Symplectic Structures on Integral Manifolds

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Let (M^{2n}, ω) be a symplectic manifold, $H : M \rightarrow \mathbf{R}$ a smooth function, and X_H the associated Hamiltonian system, *i.e.*, the unique tangent vectorfield determined by $X_H \lrcorner \omega = dH$. The Poisson bracket $\{f, g\}$ of two smooth functions f and g is defined by $\{f, g\} = \omega(X_f, X_g)$. A first integral for X_H is a smooth function f satisfying $\{f, H\} = 0$ (equivalently, f is constant on the trajectories of X_H). A set of first integrals $H_1 = H, H_2, \dots, H_k$ for X_H which are independent (*i.e.*, $dH_1, \dots, dH_k$ are everywhere linearly independent) define a codimension k foliation of M into 'regular integral manifolds'. Moreover, if $H_1, \dots, H_k$ are in involution, *i.e.*, $\{H_i, H_j\} = 0$ for all $i, j = 1, \dots, k$, then all X_{H_i} 's are tangent to the regular integral manifolds; it follows that in this case $k \leq n$. Note, furthermore, that a locally-free action of $\mathbf{R}^k$ on a regular integral manifold I is obtained if the vectorfields X_{H_i} are complete.

Our main result is:

Theorem. *Let (M^{2n}, ω) be a symplectic manifold, $\{H_1, \dots, H_{2r}\}$ an even number of functions in involution each generating a complete Hamiltonian vectorfield, and I a regular integral manifold. Then I admits a symplectic structure $\bar{\omega}$ relative to which each X_{H_i} is symplectic, *i.e.*, $X_{H_i} \lrcorner \bar{\omega}$ is closed.*

In the case of just two integrals, $H = H_1$ and H_2 , for a Hamiltonian system X_H the involution property is automatic. Thus for example:

Corollary. *In the planar n -body problem of Newtonian mechanics, every regular energy-angular momentum level I admits a symplectic structure $\bar{\omega}$ relative to which $X_H \mid I$ is a symplectic vectorfield.*

It should be pointed out that the incompleteness of the flow in the above example causes no difficulty. Although the forthcoming proof assumes completeness for the sake of elegance, the arguments also hold for a local one-parameter group of diffeomorphisms. The energy-angular momentum levels of the n -body problem have been explicitly described by Smale [3].

Before proving the theorem, we point out the following generalization of Darboux's theorem:

Lemma. If $H_1, \dots, H_k$ are independent functions in involution on (M^{2n}, ω) and $x \in M$, there are local coordinates $(H_1, \dots, H_k, H_{k+1}, \dots, H_n; x_1, \dots, x_n)$ around x so that $\omega = \sum dH_i \wedge dx_i$ and $X_{H_i} = \partial/\partial x_i, i \leq k$.

A proof of the above may be obtained by combining Darboux's theorem, Hamiltonian flow-box theorem [1], and an induction [2].

Proof of the theorem. Let ω_1 be the two-form on I defined by $\omega_1 = \omega|_I$. As $H_1, \dots, H_{2r}$ determine the integral manifold, it follows from the above lemma that a local expression for ω_1 is $\sum_{i>2r} dH_i \wedge dx_i$, and hence the rank of ω_1 is $2n - 4r$. We define a second two-form ω_2 on I as follows. Regarding the commuting complete vectorfields $X_{H_1}, \dots, X_{H_{2r}}$ as an action $\mathbf{R} \times I \rightarrow I$, let L be an orbit of the action. Since this action on L is transitive and locally-free, we may in fact regard it as an action $A : F \times L \rightarrow L$, where $F \approx \mathbf{R}^s \times T^{2r-s}$, $0 \leq s \leq 2r$, is the quotient group of $\mathbf{R}^{2r}$ by the discrete stabilizer of a point in L . Note also that F inherits a translation-invariant symplectic form ω_0 from the standard symplectic form on $\mathbf{R}^{2r}$. For each $x \in L$, the diffeomorphism $\varphi_x : F \rightarrow L$ defined by $\varphi_x(g) = A(g, x)$ then induces a symplectic 2-form $(\omega_2)_L$ on L . We show that $(\omega_2)_L$ is independent of $x \in L$. If $y \in L$ and φ_y is the corresponding diffeomorphism, a unique element $h \in F$ is defined by $(\varphi_y^{-1} \circ \varphi_x)(0) = h$, or equivalently, $x = \varphi_y(h)$. Now if g is an arbitrary element of F , $\varphi_x(g) = A(g, x) = A(g, \varphi_y(h)) = A(g, A(h, y)) = A(g + h, y)$, or $(\varphi_y^{-1} \circ \varphi_x)(g) = g + h$. Therefore $\varphi_y^{-1} \circ \varphi_x : F \rightarrow F$, being a translation, leaves ω_0 invariant, and hence $(\omega_2)_L$ is independent of $x \in L$.

The two-form ω_2 on I is then defined via its restrictions $(\omega_2)_L$ to the disjoint orbits L . The smoothness of ω_2 follows from the smoothness of the action of $\mathbf{R}^{2r}$ and the resulting foliation into orbits.

We next show that $\bar{\omega} = \omega_1 + \omega_2$ is the desired symplectic form on I . That $\bar{\omega}$ is closed is trivial. The ranks of ω_1 and ω_2 are respectively $2r$ and $2n - 4r$ which add up to the dimension of I . It then suffices to show that the kernels of the maps $X \mapsto i_X \omega_1$ and $X \mapsto i_X \omega_2$ are disjoint. The local expression $\omega_1 = \sum_{i>2r} dH_i \wedge dx_i$ shows that the kernel of the first map is in the direction of the orbits of the action $\mathbf{R}^{2r} \times I \rightarrow I$, i.e., in the direction determined by $x_1, \dots, x_{2r}$. But ω_2 is of maximal rank on the orbits, therefore the claim is proved.

Finally, to show each $X_{H_i}, i = 1, \dots, 2r$, is a symplectic vectorfield on $A(I, \bar{\omega})$, first note that $L_{X_{H(i)}} \omega_1 = 0$ by the local expression of ω_1 , where $H(i) = H_i$. Further, if $x \in I$ and L is the orbit through x , the symplectic diffeomorphism $\varphi_x^{-1} : (L, (\omega_2)_L) \rightarrow (F, \omega_0)$ takes the flow of X_{H_i} to a (symplectic) translation flow on F , and hence $L_{X_{H(i)}} \omega_2 = 0$. This finishes the proof.

Remarks.

(1) Let I be an integral manifold as in the theorem, and consider the equivalence relation ' $\sim$ ' on I of belonging to the same orbit of the action of $\mathbf{R}^{2r}$. The quotient space $\tilde{I} = I/\sim$, known as the 'reduced integral manifold', sometimes admits a natural manifold structure with the projection $p : I \rightarrow \tilde{I}$ a smooth locally trivial submersion. In this case the above proof has a more geometric

interpretation. By $\text{Diff}(F, \Omega)$ we shall mean the group of symplectic diffeomorphisms of a symplectic manifold (F, Ω) .

Lemma. *Let $p : E \rightarrow B$ be a smooth fiber bundle with fiber F and structural group G . If B and F admit symplectic structures ω_1 and ω_2 respectively, and if $G \subset \text{Diff}(F, \omega_2)$, then E admits a natural symplectic structure.*

Proof. For each $x \in B$, let E_x be the fiber over x . As $G \subset \text{Diff}(F, \omega_2)$, we may choose diffeomorphisms $\varphi_x : E_x \rightarrow F$ so that for all $x, y \in B$, $(\varphi_y \circ \varphi_x^{-1})^* \omega_2 = \omega_2$. φ_x then gives rise to a well-defined symplectic form on E_x that varies smoothly with x . We then obtain a smooth two-form ω'_2 on E . It is easy to check that $\omega = \omega'_2 + p^* \omega_1$ is symplectic on E .

Now consider the case where $p : I \rightarrow \tilde{I}$ is a smooth fiber bundle, and assume without the loss of generality that $\tilde{I}$ is connected. The first lemma of this paper shows that $(H_{2r+1}, \dots, H_n; x_{2r+1}, \dots, x_n)$ is a set of local coordinates for $\tilde{I}$, and $\tilde{I}$ admits a symplectic form $\sum_{i>2r} dH_i \wedge dx_i = \omega_1$. Moreover, our proof that $(\omega_2)_L$ is independent of $x \in L$ (see the proof of the theorem) can be interpreted as the reduction of the structural group of $p : I \rightarrow \tilde{I}$ to the group of (symplectic) translations of the fiber. Thus the resulting form $\bar{\omega}$ agrees with that given by the above lemma.

(2) Still in the above setting, if the integrals $H_1, \dots, H_{2r}$ arise from an action of the Lie group $F = \mathbb{R}^s \times T^{2r-s}$, $0 \leq s \leq 2r$, and if the orbits are homeomorphic to F , the proof actually shows that $p : I \rightarrow \tilde{I}$ has the structure of a principal fiber bundle. In particular for $s = 2r$, the fiber type is a solid space [4] and hence $p : I \rightarrow \tilde{I}$ admits a global section making it a trivial principal bundle.

(3) The methods above yield other results of similar nature, e.g., the total space of a smooth orientable two-plane bundle over a symplectic manifold admits a symplectic structure since $GL^+(2, \mathbb{R})/SL(2, \mathbb{R})$ is homeomorphic to the space of positive real numbers which is contractible; hence the structural group can be reduced [4].

(4) One cannot, in general, expect the restriction $X_{H_i} | I$ to be globally Hamiltonian on $(I, \bar{\omega})$. If I is compact, any smooth function on I has a critical point so that the corresponding Hamiltonian system cannot belong to an independent set of vectorfields. For example, in the case of $2r$ uncoupled harmonic oscillators with Hamiltonian functions $H_1, \dots, H_{2r}$, I is generically a $2r$ -dimensional torus with $X_{H_i} | I$ merely locally Hamiltonian.

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