A Note on Non-Quasitriangular Operators II

L. A. FIALKOW

Communicated by the Editors

1. Introduction. Let $\mathcal{H}$ be a fixed, separable, infinite dimensional, complex Hilbert space, and let $\mathcal{L}(\mathcal{H})$ denote the algebra of all bounded linear operators on $\mathcal{H}$. Let $\mathcal{O}$ denote the directed set of all finite rank projections in $\mathcal{L}(\mathcal{H})$ under the usual ordering, and for each T in $\mathcal{L}(\mathcal{H})$ define

$$q(T) = \liminf_{P \in \mathcal{O}} \|(1 - P)TP\| \quad \text{and} \quad Q(T) = \limsup_{P \in \mathcal{O}} \|(1 - P)TP\|.$$

In [8], Halmos initiated the study of quasitriangular operators and proved that an operator T is quasitriangular if and only if $q(T) = 0$. In [3], Douglas and Pearcy employed the η -function of Brown and Pearcy (see [11], [3]) to prove that T is a *thin* operator (*i.e.*, an operator that is the sum of a scalar and a compact operator) if and only if $Q(T) = 0$. The functions q and Q were studied, respectively, by Apostol in [1] and by Foiaş and Zsidó in [6]. We appreciatively acknowledge access to preliminary versions of [1] and [6].

Let (QT) denote the set of all quasitriangular operators in $\mathcal{L}(\mathcal{H})$. One purpose of this note is to prove (in Section 2) the conjecture of Pearcy which states that if T is in $\mathcal{L}(\mathcal{H})$, then $q(T)$ is equal to the distance from T to the set (QT) , *i.e.*, $q(T) = \inf_{S \in (QT)} \|T - S\|$. In Section 3 we prove that if T is in $\mathcal{L}(\mathcal{H})$ and S is the compression of T to a subspace of finite codimension in $\mathcal{H}$, then $q(T) = q(S)$ and $Q(T) = Q(S)$. A consequence is that if U is a nonunitary isometry in $\mathcal{L}(\mathcal{H})$, then the compression of U to a subspace of finite codimension in $\mathcal{H}$ is unitarily equivalent to a compact perturbation of U . Let (J) denote the ideal in $\mathcal{L}(\mathcal{H})$ of all compact operators, and let π denote the natural mapping of $\mathcal{L}(\mathcal{H})$ onto the quotient algebra $\mathcal{L}(\mathcal{H})/(J)$ (which is commonly known as the Calkin algebra). We prove that similarity in the Calkin algebra preserves quasitriangularity (a result which generalizes [4, Theorem 9]), and that unitary equivalence in the Calkin algebra preserves values of q . In Section 4 we prove that if T is an invertible operator such that $q(T) < 1/\|T^{-1}\|$, then $q(T^{-1}) \leq \|T^{-1}\|^2 \cdot q(T)/(1 - q(T)\|T^{-1}\|)$.

The following two lemmas will be used repeatedly in the sequel.

Lemma A. (*Apostol* [1]). *If A and B are in $\mathcal{L}(\mathcal{H})$, then $|q(A) - q(B)| \leq \|A - B\|$.*

Lemma B. (Foiş and Zsidó [6]). If T is in $\mathcal{L}(\mathcal{H})$ and A is a thin operator, then $q(T + A) = q(T)$.

2. The distance from an operator to the set (QT). If T is in $\mathcal{L}(\mathcal{H})$, let $d(T) = \inf_{Q \in (QT)} \|T - Q\|$. In Theorem 2.4 we prove that if T is in $\mathcal{L}(\mathcal{H})$, then $q(T) = d(T)$, and we thereby conclude that $q(T)$ is equal to the distance from T to the set of all quasitriangular operators in $\mathcal{L}(\mathcal{H})$. It is easy to prove that $q(T) \leq d(T)$ (see Section 2 of [5]). The proof of the reverse inequality depends on the following theorem of Apostol.

Theorem 2.1. (Apostol [1]). If T is in $\mathcal{L}(\mathcal{H})$, then there exists an increasing sequence $\{P_n\} \subset \mathcal{O}$, which converges to 1 in the strong operator topology, such that $q(T) = \lim \|(1 - P_n)TP_n\|$.

The proof of the following well-known result will be omitted.

Lemma 2.2. If K is a compact operator and $\{S_n\}$ is a sequence which converges to S in the strong operator topology, then $\{S_n K\}$ converges to SK in the uniform operator topology.

Lemma 2.3. Let ϵ be a fixed positive number. Let T be in $\mathcal{L}(\mathcal{H})$ and let $\{P_k\}$ be a sequence given by Theorem 2.1 such that $P_0 = 0$ and $|||(1 - P_1)TP_1| - q(T)| < \epsilon$. Then there is an increasing subsequence $\{P_{k_n}\}$ such that $P_{k_0} = 0$, $P_{k_1} = P_1$, and such that for each integer $n \geq 2$ the following inequalities are satisfied:

- i) $|||(P_{k_n} - P_{k_{n-1}})T(P_{k_{n-1}} - P_{k_{n-2}})| - q(T)| < \epsilon$.
- ii) $|||(1 - P_{k_n})T(P_{k_{n-1}} - P_{k_{n-2}})| < \epsilon/2^n$.
- iii) $|||(1 - P_{k_n})TP_{k_n}| - q(T)| < \epsilon/2^n$.

Proof. The proof is by induction on n . For each integer $j > 0$, we have

$$\begin{aligned} & | |(P_j - P_{k_1})TP_{k_1}| - q(T) | \\ & \leq | |(P_j - P_{k_1})TP_{k_1}| - |(1 - P_{k_1})TP_{k_1}| | + | |(1 - P_{k_1})TP_{k_1}| - q(T) | \\ & \leq | |(P_j - P_{k_1}) - (1 - P_{k_1})TP_{k_1}| | + | |(1 - P_{k_1})TP_{k_1}| - q(T) | \\ & = | |(1 - P_j)TP_{k_1}| | + | |(1 - P_{k_1})TP_{k_1}| - q(T) |. \end{aligned}$$

These inequalities, the hypothesis on P_1 , and an application of Lemma 2.2 to the finite rank operator TP_{k_1} , imply that there is an integer $j_1 > 0$ such that if $j \geq j_1$, then $|||(P_j - P_{k_1})TP_{k_1}| - q(T)| < \epsilon$ and $|||(1 - P_j)TP_{k_1}| < \epsilon/4$. The hypothesis also implies that there is an integer $j_2 > 0$ such that if $j \geq j_2$, then $|||(1 - P_j)TP_j| - q(T)| < \epsilon/4$. Therefore we may define $k_2 = \max \{j_1, j_2, k_1 + 1\}$.

Suppose that for an integer $n \geq 2$ the integers $k_1 < \dots < k_n$ have been determined so that the projections $\{P_{k_i}\}_{i=1}^n$ satisfy i)–iii). Now

$$\begin{aligned}
& | |(1 - P_{k_n})T(P_{k_n} - P_{k_{n-1}})| - q(T) | \\
& \leq | |(1 - P_{k_n})T(P_{k_n} - P_{k_{n-1}})| - |(1 - P_{k_n})TP_{k_n}| | \\
& \quad + | |(1 - P_{k_n})TP_{k_n}| - q(T) | \\
& \leq |(1 - P_{k_n})TP_{k_{n-1}}| + \epsilon/2^n \\
& = |(1 - P_{k_n})T(P_{k_1} + (P_{k_2} - P_{k_1}) + \cdots + (P_{k_{n-1}} - P_{k_{n-2}}))| + \epsilon/2^n \\
& \leq \sum_{j=2}^n |(1 - P_{k_j})T(P_{k_{j-1}} - P_{k_{j-2}})| + \epsilon/2^n \\
& \leq \sum_{j=2}^n \epsilon/2^j + \epsilon/2^n = \epsilon/2.
\end{aligned}$$

Next, for each integer $m > 0$, we have

$$\begin{aligned}
& | |(P_m - P_{k_n})T(P_{k_n} - P_{k_{n-1}})| - q(T) | \\
& \leq | |(P_m - P_{k_n})T(P_{k_n} - P_{k_{n-1}})| - |(1 - P_{k_n})T(P_{k_n} - P_{k_{n-1}})| | \\
& \quad + | |(1 - P_{k_n})T(P_{k_n} - P_{k_{n-1}})| - q(T) | \\
& \leq |(1 - P_m)T(P_{k_n} - P_{k_{n-1}})| + | |(1 - P_{k_n})T(P_{k_n} - P_{k_{n-1}})| - q(T) |.
\end{aligned}$$

The first set of inequalities above proves that $|||(1 - P_{k_n})T(P_{k_n} - P_{k_{n-1}})| - q(T)| < \epsilon$. Therefore, since $P_{k_n} - P_{k_{n-1}}$ is a finite rank operator, Lemma 2.2 and the last set of inequalities prove that there is an integer $m_1 > 0$ such that if $m \geq m_1$, then $|||(P_m - P_{k_n})T(P_{k_n} - P_{k_{n-1}})| - q(T)| < \epsilon$. Lemma 2.2 and the hypothesis also imply that there is an integer $m_2 > 0$ such that if $m \geq m_2$, then $|(1 - P_m)T(P_{k_n} - P_{k_{n-1}})| < \epsilon/2^{n+1}$ and $|||(1 - P_m)TP_m| - q(T)| < \epsilon/2^{n+1}$. The proof is completed by letting $k_{n+1} = \max \{m_1, m_2, k_n + 1\}$.

Recall that an operator T in $\mathcal{L}(\mathcal{H})$ is triangular if there exists an increasing sequence $\{P_k\} \subset \mathcal{P}$, which converges to 1 in the strong operator topology, such that for each k we have $P_k TP_k = TP_k$. Lemma B implies that if T is triangular and K is a compact operator, then $T + K$ is quasitriangular.

Theorem 2.4. *If T is in $\mathcal{L}(\mathcal{H})$, then $q(T) = d(T)$.*

Proof. In view of previous remarks, it suffices to prove that if $\epsilon > 0$, then there exists an operator Q in (QT) such that $\|T - Q\| \leq q(T) + \epsilon$. Let $\epsilon > 0$ and let $\{P_{k_n}\}$ be a sequence determined by Lemma 2.3. For each positive integer n , define the finite rank operator $F_n = \sum_{i=2}^n (1 - P_{k_i})T(P_{k_{i-1}} - P_{k_{i-2}})$. If $m > n$, then Lemma 2.3 implies that $\|F_m - F_n\| < \sum_{i=n+1}^m \epsilon/2^i$, and it follows that there is a compact operator K such that $\lim \|F_n - K\| = 0$. Now for each integer $j \geq 2$, define $D_j = (P_{k_j} - P_{k_{j-1}})T(P_{k_{j-1}} - P_{k_{j-2}})$. Lemma 2.3 implies that $\|D_j\| \leq q(T) + \epsilon$. Since the increasing sequence $\{P_{k_n}\}$ converges to 1 in the strong operator topology, it follows that if x is a vector in $\mathcal{H}$, then $x = \sum_{i=2}^{\infty} (P_{k_{i-1}} - P_{k_{i-2}})x$ and $\sum_{i=2}^{\infty} \|D_i x\|^2 \leq (\epsilon + q(T))^2 \|x\|^2$. If for each integer $i > 2$ we let $S_i = \sum_{j=2}^i D_j$, then the sequence $\{S_i\}$ converges in the strong

operator topology to an operator S such that $\|S\| \leq q(T) + \epsilon$. It follows that $Q_1 = T - (S + K)$ is a triangular operator. Therefore $Q = Q_1 + K$ is in (QT) and $\|T - Q\| = \|S\| \leq \epsilon + q(T)$.

Remark. The decomposition used in the proof of Theorem 2.4 may be considered a generalization of a technique, due to Halmos (see [9] and [10]), which is used to prove that if R is in (QT) and $\epsilon > 0$, then $R = T + K$, where T is a triangular operator and K is a compact operator such that $\|K\| < \epsilon$.

We further note that an adaptation of Theorem 2.4 can be used to prove that if T is in $\mathcal{L}(\mathcal{H})$, then $\liminf_{P \in \mathcal{P}} \|TP - PT\|$ is equal to the distance from T to the set of all quasidiagonal operators in $\mathcal{L}(\mathcal{H})$ (see [9]). Using a different approach, we can also prove that $\limsup_{P \in \mathcal{P}} \|TP - PT\| = Q(T)$. The details of the proofs of these results appear in the author's thesis.

3. The compression of an operator to a subspace of finite codimension.

Let $\mathfrak{M}$ be a subspace of $\mathcal{H}$ and let P be the projection onto $\mathfrak{M}$. If T is in $\mathcal{L}(\mathcal{H})$, let $T_{\mathfrak{M}}$ denote the compression of T to $\mathfrak{M}$, i.e., $T_{\mathfrak{M}}$ is the restriction of PT to $\mathfrak{M}$. Since $T_{\mathfrak{M}}$ is in $\mathcal{L}(\mathfrak{M})$, $q(T_{\mathfrak{M}})$, $d(T_{\mathfrak{M}})$, and $Q(T_{\mathfrak{M}})$ are defined.

Lemma 3.1. *If T is in $\mathcal{L}(\mathcal{H})$ and $\mathfrak{M}$ is a subspace of $\mathcal{H}$ such that $\dim \mathfrak{M}^\perp < \aleph_0$, then $q(T_{\mathfrak{M}}) = q(T)$.*

Proof. If $\epsilon > 0$, Theorem 2.4 implies that there is an operator S in (QT) such that $\|T - S\| < d(T) + \epsilon$. Since $\dim \mathfrak{M}^\perp < \aleph_0$, Theorem 8 of [4] implies that $S_{\mathfrak{M}}$ is quasitriangular, and therefore $d(T_{\mathfrak{M}}) \leq \|T_{\mathfrak{M}} - S_{\mathfrak{M}}\| \leq \|T - S\| < d(T) + \epsilon$. Since ϵ is arbitrary, we have $d(T_{\mathfrak{M}}) \leq d(T)$. To prove the reverse inequality, let $\delta > 0$, and let R denote a quasitriangular operator in $\mathcal{L}(\mathfrak{M})$ such that $\|T_{\mathfrak{M}} - R\| \leq d(T_{\mathfrak{M}}) + \delta$. With respect to the decomposition $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$, the operator $R \oplus 0$ is in (QT) , and $d(T_{\mathfrak{M}} \oplus 0) \leq \|T_{\mathfrak{M}} \oplus 0 - R \oplus 0\| = \|T_{\mathfrak{M}} - R\| \leq d(T_{\mathfrak{M}}) + \delta$. Since δ is arbitrary, we have $d(T_{\mathfrak{M}} \oplus 0) \leq d(T_{\mathfrak{M}})$. Since $q(T_{\mathfrak{M}} \oplus 0) = q(T)$, Theorem 2.4 implies that $q(T_{\mathfrak{M}}) = q(T)$.

The following result is the analogue of [1, Proposition 2.1] for the function Q and the proof is omitted.

Lemma 3.2. *Let T be in $\mathcal{L}(\mathcal{H})$. There is an increasing sequence $\{P_n\} \subset \mathcal{P}$ which converges to 1 in the strong operator topology, such that $Q(T) = \lim \|(1 - P_n)TP_n\|$. Moreover, $Q(T) = \sup \{ \liminf \|(1 - P_n)TP_n\| \mid \{P_n\} \subset \mathcal{P} \text{ converges to 1 in the strong operator topology} \}$.*

Lemma 3.3. *If T is in $\mathcal{L}(\mathcal{H})$ and $\mathfrak{M}$ is a subspace of $\mathcal{H}$ such that $\dim \mathfrak{M}^\perp < \aleph_0$, then $Q(T_{\mathfrak{M}}) = Q(T)$.*

Proof. Lemma 3.2 implies that there exists a sequence $\{P_n\} \subset \mathcal{L}(\mathfrak{M})$, whose members are finite rank projections which converge to 1 in the strong operator topology, such that $Q(T_{\mathfrak{M}}) = \lim \|(1 - P_n)TP_n\|$. With respect to the decomposition $\mathcal{H} = \mathfrak{M} \oplus \mathfrak{M}^\perp$, let $R_n = P_n \oplus 1$. Then the sequence $\{R_n\} \subset \mathcal{P}$ converges to 1 in the strong operator topology, and $\lim \|(1 - R_n)(T_{\mathfrak{M}} \oplus 0)R_n\| = Q(T_{\mathfrak{M}})$.

Since $Q(T) = Q(T_{\mathfrak{M}} \oplus 0)$, Lemma 3.2 implies that $Q(T_{\mathfrak{M}}) \leq Q(T)$. Now let $\{P_n\} \subset \mathcal{O}$ be a sequence which converges to 1 in the strong operator topology. Lemma 1.5 of [1] states that there is a sequence $\{S_n\} \subset \mathcal{O}$ such that $\lim \|P_n - S_n\| = 0$ and such that for each n we have $S_n \geq P$, where P is the projection onto $\mathfrak{M}^\perp$. Then $S_n = M_n \oplus 1$, where M_n is a finite rank projection in $\mathcal{L}(\mathfrak{M})$ and the sequence $\{M_n\}$ converges to 1 in the strong operator topology. Now

$$\begin{aligned}
& \|(1 - P_n)(T_{\mathfrak{M}} \oplus 0)P_n\| \\
&= \|((1 - S_n) - (P_n - S_n))(T_{\mathfrak{M}} \oplus 0)(S_n + (P_n - S_n))\| \\
&\leq \|(1 - S_n)(T_{\mathfrak{M}} \oplus 0)S_n\| + \|(1 - S_n)(T_{\mathfrak{M}} \oplus 0)(P_n - S_n)\| \\
&\quad + \|(P_n - S_n)(T_{\mathfrak{M}} \oplus 0)P_n\| \\
&\leq \|(1 - S_n)(T_{\mathfrak{M}} \oplus 0)S_n\| + 2\|P_n - S_n\| \|T_{\mathfrak{M}} \oplus 0\| \\
&= \|(1 - M_n)TM_n\| + 2\|P_n - S_n\| \|T_{\mathfrak{M}} \oplus 0\|.
\end{aligned}$$

Therefore, $\liminf \|(1 - P_n)(T_{\mathfrak{M}} \oplus 0)P_n\| \leq \liminf \|(1 - M_n)TM_n\| \leq Q(T_{\mathfrak{M}})$. Since the sequence $\{P_n\}$ is arbitrary, Lemma 3.2 implies that $Q(T_{\mathfrak{M}} \oplus 0) \leq Q(T_{\mathfrak{M}})$. Finally, since $\dim \mathfrak{M}^\perp < \aleph_0$, we have $Q(T_{\mathfrak{M}} \oplus 0) = Q(T)$, and the proof is complete.

Remark. It is possible to give a shorter proof of the preceding lemma by employing the relatively difficult Theorem 4 of [6]. Similarly, it is possible to give a longer, if more elementary, proof of Lemma 3.1 without recourse to Theorem 2.4.

In [2], Douglas proved that if U is a nonunitary isometry in $\mathcal{L}(\mathcal{H})$, then $U = S + K$, where S is a unilateral shift of suitable multiplicity and K is a compact operator. The following proposition is a mild extension of this result.

Proposition 3.4. *If U is a nonunitary isometry in $\mathcal{L}(\mathcal{H})$ and $\mathfrak{M}$ is a subspace of finite codimension in $\mathcal{H}$, then $U_{\mathfrak{M}} = S + K$, where S is a unilateral shift in $\mathcal{L}(\mathfrak{M})$, and K is a compact operator.*

Proof. From Lemma 3.1 we have $1 = q(U) = q(U_{\mathfrak{M}}) \leq \|U_{\mathfrak{M}}\| \leq 1$. Since $q(U_{\mathfrak{M}}) = \|U_{\mathfrak{M}}\| = 1$, Theorem 2 of [6] implies that $U_{\mathfrak{M}} = V + L$, where V is a nonunitary isometry and L is a compact operator. The above-mentioned result of Douglas states that $V = S + J$, where S is a unilateral shift and J is compact. Since $L + J$ is compact, the proof is complete.

Corollary 3.5. *If U is a nonunitary isometry (or a nonunitary coisometry) in $\mathcal{L}(\mathcal{H})$, and $\mathfrak{M}$ is a subspace of finite codimension in $\mathcal{H}$, then $U_{\mathfrak{M}}$ is unitarily equivalent to a compact perturbation of U .*

Proof. Since U is a nonunitary isometry, the preceding proposition implies that there exist unilateral shifts S in $\mathcal{L}(\mathcal{H})$ and T in $\mathcal{L}(\mathfrak{M})$ such that $U - S$ and $U_{\mathfrak{M}} - T$ are compact. Since $(U - S)_{\mathfrak{M}} = U_{\mathfrak{M}} - S_{\mathfrak{M}}$, it follows that $S_{\mathfrak{M}} - T$ is compact. Since T is a semi-Fredholm operator, it follows that $S_{\mathfrak{M}}$ is also

semi-Fredholm and index $T = \text{index } S_{\mathfrak{M}}$. The assumption that the codimension of $\mathfrak{M}$ is finite implies that $\text{index } S_{\mathfrak{M}} = \text{index } S$. Thus $\text{index } S = \text{index } T$, and the result follows directly.

The following result generalizes a theorem due to W. B. Arveson (see [4]) which states that every operator that is similar to a quasitriangular operator is also quasitriangular.

Proposition 3.6. *Let T and S be in $\mathcal{L}(\mathcal{H})$. If $q(T) = 0$ and $\pi(T)$ is similar to $\pi(S)$, then $q(S) = 0$.*

Proof. Let X be an invertible element of the Calkin algebra such that $X\pi(T) = \pi(S)X$. Problem 142 of [7] implies that there is a Fredholm operator Y such that $\pi(Y) = X$. Let K be a compact operator such that $YT - SY = K$. If the index of Y is zero, then there is a finite rank operator F such that $Y + F$ is invertible. If we set $J = Y + F$, then $JT = SJ - SF + FT + K$, and therefore $S - J TJ^{-1}$ is a compact operator. Then Theorem 9 of [4] and Lemma B imply that $q(S) = q(J TJ^{-1}) = q(T) = 0$. If the index of Y is negative, then there exists a finite rank operator G and an operator L such that $L(Y + G) = 1$. Let $\mathfrak{M}$ denote the range of $Y + G$. It follows that $\mathfrak{M}$ is a subspace of finite codimension in $\mathcal{H}$. Let P denote the projection onto $\mathfrak{M}$. Now $(Y + G)T = SY + GT + K$, and then

$$\begin{aligned} T &= LSY + LGT + LK \\ &= LS(Y + G) - LSG + LGT + LK \\ &= LPS(Y + G) + L(1 - P)S(Y + G) - LSG + LGT + LK. \end{aligned}$$

Therefore, we have $0 = q(T) = q(LPS(Y + G))$, and since $LPS(Y + G)$ is similar to $S_{\mathfrak{M}}$, Theorem 9 of [4] implies that $q(S_{\mathfrak{M}}) = 0$. Since $\dim \mathfrak{M}^{\perp} < \aleph_0$, Lemma 3.1 implies that $q(S) = q(S_{\mathfrak{M}}) = 0$. If the index of Y is positive, then there exists a finite rank operator H and an operator R such that $(Y + H)R = 1$. Let $\mathfrak{N}$ denote the range of R . It follows that $\mathfrak{N}$ is a subspace of finite codimension in $\mathcal{H}$. Let N denote the projection onto $\mathfrak{N}$. Now $YT + SH = S(Y + H) + K$, and therefore

$$\begin{aligned} S &= YTR + SHR - KR \\ &= (Y + H)TR - HTR + SHR - KR \\ &= (Y + H)NTR + (Y + H)(1 - N)TR - HTR + SHR - KR. \end{aligned}$$

Since $q(T) = 0$ and $\dim \mathfrak{N}^{\perp} < \aleph_0$, we have $q(T_{\mathfrak{N}}) = 0$, and since $(Y + H)NTR$ is similar to $T_{\mathfrak{N}}$, Theorem 9 of [4] implies that $q(S) = q((Y + H)NTR) = q(T) = 0$.

Theorem 3.7. *Let T and S be in $\mathcal{L}(\mathcal{H})$. If $\pi(T)$ is unitarily equivalent to $\pi(S)$ in the Calkin algebra, then $q(T) = q(S)$ and $Q(T) = Q(S)$.*

Proof. Let U be an operator in $\mathcal{L}(\mathcal{H})$ such that $\pi(U)$ is unitary and $\pi(U)\pi(T) = \pi(S)\pi(U)$. Now U is a Fredholm operator and [2, Theorem 1] implies that

U is a compact perturbation of a unitary operator, a nonunitary isometry, or a nonunitary coisometry, according as, respectively, $\text{index } U = 0$, $\text{index } U < 0$, or $\text{index } U > 0$. We will give the proof only in the case where $\text{index } U < 0$, since the other cases are proven similarly.

Let K be a compact operator such that $V = U + K$ is a nonunitary isometry. If P denotes the projection onto the range $\mathfrak{M}$ of V , then $1 - P$ is a finite rank projection, and it follows as in the proof of the preceding result that $q(T) = q(V^*PSV)$. Since V^*PSV is unitarily equivalent to $S_{\mathfrak{M}}$, Lemmas 3.1 and 3.3 imply that $q(T) = q(S_{\mathfrak{M}}) = q(S)$ and $Q(T) = Q(S_{\mathfrak{M}}) = Q(S)$.

Corollary 3.8. *If T is a Fredholm operator in $\mathcal{L}(\mathfrak{H})$ and $T = UP$ denotes the polar decomposition of T , then $q(UP) = q(PU)$ and $Q(UP) = Q(PU)$.*

4. Invertible operators. In [4], Douglas and Percy proved that if T is in $\mathcal{L}(\mathfrak{H})$, $\ker T^* \neq \{0\}$, and $\inf_{\|x\|=1} \|Tx\| > 0$, then $q(T) \geq \inf_{\|x\|=1} \|Tx\|$. If T is invertible, then $\inf_{\|x\|=1} \|Tx\| = 1/\|T^{-1}\|$, and it can happen that $q(T) < 1/\|T^{-1}\|$. For example, if $\alpha > 2$ and U is a unilateral shift of multiplicity one, then $1/\|(U - \alpha)^{-1}\| \geq \alpha - 1 > 1 = q(U - \alpha)$.

Proposition 4.1. *If T is an invertible operator in $\mathcal{L}(\mathfrak{H})$ and $q(T) < 1/\|T^{-1}\|$, then $q(T^{-1}) \leq \|T^{-1}\|^2 q(T)/(1 - q(T)\|T^{-1}\|)$.*

Proof. Theorem 2.4 implies that there exists a sequence $\{Q_n\} \subset (QT)$ such that for each integer $n > 0$, we have $\|T - Q_n\| < q(T) + 1/n$. The hypothesis implies that by passing, if necessary, to a subsequence, we may assume that if $n > 0$, then $q(T) + 1/n < 1/\|T^{-1}\|$. It follows that each Q_n is invertible, and Theorem 2.4 of [10] implies that Q_n^{-1} is in (QT) . The proof of [7, Problem 84] shows that

$$\begin{aligned} \|T^{-1} - Q_n^{-1}\| &\leq \|T^{-1}\| \|T - Q_n\| \|Q_n^{-1}\| \\ &\leq \|T^{-1}\| (q(T) + 1/n) \|T^{-1}\|/(1 - \|T - Q_n\| \|T^{-1}\|). \end{aligned}$$

Therefore, we have $q(T) = d(T) \leq \liminf \|T^{-1} - Q_n^{-1}\| \leq \|T^{-1}\|^2 q(T)/(1 - q(T)\|T^{-1}\|)$.

The following example provides a class of operators which satisfy the hypothesis of the preceding proposition.

Example 4.2. Let T be an invertible operator in $\mathcal{L}(\mathfrak{H})$ such that $\|T\| \|T^{-1}\| < 3$. We assert that $q(T) < 1/\|T^{-1}\|$. Let $T = UP$ denote the polar decomposition of T . Then U is unitary and the spectrum of P is contained in the interval $[1/\|P^{-1}\|, \|P\|]$. For each number α we have

$$\begin{aligned} q(T) &\leq q(\alpha U) + \|UP - \alpha U\| \\ &= \sup_{1/\|P^{-1}\| \leq t \leq \|P\|} |t - \alpha|. \end{aligned}$$

Since α is arbitrary, it follows that $q(T) \leq (\|P\| - 1/\|P^{-1}\|)/2 = (\|T\| - 1/\|T^{-1}\|)/2$. Since $\|T\| \|T^{-1}\| < 3$, we have $(\|T\| - 1/\|T^{-1}\|)/2 < 1/\|T^{-1}\|$, and the proof of the assertion is complete.

REFERENCES

1. C. APOSTOL, *Quasitriangularity in Hilbert Space*, Indiana Univ. Math. J. **22** (1973), 817–825.
2. L. G. BROWN, R. G. DOUGLAS & P. A. FILLMORE, *Unitary equivalence modulo the compact operators and extensions of C^* -algebras*, Proceedings of a Conference on Operator Theory, Springer-Verlag, New York, (to appear).
3. R. G. DOUGLAS & CARL PEARCY, *A characterization of thin operators*, Acta Sci. Math. (Szeged) **29** (1968), 295–297.
4. R. G. DOUGLAS & C. PEARCY, *A note on quasitriangular operators*, Duke Math. J. **37** (1970), 177–188.
5. L. A. FIALKOW, *A note on non-quasitriangular operators*, Acta Sci. Math. (Szeged) (to appear).
6. C. FOIAŞ & L. ZSIDÓ, *Some results on non-quasitriangular operators*, preprint.
7. P. R. HALMOS, *A Hilbert space problem book*, Van Nostrand, Princeton, 1967.
8. P. R. HALMOS, *Quasitriangular operators*, Acta Sci. Math. (Szeged) **29** (1968), 283–294.
9. P. R. HALMOS, *Ten problems in Hilbert space*, Bull. Amer. Math. Soc. **76** (1970), 887–933.
10. C. PEARCY & N. SALINAS, *An invariant subspace theorem*, Michigan Math. J. **20** (1973), 21–31.
11. N. SALINAS, *On the η -function of Brown and Percy and the numerical function of an operator*, Canad. J. Math. **23** (1971), 565–578.

This paper constitutes part of the author's Ph.D. thesis written at the University of Michigan under the direction of Professor Carl Pearcy.

University of Michigan

Date communicated: DECEMBER 29, 1972