \epsilon)$ -strictly-uniform (shortly $(\underline{A}, \epsilon)$ -s-uniform) iff both B and B^* are $(\underline{A}, \epsilon)$ -uniform. Sequence B is said to be $(\underline{A}, a, b, \epsilon)$ -uniform iff B can be factorized as a product $\prod_{i=1}^p B_i$ of $(\underline{A}, \epsilon)$ -uniform blocks, in such a way that, for all $i \in \{1, 2, \dots, p\}$, $a \leq |B_i| \leq b$. Sequence B is said to be $(\underline{A}, a, b, \epsilon)$ -strictly-uniform (shortly $(\underline{A}, a, b, \epsilon)$ -s-uniform) iff B can be factorized as a product $\prod_{i=1}^p B_i$ of blocks, in such a way that

- (i) for all $i \in \{1, 2, \dots, p\}$, $a \leq |B_i| \leq b$ and B_i is $(\underline{A}, \epsilon)$ -uniform,
- (ii) $a + 1 \leq |B_p| \leq b$, and B_p is $(\underline{A}, \epsilon)$ -s-uniform.

It is clear that, if B is $(\underline{A}, a, b, \epsilon)$ -s-uniform, both B and B^* are $(\underline{A}, a, b, \epsilon)$ -uniform.

Proposition 1. Let $\underline{A} \in \bigcup_{k \in \mathbb{N}^*} \mathcal{G}^k$, and let $B_1, B_2, \dots, B_n$ be n $(\underline{A}, a, b, \epsilon)$ -uniform sequences; if B_n is $(\underline{A}, a, b, \epsilon)$ -s-uniform, so is $B = \prod_{i=1}^n B_i$.

Proof. Obvious.

2. Construction of $(\underline{A}, \epsilon)$ -uniform sequences.

2.1. Let $K \in \mathcal{G}^*$ be a bounded subset of W , and let $\underline{A} \in \mathcal{G}^k$; let us define on K an equivalence relation, denoted by $\sim$, by

$$w \sim w' \Leftrightarrow \begin{cases} U_K(w) = U_K(w'), \\ (\forall A \in \sigma(\underline{A})) (\forall i \in \{0, 1, \dots, U_K(w) - 1\}) : 1_A(T^i(w)) = 1_A(T^i(w')). \end{cases}$$

(We remark that, if $w \sim w'$ and $U_K(w) \geq j$, then for all $A \in \sigma(\underline{A})$, $f_{K(w)}^{A,i} = f_{K(w')}^{A,i}$).

This relation induces a finite partition of K into measurable equivalence classes. Let $F_1, F_2, \dots, F_m$ be those, among these equivalence classes, which have a strictly positive measure. For all $i \in \{1, 2, \dots, m\}$,

- we put $L_i = K(F_i)$,
- we denote by l_i the value on F_i of the function U_K ,
- we pick an element $w_i \in F_i$, and we put $B_i = K(w_i)$.

Remark. Let $A \in \sigma(\underline{A})$; T being measure preserving, we get $P^{L_i}(A) = P(L_i \cap A)/P(L_i) = f_{B_i}^{A,1}$; moreover, if K is (a, b) -bounded, we get, for all $i \in \{1, 2, \dots, m\}$,

$$f_{B_i}^{A,k} \leq P^{L_i}(A) \leq f_{B_i}^{A,k} + \frac{k-1}{a}.$$

2.2. Lemma 1. Let $K \in \mathcal{G}^*$ be (a, b) -bounded; let $\underline{A} \in \mathcal{G}^k$ and let $\epsilon > 0$. If $(k-1)/a < \epsilon$, there exists a sequence B , which is a product of K -blocks and is $(\underline{A}, \epsilon)$ -uniform.

Proof. Let $p_1, p_2, \dots, p_m$ be a finite sequence of positive integers, and let us put $B = \prod_{i=1}^m B_i^{p_i}$. We get $|B| = \sum_{i=1}^m p_i l_i$.

Let $A \in \sigma(\underline{A})$; we obtain the inequalities

$$\sum_{i=1}^m p_i l_i f_{B_i}^{A,k} \leq |B| f_B^{A,k} \leq \sum_{i=1}^m p_i l_i f_{B_i}^{A,k} + (k-1) \sum_{i=1}^m p_i ,$$

and, according to the above remark, we get

$$\frac{\sum_{i=1}^m p_i l_i P^{L_i}(A)}{\sum_{i=1}^m p_i l_i} - \frac{k-1}{a} \leq f_B^{A,k} \leq \frac{\sum_{i=1}^m p_i l_i P^{L_i}(A)}{\sum_{i=1}^m p_i l_i} + \frac{k-1}{a} .$$

$p_1, p_2, \dots, p_m$ can be chosen in order that

$$\sum_{i=1}^m \left| \frac{p_i l_i}{\sum_{i=1}^m p_i l_i} - P(L_i) \right| < \epsilon - \frac{k-1}{a} ;$$

with such a choice, we get for all $A \in \sigma(\underline{A})$, $|f_B^{A,k} - P(A)| < \epsilon$.

2.3. Proposition 2. *Let $\underline{A} \in \mathcal{Q}^k$, and let $B \in W^n$ be an $(\underline{A}, \epsilon)$ -uniform sequence. Then, for all $h \in \mathbf{N}^*$, B^h is an $(\underline{A}, \epsilon + (k-1)/n)$ -uniform sequence.*

Proof. Let $A \in \sigma(\underline{A})$. We obtain the inequalities

$$h n f_B^{A,k} \leq h n f_{B^h}^{A,k} \leq h n f_B^{A,k} + h(k-1).$$

As, by hypothesis, $|f_B^{A,k} - P(A)| < \epsilon$, we get $|f_{B^h}^{A,k} - P(A)| < \epsilon + (k-1)/n$.

2.4. Proposition 3. *Let $\underline{A} \in \mathcal{Q}^k$, let $B \in W^n$ be an $(\underline{A}, \epsilon)$ -uniform sequence, and let $B' \in W^{n'}$. Then $B \times B'$ is an $(\underline{A}, \epsilon + n'/n)$ -uniform sequence.*

Proof. Let $A \in \sigma(\underline{A})$. We obtain the inequalities

$$n f_B^{A,k} \leq (n + n') f_{B \times B'}^{A,k} \leq n f_B^{A,k} + n' ,$$

whence

$$f_B^{A,k} - n'/n \leq f_{B \times B'}^{A,k} \leq f_B^{A,k} + n'/n; \text{ then } |f_{B \times B'}^{A,k} - P(A)| < \epsilon + n'/n.$$

2.5. Proposition 4. *Let $K \in \mathcal{Q}^*$ be (a, b) -bounded; let $\underline{A} \in \mathcal{Q}^k$ and let $\epsilon > 0$. If $(k-1)/a < \epsilon/2$, there exists $q \in \mathbf{N}$ and, for all $m \geq q$, there exists a sequence $B_m \in W^m$, such that the following properties hold*

- (i) B_m is a product of K -blocks and reduced K -blocks, whose last factor is a K -block,
- (ii) B_m is $(\underline{A}, \epsilon)$ -strictly-uniform.

Proof. According to Lemma 1, we can find a sequence B , product of K -blocks, and $(\underline{A}, \epsilon/2)$ -uniform; let l be its length ($l \geq a$).

Besides, let $w_0 \in K$; let us put $B_0 = K(w_0)$ and $l_0 = |B_0|$.

Let us then consider, for all $(h, \alpha, \beta) \in \mathbf{N}^* \times \mathbf{N} \times \mathbf{N}$, the sequence

$$D = D(h, \alpha, \beta) = B^h \times (B_0^*)^\alpha \times B_0^\beta \times B_0 .$$

Then $|D| = hl + \alpha(l_0 - 1) + (\beta + 1)l_0$. It follows by Proposition 2 that B^h is $(\underline{A}, \epsilon/2 + (k - 1)/l)$ -uniform. As $(k - 1)/l \leq (k - 1)/a < \epsilon/2$, there exists $\epsilon' < \epsilon$ such that, for all $h \in \mathbf{N}^*$, B^h is $(\underline{A}, \epsilon')$ -uniform. Then it follows by Proposition 3 that sequences D and D^* are $(\underline{A}, \epsilon' + (\alpha(l_0 - 1) + (\beta + 1)l_0)/hl)$ -uniform. But, if q is large enough, we can associate to each integer $m \geq q$, $(h_m, \alpha_m, \beta_m) \in \mathbf{N}^* \times \mathbf{N} \times \mathbf{N}$ in such a way that

$$m = h_m l + \alpha_m(l_0 - 1) + (\beta_m + 1)l_0,$$

$$\frac{\alpha_m(l_0 - 1) + (\beta_m + 1)l_0}{h_m l} < \epsilon - \epsilon'.$$

Let us put, for all $m \geq q$, $B_m = D(h_m, \alpha_m, \beta_m)$. B_m is a sequence with the required properties.

3. Corresponding subsets.

3.1. Definition. Let $\underline{A} = (A_0, A_1, \dots, A_{k-1}) \in \mathcal{A}^k$ and $\underline{C} = (C_0, C_1, \dots, C_{k-1}) \in \mathcal{A}^k$. Let us denote by $\mathcal{R}$ the binary relation defined on W by

$$w\mathcal{R}w' \Leftrightarrow (\forall (i, j) \in \{0, 1, \dots, k-1\}^2): 1_{A_i}(T^i(w)) = 1_{C_j}(T^j(w')).$$

Let $A \in \sigma(\underline{A})$ and $C \in \sigma(\underline{C})$. A and C are said to be *corresponding subsets* iff $(\forall (w, w') \in W^2) w\mathcal{R}w' \Rightarrow 1_A(w) = 1_C(w')$. We denote by $\mathcal{C}(\underline{A}, \underline{C})$ the set of all couples $(A, C) \in \sigma(\underline{A}) \times \sigma(\underline{C})$ of corresponding subsets.

3.2. Notation. Let $(\underline{A}, \underline{C}) \in (\mathcal{A}^k)^2$; we put

$$\underline{A} \triangle \underline{C} = \bigcup_{i=0}^{k-1} \bigcup_{j=0}^{k-1} [T^{-i}(A_i) \triangle T^{-j}(C_j)]$$

3.3. Properties of corresponding subsets. In the next proposition, we gather the properties of corresponding subsets that we shall need afterwards.

Proposition 5. Let $(\underline{A}, \underline{C}) \in (\mathcal{A}^k)^2$.

- 1°. For all $(i, j) \in \{0, 1, \dots, k-1\}^2$, $(T^{-i}(A_i), T^{-j}(C_j)) \in \mathcal{C}(\underline{A}, \underline{C})$;
- 2°. If $(A, C) \in \mathcal{C}(\underline{A}, \underline{C})$, then $A \triangle C \subset \underline{A} \triangle \underline{C}$;
- 3°. Let $m \leq k$; we get $\mathcal{C}({}^m \underline{A}, {}^m \underline{C}) \subset \mathcal{C}(\underline{A}, \underline{C})$;
- 4°. Let (A, C) and (A', C') belong to $\mathcal{C}(\underline{A}, \underline{C})$; then, so do (A^c, C^c) and $(A \cup A', C \cup C')$;
- 5°. Let $m < k$ and $(A, C) \in \mathcal{C}({}^m \underline{A}, {}^m \underline{C})$; then $(T^{-1}(A), T^{-1}(C)) \in \mathcal{C}({}^{m+1} \underline{A}, {}^{m+1} \underline{C})$;
- 6°. For all $A \in \sigma(\underline{A})$, there exists $C \in \sigma(\underline{C})$, such that $(A, C) \in \mathcal{C}(\underline{A}, \underline{C})$.

Proof. Properties 1°, ..., 5° are easily deduced from the definition of corresponding subsets; they imply property 6°.

4. Modifications.

4.1. Definition. Let $K \in \mathcal{A}$ and $F \in \mathcal{A}$, such that $F \subset K$. Let

$$\underline{A} = (A_0, A_1, \dots, A_{k-1}) \in \mathcal{A}^k \quad \text{and} \quad \underline{C} = (C_0, C_1, \dots, C_{k-1}) \in \mathcal{A}^k.$$

$\underline{C}$ is said to be a (K, F) -modification of $\underline{A}$ iff for all $i \in \{0, 1, \dots, k-1\}$, we have $A_i \cap K(F)^c = C_i \cap K(F)^c$.

4.2. Proposition 6. *Let $K \in \mathcal{A}$ be (a, b) -bounded, and let $F \in \mathcal{A}$, such that $F \subset K$. Let $\underline{A} \in \mathcal{A}^k$, and let $\underline{C} \in \mathcal{A}^k$ be a (K, F) -modification of $\underline{A}$. Then*

$$\forall (A, C) \in \mathcal{C}(\underline{A}, \underline{C}) : P(A \triangle C) \leq P(K(F)) + \frac{k-1}{a}.$$

Proof. Let $(i, j) \in \{0, 1, \dots, k-1\}^2$. The definition of (K, F) -modification implies that $A_i \triangle C_i \subset K(F)$. It follows that

$$T^{-i}(A_i) \triangle T^{-i}(C_i) \subset K(F) \cup \left(\bigcup_{i=0}^{k-1} T^{-i}(F) \right),$$

whence

$$\underline{A} \triangle \underline{C} \subset K(F) \cup \left(\bigcup_{i=0}^{k-1} T^{-i}(F) \right).$$

K being (a, b) -bounded, $P(F) \leq P(K) \leq 1/a$, and then

$$P(\underline{A} \triangle \underline{C}) \leq P(K(F)) + (k-1)/a.$$

The conclusion follows by Proposition 5.2°.

5. Corresponding subsets and uniformity.

5.1. Definition. Let $\underline{A} = (A_0, A_1, \dots, A_{k-1}) \in \mathcal{A}^k$, and $B = (w_0, w_1, \dots, w_{m-1}) \in W^m$. Let the mapping $M(\underline{A}, B) : \{0, 1, \dots, k-1\} \times \{0, 1, \dots, m-1\} \rightarrow \{0, 1\}$ be defined by $M(\underline{A}, B)[(i, j)] = 1_{A_i}(w_j)$.

Remark. Let us use the same notations as in 4.1; it is clear that, if, for all $w \in K \setminus F$, $M(\underline{C}, K(w)) = M(\underline{A}, K(w))$, then $\underline{C}$ is a (K, F) -modification of $\underline{A}$.

5.2. Proposition 7. *Let $(\underline{A}, \underline{C}) \in (\mathcal{A}^k)^2$, and let $B = \prod_{i=1}^n B_i$ be a product of n blocks and be $(\underline{A}, \epsilon)$ -uniform (respect. $(\underline{A}, \epsilon)$ -s-uniform). Let B' be a block which has the same length as sequence B . Let us assume that*

- 1) $M(\underline{A}, B) = M(\underline{C}, B')$,
- 2) $\forall (A, C) \in \mathcal{C}(\underline{A}, \underline{C}) : P(A \triangle C) < \epsilon'$.

Then

1°. *If for all $i \in \{1, 2, \dots, n-1\}$, the inequality $(k-1)/|B_i| < \epsilon''$ holds, then B' is $(\underline{C}, \epsilon + \epsilon' + \epsilon'')$ -uniform (respect. $(\underline{C}, \epsilon + \epsilon' + \epsilon'')$ -s-uniform);*

2°. *If B itself is a block (that is, if $n = 1$), B' is $(\underline{C}, \epsilon + \epsilon')$ -uniform (respect. $(\underline{C}, \epsilon + \epsilon')$ -s-uniform).*

Proof.

1°. Let $B = (w_i)_{0 \leq i \leq p-1}$ and $B' = (w'_i)_{0 \leq i \leq p-1}$. For all $i \in \{1, 2, \dots, n\}$, let w_{k_i} be the first term of sequence B_i . It follows by assumption 1) and the definition of corresponding subsets that

$$(\forall (A, C) \in \mathcal{C}(\underline{A}, \underline{C})) (\forall i \in \{1, 2, \dots, n\}) (\forall j \in \{0, 1, \dots, |B_i| - k\}):$$

$$1_A(T^j(w_{k_i})) = 1_C(T^j(w'_{k_i})).$$

Hence, we get the inequalities

$$(1) \quad |B| f_B^{A,k} - (n-1)(k-1) \leq |B'| f_{B'}^{C,k} \leq |B| f_B^{A,k} + (n-1)(k-1)$$

and, in the same way,

$$(1a) \quad |B^*| f_{B^*}^{A,k} - (n-1)(k-1) \leq |B'^*| f_{B'^*}^{C,k} \leq |B^*| f_{B^*}^{A,k} + (n-1)(k-1)$$

Besides, $(n-1)(k-1)/|B^*| < \epsilon''$; then, we get

$$(2) \quad f_B^{A,k} - \epsilon'' \leq f_{B'}^{C,k} \leq f_B^{A,k} + \epsilon''$$

and, in the same way,

$$(2a) \quad f_{B^*}^{A,k} - \epsilon'' \leq f_{B'^*}^{C,k} \leq f_{B^*}^{A,k} + \epsilon''.$$

If B is $(\underline{A}, \epsilon)$ -uniform, it follows from inequalities (2) and assumption 2) that $|f_{B'}^{C,k} - P(C)| < \epsilon + \epsilon' + \epsilon''$; therefore, B' is $(\underline{C}, \epsilon + \epsilon' + \epsilon'')$ -uniform. Moreover, if B is $(\underline{A}, \epsilon)$ -s-uniform, it follows from inequalities (2a) that we also get $|f_{B'^*}^{C,k} - P(C)| < \epsilon + \epsilon' + \epsilon''$; therefore, B' is $(\underline{C}, \epsilon + \epsilon' + \epsilon'')$ -s-uniform.

2°. If B is a block, that is to say if $n = 1$, relations (1) and (1a) are equalities, and we get $f_{B'}^{C,k} = f_B^{A,k}$ and $f_{B'^*}^{C,k} = f_{B^*}^{A,k}$; the end of the proof follows as before.

5.3. Proposition 8. *Let $(\underline{A}, \underline{C}) \in (\mathcal{Q}^k)^2$. Let B be a finite sequence in W which is $(\underline{A}, a, b, \epsilon)$ -s-uniform. Let B' be a block, which has the same length as sequence B . Let us assume that*

- 1) $M(\underline{A}, B) = M(\underline{C}, B')$
 - 2) $\forall (A, C) \in \mathcal{C}(\underline{A}, \underline{C}) : P(A \triangle C) < \epsilon'$.
- Then, B' is $(\underline{C}, a, b, \epsilon + \epsilon')$ -s-uniform.*

Proof. B being $(\underline{A}, a, b, \epsilon)$ -s-uniform, there exists a finite sequence of blocks $(B_i)_{1 \leq i \leq n}$ such that

- (i) $B = \prod_{i=1}^n B_i$,
- (ii) $(\forall i \in \{1, 2, \dots, n-1\}) a \leq |B_i| \leq b$ and B_i is $(\underline{A}, \epsilon)$ -uniform,
- (iii) $a+1 \leq |B_n| \leq b$, and B_n is $(\underline{A}, \epsilon)$ -s-uniform.

The length of block B' being the same as that of sequence B , there exists a sequence $(B'_i)_{1 \leq i \leq n}$ of blocks, such that

- (a) $B' = \prod_{i=1}^n B'_i$,
- (b) $\forall i \in \{1, 2, \dots, n-1\} : |B'_i| = |B_i|$, and then $a \leq |B'_i| \leq b$,
- (c) $|B'_n| = |B_n|$, and then $a+1 \leq |B'_n| \leq b$,
- (d) $\forall i \in \{1, 2, \dots, n\} : M(\underline{C}, B'_i) = M(\underline{A}, B_i)$.

Then, it follows by Proposition 7.2° that, for all $i \in \{1, 2, \dots, n-1\}$, B'_i is $(\underline{C}, \epsilon + \epsilon')$ -uniform and that B'_n is $(\underline{C}, \epsilon + \epsilon')$ -s-uniform. Therefore B' is $(\underline{C}, a, b, \epsilon + \epsilon')$ -s-uniform.

6. Creation of uniformities.

6.1. The next proposition is very close to a result established by R. I. Jewett [3].

Proposition 9. *Let $(D_n)_{n \in \mathbf{N}}$ be an increasing sequence of measurable subsets, such that $\lim_{n \rightarrow \infty} P(D_n) = 1$. Let $K \in \mathcal{A}^*$ be (a, b) -bounded. Then, for all integers $s \geq b$, and all $\eta > 0$, there exist $K' \in \mathcal{A}^*$ and $n \in \mathbf{N}$, such that the following properties hold*

- 1) $K' \subset K$,
- 2) $n \geq 2s + b$,
- 3) K' is $(s, n + b)$ -bounded and, precisely, for all $w \in K'$, we have
—either $n \leq U_{K'}(w) < n + b$, and then $w \in D_n$,
—or $s \leq U_{K'}(w) < 2s + b$,
- 4) Let $F' = \{w \in K' / s \leq U_{K'}(w) < 2s + b\}$; then $P(K'(F')) < \eta$.

Proof.

1°. Let $n \in \mathbf{N}$, such that

$$(1) \quad P(D_n) > 1 - \frac{\eta}{2(2s + b)},$$

$$(2) \quad P(D_n \cap K) > 0,$$

$$(3) \quad n \geq 2s + b.$$

Let $m \in \mathbf{N}$, such that

$$(4) \quad \frac{n + s + b}{m} < \frac{\eta}{2} \quad \text{and} \quad m \geq s.$$

P being supposed non-atomic (see 1.1), we can find a measurable subset of $D_n \cap K$, say E , such that

$$(5) \quad 0 < P(E) < \frac{1}{m}.$$

Let us put $G = E \setminus \bigcup_{i=1}^{m-1} T^{-i}(E)$. It follows by (5) and ergodicity of T that

$$(6) \quad 0 < P(G) < \frac{1}{m}.$$

Besides, the construction implies that, for all $w \in G$, we get $U_G(w) \geq m$.

2°. For all $w' \in W$, let us denote by w'_K the return point in K from w' , that is to say the element of K defined by $w'_K = T^{U_{K'}(w')}(w')$.

We intend to build a measurable subset K' , such that $G \subset K' \subset K$, whence $P(K') \geq P(G) > 0$; in order to determine K' , we build inductively, for all $w \in G$, the subsequence of $G(w)$, let us say $(w_k)_{0 \leq k < l}$ (with $l \in \mathbf{N}^*$) which is made of the terms of $G(w)$ which are to be elements of K' ; the induction goes in the following way, which includes the verification that, for all w_k , the inequality $U_G(w_k) \geq s$ is satisfied:

- $w_0 = w$ (we have $U_G(w_0) \geq m \geq s$);
- suppose w_k has been built; then
 - a) if $U_G[(T^{s-1}(w_k))_K] < s$, w_k is the last term of the subsequence;
 - b) if $U_G[(T^{s-1}(w_k))_K] \geq s$, we exhibit w_{k+1}

- (i) if $w_k \notin D_n$, then $w_{k+1} = (T^{s-1}(w_k))_K$;
 (ii) if $w_k \in D_n$, then
 —if $U_G(w_k) \geq n$ and $U_G(T^{m-1}(w_k))_K \geq s$, we put $w_{k+1} = (T^{n-1}(w_k))_K$;
 —if not, we put $w_{k+1} = (T^{s-1}(w_k))_K$.
 3°. It is obvious that $K' \in \mathcal{A}$ (and even $K' \in \mathcal{A}^*$) and that properties 1), 2 and 3) of the proposition are satisfied; let us prove property 4); by construction of K' , if $w \in F'$, that is to say if $s \leq U_{K'}(w) < 2s + b$, we get
 —either $w \notin D_n$,
 —or $w \in D_n$ and $U_G(w) \leq n + s + b$;
 hence

$$K'(F') \subset \left[\bigcup_{i=0}^{2s+b-1} T^i(D_n^c) \right] \cup \left[\bigcup_{i=1}^{n+s+b} T^{-i}(G) \right],$$

whence, according to (1), (4) and (6),

$$P(K'(F')) \leq (2s + b)P(D_n^c) + (n + s + b)P(G) < \eta.$$

6.2. Definition. Let $K \in \mathcal{A}$, $\underline{A} \in \mathcal{A}^k$, $(a, b) \in \mathbf{N}^2$, with $a \leq b$, and $\epsilon > 0$. K is said to be $(\underline{A}, a, b, \epsilon)$ -strictly-uniform iff so is every K -block.

6.3. Lemma 2. Let $(A_k)_{k \in \mathbf{N}}$ be a sequence of measurable subsets, and let $(\epsilon_k)_{k \in \mathbf{N}}$ be a decreasing sequence of strictly positive real numbers. For all $(i, j) \in \mathbf{N}^2$, such that $i \leq j$, we put $\epsilon_i^j = \sum_{m=i}^j \epsilon_m$. Then there exists, for all $k \in \mathbf{N}$, a 4-tuple $(K_k, a_k, b_k, \underline{A}^{(k)})$, where $K_k \in \mathcal{A}^*$, $(a_k, b_k) \in \mathbf{N}^2$, $\underline{A}^{(k)} \in \mathcal{A}^{k+1}$ and the last term of $\underline{A}^{(k)}$ is A_k , and such that the following properties hold

- $k/a_k < \epsilon_{k+1}/8$ and K_k is $(a_k + 1, b_k)$ -bounded,
- if $k \neq 0$, $K_k \subset K_{k-1}$,
- if $k \neq 0$, then for all $(A, C) \in \mathcal{C}(\underline{A}^{(k-1)}, {}^k \underline{A}^{(k)})$, we get $P(A \triangle C) < \epsilon_k/2$,
- for all $i \in \{1, 2, \dots, k\}$, K_k is $({}^i \underline{A}^{(k)}, a_i, b_i, \epsilon_i^k)$ -s-uniform.

Proof.

1°. Let $K_0 \in \mathcal{A}^*$ be an (a, b) -bounded subset, where $a \geq 2$ (such a subset exists, according to Proposition 9, in which we put, for all n , $D_n = W$ and $K = W$). Let us put $a_0 = a - 1$, $b_0 = b$, $\underline{A}^{(0)} = (A_0)$; then the properties of the lemma hold for $k = 0$.

Let us suppose the lemma has been established up to order $k - 1$; we are now to exhibit $K_k \subset K_{k-1}$, a subset $F_k \subset K_k$ and a sequence $\mathcal{C} = (C_0, C_1, \dots, C_{k-1})$, which is to be a (K_k, F_k) -modification of $\underline{A}^{(k-1)}$. We shall then put $\underline{A}^{(k)} = (\mathcal{C}, A_k)$.

2°. As $(k - 1)/a_{k-1} < \epsilon_k/8$, we can by Proposition 4, where $\underline{A} = \underline{A}^{(k-1)}$, determine an integer q and a family $(B_m)_{m \geq q}$ of sequences, in such a way that, for all $m \geq q$, the following properties hold

- the length of B_m is m ,
- B_m is a product of K_{k-1} -blocks and reduced K_{k-1} -blocks, whose last factor is not reduced,
- B_m is $(\underline{A}^{(k-1)}, \epsilon_k/4)$ -s-uniform.

3°. Let us put, for all integers $n \geq 2$,

$$D_n = \left\{ w \in W / (\forall A \in \sigma(\underline{A}^{(k-1)})) (\forall m \geq n-1) \right. \\ \left. : \left| \frac{1}{m} \sum_{i=0}^{m-k} 1_A(T^i(w)) - P(A) \right| < \frac{\epsilon_k}{4} \right\}$$

Sequence $(D_n)_{n \geq 2}$ is increasing and, by ergodicity, we get $\lim_{n \rightarrow \infty} P(D_n) = 1$.

(**Remark.** If $w \in D_n$ and $m \geq n$, then sequence $(T^i(w))_{0 \leq i \leq m}$ is $(\underline{A}^{(k-1)}, \epsilon_k/4)$ -s-uniform.)

Let s be an integer such that the following inequalities hold

$$(1) \quad s \geq \sup(b_{k-1}, q),$$

$$(2) \quad \frac{k}{s-1} < \frac{\epsilon_{k+1}}{8}.$$

It follows by Proposition 9 (in which we put $K = K_{k-1}$, $\eta = \epsilon_k/4$, $a = a_{k-1}$, $b = b_{k-1}$) that we can exhibit $K_k \in \mathcal{G}^*$, a measurable subset $F_k \subset K_k$, and $n \in \mathbb{N}$, in such a way that the following properties hold:

$$(3) \quad K_k \subset K_{k-1}; \quad F_k = \{w \in K_k / s \leq U_{K_k}(w) < 2s + b_{k-1}\}$$

$$(4) \quad n \geq 2s + b_{k-1};$$

K_k is $(s, n + b_{k-1})$ -bounded and, precisely, for all $w \in K_k$, we get

$$(5) \quad \text{—either } w \in D_n \text{ and } n \leq U_{K_k}(w) < n + b_{k-1}$$

$$(6) \quad \text{—or } s \leq U_{K_k}(w) < 2s + b_{k-1};$$

$$(7) \quad P(K_k(F_k)) < \frac{\epsilon_k}{4}.$$

We put

$$(8) \quad a_k = s - 1$$

$$(9) \quad b_k = n + b_{k-1}.$$

4°. For all $w \in F_k$, we get $U_{K_k}(w) \geq q$; hence sequence $B_{U_{K(k)}(w)}$ exists, where $K(k) = K_k$. Then we define $\mathcal{C} = (C_0, C_1, \dots, C_{k-1})$ to be the (K_k, F_k) -modification of $\underline{A}^{(k-1)}$ satisfying the following condition

$$(10) \quad \forall w \in F_k : M(\mathcal{C}, K_k(w)) = M(\underline{A}^{(k-1)}, B_{U_{K(k)}(w)})$$

and we put $\underline{A}^{(k)} = (\mathcal{C}, A_k)$.

5°. Let us show that $(K_k, a_k, b_k, \underline{A}^{(k)})$ fulfills conditions a), b), c), d) of the lemma.

a) and b) are fulfilled according to (2), (3), (5), (6), (8), (9).

c) is fulfilled according to Proposition 6 (in which we put $K = K_k$, $\underline{A} = \underline{A}^{(k-1)}$, $a = a_k$), inequality (7), and condition a) whose fulfillment is yet established.

Let us prove condition d). Let $w \in K_k$. We know that

$$(11) \quad \text{if } w \in F_k, \text{ then } M(\mathcal{C}, K_k(w)) = M(\underline{A}^{(k-1)}, B_{U_{K(k)}(w)}),$$

$$(12) \quad \text{if } w \in K_k \setminus F_k, \text{ then } M(\mathcal{C}, K_k(w)) = M(\underline{A}^{(k-1)}, K_k(w))$$

(for this last case see Remark in §5.1).

Sequences $B_{U_{K(k)}(w)}$, where $w \in F_k$, and sequences $K_k(w)$, where $w \in K_k \setminus F_k$, satisfy the following properties:

A) They are products of K_{k-1} -blocks and reduced K_{k-1} -blocks, whose last factor is not reduced:

—for sequences $B_{U_{K(k)}(w)}$, this follows by 2° β above;

—for sequences $K_k(w)$, this follows by inclusion $K_k \subset K_{k-1}$.

B) For all $i \in \{1, 2, \dots, k-1\}$, they are $(\underline{A}^{(k-1)}, a_i, b_i, \epsilon_i^{k-1})$ -s-uniform; this follows by A) above, the induction hypothesis, and Proposition 1.

C) They are $(\underline{A}^{(k-1)}, \epsilon_k/4)$ -s-uniform:

—for sequences $B_{U_{K(k)}(w)}$, this follows by 2° γ above;

—for sequences $K_k(w)$, this follows by inclusion $K_k \setminus F_k \subset D_n$, the definition of D_n , and (5).

Let $i \in \{1, 2, \dots, k-1\}$, and $w \in K_k$. It follows by Proposition 5.3°, by Proposition 8, by (11) and (12), by condition c) which has yet been verified, and by B), that sequence $K_k(w)$ is $(\mathcal{C}, a_i, b_i, \epsilon_i^{k-1} + \epsilon_k/2)$ -s-uniform. But $\mathcal{C} = \underline{A}^{(k)}$, and $\epsilon_i^k = \epsilon_i^{k-1} + \epsilon_k$. Therefore, K_k is a $(\underline{A}^{(k)}, a_i, b_i, \epsilon_i^k)$ -s-uniform subset.

Let $i = k$ and $w \in K_k$. It follows by Proposition 7, by (11) and (12), by A) and C), by condition c) which has yet been verified, and by the fact that, according to the induction hypothesis, for any K_{k-1} -block or reduced K_{k-1} -block, let us say B , we get $(k-1)/|B| \leq (k-1)/a_{k-1} < \epsilon_k/8$, the sequence $K_k(w)$ is $(\mathcal{C}, \epsilon_k/4 + \epsilon_k/2 + \epsilon_k/8)$ -s-uniform. But K_k is $(a_k + 1, b_k)$ -bounded (according to condition a) yet verified), $\mathcal{C} = \underline{A}^{(k)}$, and $\epsilon_k^k = \epsilon_k$. Therefore, K_k is a $(\underline{A}^{(k)}, a_k, b_k, \epsilon_k^k)$ -s-uniform subset.

6.4. Lemma 2a. *Let $(A_k)_{k \in \mathbf{N}}$ be a sequence of measurable subsets, and let $(\epsilon_k)_{k \in \mathbf{N}}$ be a decreasing sequence of strictly positive real numbers.*

Then, there exists a sequence of 4-tuples $(K_k, a_k, b_k, \underline{A}^{(k)})_{k \in \mathbf{N}}$, where, for all $k \in \mathbf{N}$, $K_k \in \mathcal{Q}^$, $(a_k, b_k) \in \mathbf{N}^2$, $\underline{A}^{(k)} \in \mathcal{Q}^{k+1}$ and the last term of $\underline{A}^{(k)}$ is A_k , and such that the following properties hold:*

a) $(\forall k \in \mathbf{N}^*)(\forall (A, C) \in \mathcal{C}(\underline{A}^{(k-1)}, \underline{A}^{(k)})) : P(A \triangle C) < \epsilon_k$;

b) $(\forall i \in \mathbf{N}^*)(\forall k \geq i)(\forall A \in \sigma(\underline{A}^{(k)})) : K_k \text{ is } ((A), a_i, b_i, 2(\sum_{m \geq i} \epsilon_m))$ -uniform.

Proof. Let $(K_k, a_k, b_k, \underline{A}^{(k)})_{k \in \mathbf{N}}$ be a sequence associated, according to Lemma 2, to sequences $(A_k)_{k \in \mathbf{N}}$ and $(\epsilon_k)_{k \in \mathbf{N}}$.

Conclusion a) of lemma is an immediate consequence of conclusion c) of Lemma 2.

Let us prove conclusion b). Let $i \in \mathbf{N}^*$, $k \geq i$, $A \in \sigma(\underline{A}^{(k)})$ and $w \in K_k$. It follows by conclusion d) of Lemma 2 and the definition of $(\underline{A}^{(k)}, a_i, b_i, \epsilon_i^k)$ -uni-

formity, that block $K_k(w)$ is a product of blocks B , the length of which is between a_i and b_i , and such that $|f_B^{A,i} - P(A)| < \epsilon_i^k$. But (see 1.4), $|f_B^{A,i} - f_B^{A,1}| \leq (i-1)/B$; hence, according to conclusion a) of Lemma 2, we get

$$|f_B^{A,i} - f_B^{A,1}| \leq \frac{i-1}{a_i} < \frac{\epsilon_{i+1}}{8}.$$

Therefore $|f_B^{A,1} - P(A)| \leq \epsilon_i^k + \epsilon_{i+1}/8 < 2(\sum_{m \geq i} \epsilon_m)$. So K_k is $((A), a_i, b_i, 2(\sum_{m \geq i} \epsilon_m))$ -uniform.

7. (a', ϵ) -uniform subsets.

7.1. Definition. Let $a' \in \mathbf{N}^*$ and $\epsilon > 0$. A measurable subset A of W is said to be (a', ϵ) -uniform iff, for almost all $w \in W$, and for all $n \geq a'$, we have

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} 1_A(T^i(w)) - P(A) \right| < \epsilon.$$

7.2. Proposition 10. Let $b \in \mathbf{N}^*$, and let $\epsilon > 0$. Then there exists $a' \in \mathbf{N}^*$, such that the following property holds:

In order that a measurable subset A of W be (a', ϵ) -uniform, a sufficient condition is that there exists a measurable subset $K \in \mathcal{A}^$, which is $((A), 0, b, \epsilon/2)$ -uniform.*

Proof. Let $A \in \mathcal{A}$, and let $K \in \mathcal{A}^*$, which is $((A), 0, b, \epsilon/2)$ -uniform. $P(K) > 0$, hence, as P is ergodic, almost all orbits can be partitioned into K -blocks and even, as K is supposed to be $((A), 0, b, \epsilon/2)$ -uniform, into $((A), \epsilon/2)$ -uniform blocks, the lengths of which are majorized by b .

Let w belong to such an orbit, and let $n \in \mathbf{N}^*$. Sequence $B = (T^i(w))_{0 \leq i \leq n-1}$ can be factorized as $B = \prod_{i=0}^{m-1} B_i$, where

- for all $j \in \{0, 1, \dots, m\}$, the length of B_j , denoted l_j , is lower than b ,
- for all $j \in \{1, 2, \dots, m-1\}$, B_j is $((A), \epsilon/2)$ -uniform.

Then, we get

$$\sum_{j=1}^{m-1} f_{B_j}^{A,1} l_j \leq \sum_{i=0}^{n-1} 1_A(T^i(w)) \leq \sum_{j=1}^{m-1} f_{B_j}^{A,1} l_j + 2b,$$

whence

$$\left(P(A) - \frac{\epsilon}{2} \right) \left(\frac{n-2b}{n} \right) \leq \frac{1}{n} \sum_{i=0}^{n-1} 1_A(T^i(w)) \leq P(A) + \frac{\epsilon}{2} + \frac{2b}{n}.$$

It follows that there exists a' , which depends only on ϵ and b , such that if $n \geq a'$, we get

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} 1_A(T^i(w)) - P(A) \right| < \epsilon.$$

7.3. Proposition 11. Let $(A_k)_{k \in \mathbf{N}}$ be a sequence of measurable sets, and consider a convergent series, of general term ϵ_k , strictly positive, decreasing to 0. Then, there

exists an increasing sequence of positive integers, let us say $(n_k)_{k \in \mathbf{N}^*}$ and, for all $k \in \mathbf{N}$, there exists $\underline{A}^{(k)} \in \mathcal{Q}^{k+1}$, whose last term is A_k , such that the following properties hold:

- a) $(\forall k \in \mathbf{N}^*)(\forall (A, C) \in \mathcal{C}(\underline{A}^{(k-1)}, {}^k \underline{A}^{(k)})) : P(A \triangle C) < \epsilon_k$,
- b) $(\forall i \in \mathbf{N}^*)(\forall k \geq i)(\forall A \in \sigma({}^i \underline{A}^{(k)})) : A \text{ is } (n_i, 4(\sum_{m \geq i} \epsilon_m))\text{-uniform.}$

Proof. Let $(K_k, a_k, b_k, \underline{A}^{(k)})_{k \in \mathbf{N}}$ be a sequence associated, according to Lemma 2a, to sequences $(A_k)_{k \in \mathbf{N}}$ and $(\epsilon_k)_{k \in \mathbf{N}}$.

Conclusion a) of the proposition is nothing else than conclusion a) of Lemma 2a.

Let us prove conclusion b): let $i \in \mathbf{N}^*$; for all $k \geq i$, and for all $A \in \sigma({}^i \underline{A}^{(k)})$, it follows from conclusion b) of Lemma 2a that K_k is $((A), 0, b_i, 2(\sum_{m \geq i} \epsilon_m))$ -uniform; then, there exists, according to Proposition 10, $a'_i \in \mathbf{N}^*$ such that, for all $k \geq i$, every $A \in \sigma({}^i \underline{A}^{(k)})$ is $(a'_i, 4(\sum_{m \geq i} \epsilon_m))$ -uniform; let now $(n_i)_{i \in \mathbf{N}^*}$ be an increasing sequence such that, for all $i \in \mathbf{N}^*$, $n_i \geq a'_i$; then conclusion b) holds.

8. Stable and dense subalgebra of uniform subsets. From now on, $\mathcal{Q}$ is equipped with the topology of the complete, not necessarily Hausdorff, space, associated to the pseudo-metric δ defined by

$$\forall (A, B) \in \mathcal{Q}^2 : \delta(A, B) = P(A \triangle B).$$

Let us remember that the notion of *uniform subset* has been defined in the Introduction.

8.1. Lemma 3. *Let*

- $(n_k)_{k \in \mathbf{N}^*}$ be an increasing sequence of positive integers,
- $(\eta_k)_{k \in \mathbf{N}^*}$ be a decreasing sequence of positive real numbers, converging to 0,
- $(C_k)_{k \in \mathbf{N}^*}$ be a Cauchy sequence in $\mathcal{Q}$,
- C be a limit of $(C_k)_{k \in \mathbf{N}^*}$.

Let us suppose that, for all $i \in \mathbf{N}^*$ and all for $k \geq i$, C_k is (n_i, η_i) -uniform. Then C is an uniform subset.

Proof. Let us fix $\epsilon > 0$. For all $n \in \mathbf{N}^*$, all $k \in \mathbf{N}^*$ and all $w \in W$, we have

$$(1) \quad \left| \frac{1}{n} \sum_{i=0}^{n-1} 1_C(T^i(w)) - P(C) \right| \leq \left| \frac{1}{n} \sum_{i=0}^{n-1} 1_C(T^i(w)) - \frac{1}{n} \sum_{i=0}^{n-1} 1_{C_k}(T^i(w)) \right| + \left| \frac{1}{n} \sum_{i=0}^{n-1} 1_{C_k}(T^i(w)) - P(C_k) \right| + |P(C_k) - P(C)|.$$

Let $k_0 \in \mathbf{N}^*$ be such that, for all $k \geq k_0$, we have

$$(2) \quad |P(C_k) - P(C)| < \frac{\epsilon}{3}.$$

Let $k_1 \geq k_0$, be such that $\eta_{k_1} < \epsilon/3$; for all $k \geq k_1$, C_k is a (n_{k_1}, η_{k_1}) -uniform subset; hence, for almost all w , for all $k \geq k_1$ and all $n \geq n_{k_1}$, we get

$$(3) \quad \left| \frac{1}{n} \sum_{i=0}^{n-1} 1_{C_k}(T^i(w)) - P(C_k) \right| < \frac{\epsilon}{3}.$$

It follows from (1), (2), (3) that, for almost all w , for all $k \geq k_1$, and all $n \geq n_{k_1}$, we have

$$(4) \quad \left| \frac{1}{n} \sum_{i=0}^{n-1} 1_C(T^i(w)) - P(C) \right| \leq \left| \frac{1}{n} \sum_{i=0}^{n-1} 1_C(T^i(w)) - \frac{1}{n} \sum_{i=0}^{n-1} 1_{C_k}(T^i(w)) \right| + \frac{2\epsilon}{3}.$$

But, for all $n \in \mathbf{N}^*$,

$$P\left(\left\{w \in W / \left| \frac{1}{n} \sum_{i=0}^{n-1} 1_C(T^i(w)) - \frac{1}{n} \sum_{i=0}^{n-1} 1_{C_k}(T^i(w)) \right| \geq \frac{\epsilon}{3} \right\}\right) \leq nP(C \triangle C_k).$$

Therefore, it follows by (4), where we let k converge to infinity, that, for almost all w , and for all $n \geq n_k$, we get

$$\left| \frac{1}{n} \sum_{i=0}^{n-1} 1_C(T^i(w)) - P(C) \right| < \epsilon.$$

8.2. Lemma 4. *There exists $\mathcal{Q}''$, which is a subset of $\mathcal{Q}$, such that the following properties hold:*

- 1) $\mathcal{Q}''$ is a denumerable algebra,
- 2) $T^{-1}(\mathcal{Q}'') \subset \mathcal{Q}''$,
- 3) $\mathcal{Q}''$ is dense in $\mathcal{Q}$,
- 4) every element of $\mathcal{Q}''$ is a uniform subset of W .

Proof.

1°. Let $\tilde{\mathcal{A}}$ be a subset of $\mathcal{A}$, denumerable and dense in $\mathcal{A}$. Let $(A_k)_{k \in \mathbf{N}}$ be a sequence of elements of $\tilde{\mathcal{A}}$, such that, for all $A \in \tilde{\mathcal{A}}$, $\{k \in \mathbf{N} / A_k = A\}$ is infinite. Besides, consider a convergent series, the general term of which, ϵ_k , is strictly positive and decreasing to 0.

Let us apply Proposition 11 to sequences $(A_k)_{k \in \mathbf{N}}$ and $(\epsilon_k)_{k \in \mathbf{N}}$: we get sequences $(n_k)_{k \in \mathbf{N}^*}$ and $(\underline{A}^{(k)})_{k \in \mathbf{N}}$.

For all $m \in \mathbf{N}$, let $\mathcal{G}_m$ be the set of all sequences $(C_k)_{k \in \mathbf{N}}$ such that the following condition holds

$$(\forall k \geq m) : (C_k, C_{k+1}) \in \mathcal{C}^{(m+1)} \underline{A}^{(k)}, {}^{m+1} \underline{A}^{(k+1)}.$$

Let $\mathcal{G} = \bigcup_{m \in \mathbf{N}} \mathcal{G}_m$. It follows, by Proposition 5.3° and Proposition 11a) that all elements of $\mathcal{G}$ are Cauchy sequences; let $\mathcal{Q}'''$ be the set of their limits.

2°. Let us show that $\mathcal{Q}'''$ is an algebra of uniform subsets of W , such that $T^{-1}(\mathcal{Q}''') \subset \mathcal{Q}'''$.

—Let $A \in \mathcal{Q}'''$. There exists $m \in \mathbf{N}$, and a sequence $(C_k)_{k \geq m}$ which belongs to $\mathcal{G}_m$ and has A for a limit. It follows by Proposition 11b) that, for all $i \geq m+1$ and for all $k \geq i$, C_k is $(n_i, 4 \sum_{i \geq i} \epsilon_i)$ -uniform; then, we deduce from Lemma 3 that A is an uniform subset.

—Let A and A' be two elements in $\mathcal{Q}'''$; there exist two integers, denoted m and m' , and two sequences, denoted $(C_k)_{k \geq m}$ and $(C'_k)_{k \geq m'}$, which belong respectively to $\mathcal{G}_m$ and $\mathcal{G}_{m'}$, and have A and A' for limits; it follows by Proposition 5 that sequences $(C_k \cup C'_k)_{k \geq \sup(m, m')}$, $(C_k^c)_{k \geq m}$ and $(T^{-1}(C_k))_{k \geq m+1}$ all belong to $\mathcal{G}$; therefore, $A \cup A'$, A^c and $T^{-1}(A)$ all belong to $\mathcal{Q}'''$.

3°. For all $m \in \mathbf{N}$, let $(C_k^m)_{k \geq m}$ be a sequence which belongs to $\mathcal{G}_m$ and is such that $C_m^m = A_m$ (such a sequence exists according to Proposition 5.6°).

Let A'_m be a limit of sequence $(C_k^m)_{k \geq m}$. It follows by Proposition 11a) that

$$(1) \quad P(A_m \triangle A'_m) = P(C_m^m \triangle A'_m) \leq \sum_{k \geq m} \epsilon_{k+1}.$$

Let α'' be the subalgebra of α''' which is generated by the family of subsets $(T^{-i}(A'_m))_{(i, m) \in \mathbf{N}^2}$.

Properties 1), 2) and 4) evidently hold. Let us verify property 3). Let $A \in \tilde{\alpha}$ and $\epsilon > 0$. Let n be such that $\sum_{k \geq n} \epsilon_{k+1} \leq \epsilon$. There exists $m \geq n$, such that $A_m = A$. Then, it follows by (1) that $P(A \triangle A'_m) \leq \epsilon$; therefore, α'' is dense in $\tilde{\alpha}$. As $\tilde{\alpha}$ is dense in α , so is α'' .

9. Subalgebra α' of uniform subsets, dense in α and such that $T^{-1}(\alpha') = \alpha'$.
From now on, we do not suppose P non-atomic.

Theorem 1. *Let (W, α, P, T) be a dynamical system, with the following properties:*

- (i) α is a separable σ -algebra,
- (ii) T is an invertible mapping of W onto itself, measure preserving, and ergodic.

Then, there exists α' , subset of α , such that the following properties hold:

- 1) α' is a denumerable algebra;
- 2) $T^{-1}(\alpha') = \alpha'$;
- 3) α' is dense in α ;
- 4) Every element of α' is a uniform subset of W .

Proof. As T is ergodic, probability P is either totally atomic, or non-atomic. In the first case, the theorem is obvious.

So, let us suppose that P is non-atomic; then we can use Lemma 4, and, as $T^{-1}(\alpha'') \subset \alpha''$, it follows that, for all $n \in \mathbf{N}$, we get $T^n(\alpha'') \subset T^{n+1}(\alpha'')$. Let us now put $\alpha' = \bigcup_{n \in \mathbf{N}} T^n(\alpha'')$. The image by T of an uniform subset being a uniform subset, it is easy to see that conditions 1) to 4) hold.

10. Conjugacy and isomorphism.

10.1. Definitions-notations. Let (W, α, P, T) and $(X, \mathcal{B}, Q, U)$ be two dynamical systems.

a) *Isomorphism* (see [4], p. 53). (W, α, P, T) and $(X, \mathcal{B}, Q, U)$ are said to be *isomorphic* iff there exists $A \in \alpha$, $B \in \mathcal{B}$ and φ , which is an invertible and measure-preserving mapping from A onto B , in such a way that $P(A) = 1$, $Q(B) = 1$, $T(A) = A$, $U(B) = B$ and $U = \varphi \circ T \circ \varphi^{-1}$.

b) *Conjugacy* (see [4], p. 66). Let $\tilde{\alpha}$ (respect. $\tilde{\mathcal{B}}$) be the quotient algebra α/P (respect. $\mathcal{B}/Q$); if α' (respect. $\mathcal{B}'$) is a subalgebra of α (respect. of $\mathcal{B}$), and if P' (respect. Q') is the restriction of P to α' (respect. of Q to $\mathcal{B}'$), we denote by $\tilde{\alpha}'$ (respect. $\tilde{\mathcal{B}}'$) the algebra α'/P' (respect. $\mathcal{B}'/Q'$); we remark that we may consider $\tilde{\alpha}'$ (respect. $\tilde{\mathcal{B}}'$) as a subalgebra of $\tilde{\alpha}$ (respect. $\tilde{\mathcal{B}}$).

Mappings T and U induce mappings $\tilde{T}$ (from $\tilde{\mathfrak{A}}$ to $\tilde{\mathfrak{A}}$) and $\tilde{U}$ (from $\tilde{\mathfrak{B}}$ to $\tilde{\mathfrak{B}}$). $(W, \mathfrak{A}, P, T)$ and $(X, \mathfrak{B}, Q, U)$ are said to be *conjugate* iff there exists $\tilde{\varphi}$, a mapping from $\tilde{\mathfrak{A}}$ onto $\tilde{\mathfrak{B}}$, which is an algebraic isomorphism and an isometry, and such that the equality $\tilde{U} \circ \tilde{\varphi} = \tilde{\varphi} \circ \tilde{T}$ holds.

10.2. Uniquely ergodic system conjugate to a given system.

Theorem 2. *Let $(W, \mathfrak{A}, P, T)$ be an ergodic dynamical system; $\mathfrak{A}$ is supposed to be separable, and T is supposed to be invertible and measure preserving. Then, there exists $(X, \mathfrak{B}, Q, U)$, which is a uniquely ergodic dynamical system, conjugate to $(W, \mathfrak{A}, P, T)$.*

Proof.

1°. Let $\mathfrak{A}'$ be a sub-algebra of $\mathfrak{A}$ such that the conditions of Theorem 1 hold, and let X be the Stone space associated to $\mathfrak{A}'$ (we use the notations of 10.1.b): X is the set of all maximal filters on $\mathfrak{A}'$.

To each $A' \in \mathfrak{A}'$, we associate $\varphi'(A') = \{x \in X / \tilde{A}' \in x\}$ ($\tilde{A}'$ is the equivalence class of A'); let us put $\mathfrak{B}' = \varphi'(\mathfrak{A}')$. $\mathfrak{B}'$ is an algebra of subsets of X , which is isomorphic to $\tilde{\mathfrak{A}'}$.

X , equipped with the topology which has $\mathfrak{B}'$ as a base of clopen sets, is a metrizable space (because $\mathfrak{A}'$ is denumerable), compact and totally disconnected. Probability P induces a mapping Q' from $\mathfrak{B}'$ into $[0, 1]$, such that

- (a) if $B' \neq \emptyset$, then $Q'(B') > 0$,
- (b) Q' is σ -additive on $\mathfrak{B}'$, and $Q'(X) = 1$.

It follows that Q' has an unique extension as a probability Q on the Borel σ -algebra $\mathfrak{B}$ on X (that is the σ -algebra generated by $\mathfrak{B}'$) such that, for every non-void open subset D of X , $Q(D) > 0$; hence, every $E \in \mathfrak{B}$ such that $Q(E) = 1$ is everywhere dense.

The mapping T of W onto itself induces a bijective mapping U of X onto itself (because $T(\mathfrak{A}') = \mathfrak{A}'$), such that the following properties hold:

- i) $(\forall A' \in \mathfrak{A}') : \varphi'(T(A')) = U(\varphi'(A'))$,
- ii) U is a homeomorphism of X , measure-preserving for probability Q .

We have thus defined the dynamical system $(X, \mathfrak{B}, Q, U)$.

2°. φ' induces an isometry-algebraic isomorphism, let us say $\tilde{\varphi}'$, from $\tilde{\mathfrak{A}'}$ onto $\tilde{\mathfrak{B}'}$ (which is isomorphic to $\mathfrak{B}'$), such that the following property holds

$$(\forall \tilde{A}' \in \tilde{\mathfrak{A}'} : \tilde{\varphi}'(\tilde{T}(\tilde{A}')) = \tilde{U}(\tilde{\varphi}'(\tilde{A}'))).$$

$\tilde{\mathfrak{A}'}$ is dense in $\tilde{\mathfrak{A}}$ (because $\mathfrak{A}'$ is dense in $\mathfrak{A}$), and $\tilde{\mathfrak{B}'}$ is dense in $\tilde{\mathfrak{B}}$; hence, $\tilde{\varphi}'$ can be extended as an isometry from $\tilde{\mathfrak{A}}$ to $\tilde{\mathfrak{B}}$; let us denote by $\tilde{\varphi}$, this isometry. $\tilde{\varphi}$ is an algebraic isomorphism, and $\tilde{\varphi} \circ \tilde{T} = \tilde{U} \circ \tilde{\varphi}$.

We have thus shown that dynamical systems $(W, \mathfrak{A}, P, T)$ and $(X, \mathfrak{B}, Q, U)$ are conjugate.

3°. Let us denote by $\| \cdot \|$ the semi-norms in $\mathfrak{L}^{\infty}(W, \mathfrak{A}, P)$ and $\mathfrak{L}^{\infty}(X, \mathfrak{B}, Q)$. Let $B' \in \mathfrak{B}'$, and $A' \in \mathfrak{A}'$, such that $\varphi'(A') = B'$; we get

$$\left\| P(A') - \frac{1}{n} \sum_{i=0}^{n-1} 1_{A'} \circ T^i \right\| = \left\| Q(B') - \frac{1}{n} \sum_{i=0}^{n-1} 1_{B'} \circ U^i \right\|;$$

but, $\mathfrak{A}'$ being made of uniform subsets of W , we know that

$$\lim_{n \rightarrow \infty} \left\| P(A') - \frac{1}{n} \sum_{i=0}^{n-1} 1_{A'} \circ T^i \right\| = 0;$$

hence there exists $E \in \mathfrak{B}$, such that $Q(E) = 1$ and such that the sequence of continuous function $(n^{-1} \sum_{i=0}^{n-1} 1_{B'} \circ U^i)_{n \in \mathbb{N}}$ converges, uniformly on E , towards $Q(B')$; E is everywhere dense in X ; it follows that sequence $(n^{-1} \sum_{i=0}^{n-1} 1_{B'} \circ U^i)_{n \in \mathbb{N}}$ converges everywhere towards $Q(B')$.

Then, it follows by the individual ergodic theorem (*i.e.*, Birkhoff's theorem) that, for any probability on $\mathfrak{B}$ which is preserved by U , let us say Q_1 , and for all $B' \in \mathfrak{B}'$, we get $Q_1(B') = Q(B')$. Therefore, the dynamical system $(X, \mathfrak{B}, Q, U)$ is uniquely ergodic.

10.3. Uniquely ergodic system isomorphic to a given system.

Theorem 3. *Let $(W, \mathfrak{A}, P, T)$ be an ergodic dynamical system; W is supposed to be a polish space (*i.e.*, metrizable separable and complete), whose Borel σ -algebra is $\mathfrak{A}$; T is supposed to be an invertible and measure-preserving mapping of W onto itself. Then, there exists $(X, \mathfrak{B}, Q, U)$, which is a uniquely ergodic dynamical system, isomorphic to $(W, \mathfrak{A}, P, T)$.*

Proof. The hypotheses of Theorem 2 hold; therefore, there exists $(X, \mathfrak{B}, Q, U)$, which is a uniquely ergodic dynamical system, conjugate to $(W, \mathfrak{A}, P, T)$. Both $(X, \mathfrak{B})$ and $(W, \mathfrak{A})$ are polish spaces equipped with their Borel σ -algebras; then, it follows from a theorem by von Neumann (see [4], p. 69) that $(X, \mathfrak{B}, Q, U)$ and $(W, \mathfrak{A}, P, T)$ are isomorphic.

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Faculté des Sciences de Rouen
Mont-Saint-Aignan, France

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