On The Stepanov-Almost Periodic Solution of an Abstract Differential Equation

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Communicated by the Editors

1. Let J be the interval $-\infty < t < \infty$, X a Banach space and X^* the dual space of X . A continuous function $f: J \rightarrow X$ is said to be (strongly) almost periodic if, given $\epsilon > 0$, there is a positive real number $\ell = \ell(\epsilon)$ such that any interval of the real line of length ℓ contains at least one point τ for which

$$(1.1) \quad \sup_{t \in J} \|f(t + \tau) - f(t)\| \leq \epsilon.$$

We say that a function $f: J \rightarrow X$ is weakly almost periodic if the scalar-valued function $\langle x^*, f(t) \rangle = x^*f(t)$ is almost periodic for each $x^* \in X^*$.

For $1 \leq p < \infty$, a function $f \in L^p_{loc}(J; X)$ is said to be Stepanov-bounded or S^p -bounded on J if

$$(1.2) \quad \|f\|_{S^p} = \sup_{t \in J} \left[\int_t^{t+1} \|f(s)\|^p ds \right]^{1/p} < \infty.$$

For $1 \leq p < \infty$, a function $f \in L^p_{loc}(J; X)$ is said to be Stepanov-almost periodic or S^p -almost periodic if, given $\epsilon > 0$, there exists a positive real number $\ell = \ell(\epsilon)$ such that any interval of the real line of length ℓ contains at least one point τ for which

$$(1.3) \quad \sup_{t \in J} \left[\int_t^{t+1} \|f(s + \tau) - f(s)\|^p ds \right]^{1/p} \leq \epsilon.$$

Let $\mathfrak{L}(X, X)$ be the Banach space of all bounded linear operators on X into itself, with the uniform operator topology. An operator-valued function $T: J \rightarrow \mathfrak{L}(X, X)$ is called a (strongly) continuous group if

$$(1.4) \quad T(t_1 + t_2) = T(t_1)T(t_2) \quad \text{for all } t_1, t_2 \in J;$$

$$(1.5) \quad T(0) = I = \text{the identity operator on } X;$$

$$(1.6) \quad \text{for each } x \in X, T(t)x, t \in J \rightarrow X \text{ is continuous.}$$

A group $T : J \rightarrow \mathfrak{L}(X, X)$ is said to be almost periodic if $T(t)x, t \in J \rightarrow X$ is almost periodic for each $x \in X$.

The infinitesimal generator A of a continuous group T is a closed linear operator, with domain $D(A)$ dense in X , defined by

$$(1.7) \quad Ax = \lim_{t \rightarrow 0} \frac{T(t)x - x}{t} \quad \text{for } x \in D(A)$$

(see Dunford–Schwartz [3]).

The object of this paper is to establish the following main result.

Theorem 1. Suppose $f : J \rightarrow X$ is an S^p -almost periodic continuous function ($1 \leq p < \infty$), A is the infinitesimal generator of an almost periodic group $T : J \rightarrow \mathfrak{L}(X, X)$ and $u : J \rightarrow D(A)$ is a (strong) solution of the differential equation

$$(1.8) \quad u'(t) = [A + B(t)]u(t) + f(t) \text{ on } J,$$

where $B : J \rightarrow \mathfrak{L}(X, X)$ is almost periodic with respect to the norm of $\mathfrak{L}(X, X)$. Then, if u is S^p -almost periodic, it is also almost periodic.

2. For the proof of Theorem 1, we require the following lemmas.

Lemma 1. Any solution of (1.8) admits the representation

$$(2.1) \quad u(t) = T(t)u(0) + \int_0^t T(t-s)[B(s)u(s) + f(s)] ds \quad \text{on } J.$$

Proof. See Zaidman [9].

Lemma 2. If $g : J \rightarrow X$ is almost periodic, and if T is defined as in Theorem 1, then the function $T(t)g(t)$ is almost periodic from J to X .

Proof. See Günzler–Zaidman [4, 5, Lemma 1] and Zaidman [8].

Lemma 3. If f and T are defined as in Theorem 1, then the function $T(t)f(t)$ is S^p -almost periodic from J to X .

Proof. See Günzler–Zaidman [4, 5, Lemma 2].

3. **Proof of Theorem 1.** From (2.1), we obtain

$$(3.1) \quad T(-t)u(t) = u(0) + \int_0^t T(-s)[B(s)u(s) + f(s)] ds \quad \text{on } J.$$

We write

$$(3.2) \quad w(t) = B(t)u(t) + f(t) \text{ on } J.$$

Since B is almost periodic from J to $\mathfrak{L}(X, X)$, we have

$$(3.3) \quad \sup_{t \in J} \|B(t)\| = M < \infty.$$

Further, since u is S^p -almost periodic from J to X , it is S^p -bounded on J .

Now, given $\epsilon > 0$, suppose that τ is an ϵ -almost period of B and also an ϵ - S^p -almost period of u (see pp. 10, 77 and 78, Amerio-Prouse [1]). Then we have

$$\begin{aligned}
 (3.4) \quad & \left[\int_t^{t+1} \|B(s+\tau)u(s+\tau) - B(s)u(s)\|^p ds \right]^{1/p} \\
 & \leq \left[\int_t^{t+1} \|B(s+\tau) - B(s)\|^p \|u(s+\tau)\|^p ds \right]^{1/p} \\
 & \quad + \left[\int_t^{t+1} \|B(s)\|^p \|u(s+\tau) - u(s)\|^p ds \right]^{1/p} \\
 & \leq \epsilon \|u\|_{S^p} + M\epsilon \quad \text{on } J, \text{ by (1.2) and (3.3).}
 \end{aligned}$$

Hence $B(t)u(t)$ is S^p -almost periodic from J to X . Consequently, w is S^p -almost periodic from J to X .

Evidently, $T(-t)$, $t \in J \rightarrow \mathcal{L}(X, X)$ is an almost periodic group. So, by Lemma 3, $T(-t)u(t)$ and $T(-t)w(t)$ are both S^p -almost periodic from J to X .

Now we write

$$(3.5) \quad v(t) = \int_0^t T(-s)w(s) ds \quad \text{on } J.$$

Then, by Theorem VIII, p. 79, Amerio-Prouse [1], v is uniformly continuous on J .

By (3.1), (3.2) and (3.5), we have

$$(3.6) \quad T(-t)u(t) = u(0) + v(t) \quad \text{on } J.$$

Consequently, it follows that $T(-t)u(t)$ is uniformly continuous on J . Hence, by Theorem VII, p. 78, Amerio-Prouse [1], $T(-t)u(t)$ is almost periodic from J to X . Therefore, by Lemma 2, $T(t)[T(-t)u(t)] = u(t)$ is almost periodic from J to X , which completes the proof of the theorem.

4. Note. Here we prove the following result.

Theorem 2. Suppose $f: J \rightarrow X$ is a weakly almost periodic (or an S^p -almost periodic) continuous function, and A is the infinitesimal generator of a continuous group $T: J \rightarrow \mathcal{L}(X, X)$, with the group of adjoint operators $T^*: J \rightarrow \mathcal{L}(X^*, X^*)$ being almost periodic. Then, if $u: J \rightarrow D(A)$ is an S^p -almost periodic solution of the equation (1.8), it is also weakly almost periodic (X a Banach space and $1 \leq p < \infty$).

To demonstrate this theorem, we need the following lemma.

Lemma 4. If $y(t)$ is weakly almost periodic from J to X , and if $z(t)$ is almost periodic from J to the dual space X^* , then the scalar-valued function $z(t)y(t)$ is almost periodic.

For the proof, see p. 72, Amerio-Prouse [1].

Proof of Theorem 2. By our assumption, for an arbitrary but fixed $x^* \in X^*$, $x^*T(-t) = T^*(-t)x^*$ is almost periodic from J to X^* , and hence is bounded on J . Also, by Lemma 4, $x^*T(-t)f(t)$ is almost periodic from J to the scalars.

Now proceeding on the lines of the proof of Lemma 2, Günzler-Zaidman [4], we can show that $x^*T(-t)u(t)$ and $x^*T(-t)[B(t)u(t) + f(t)]$ are both S^p -almost periodic from J to the scalars.

By (2.1), we have

$$(4.1) \quad x^*T(-t)u(t) = x^*u(0) + \int_0^t x^*T(-s)[B(s)u(s) + f(s)] ds \quad \text{on } J.$$

From (4.1), it follows that $x^*T(-t)u(t)$ is almost periodic from J to the scalars. Therefore $T(-t)u(t)$ is weakly almost periodic from J to X . Again by Lemma 4, $T(t)[T(-t)u(t)] = u(t)$ is weakly almost periodic from J to X .

By a similar argument, we obtain the desired conclusion when f is S^p -almost periodic from J to X .

It is a pleasure to thank Prof. S. Zaidman for his useful comments as well as for the financial support from his N.R.C. grant during the preparation of this paper.

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Date communicated: JUNE 5, 1972