The Cartan–Janet Theorem for Conformal Embeddings

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Introduction. The Cartan–Janet Theorem [1], [6] asserts that any real analytic n -dimensional Riemannian manifold can be locally mapped by a real analytic isometric embedding into a Euclidean space of dimension $\frac{1}{2}n(n+1)$. It is known (compare [4, Appendix 1]) that this result is dimension-wise the best possible; *i.e.*, there exist real analytic n -dimensional Riemannian manifolds which do not possess smooth local isometric embeddings in any Euclidean space of dimension strictly less than $\frac{1}{2}n(n+1)$. If we seek a conformal embedding, however, we expect to be able to take a Euclidean space of one lower dimension. We will give two proofs that this is so, the first patterned upon Janet’s original paper, the second upon Cartan’s. It is hoped that the juxtaposition of the two proofs will help to clarify the relationship between the approaches of Cartan and Janet to the isometric embedding problem.

1. Conformal embeddings: Janet’s approach. We are interested in local results and take for the n -dimensional Riemannian manifold an open set U in $\mathbf{R}^n$ together with some metric. Let H be a hypersurface in U , p a given point of H , and f a map from H or U into E^N where E^N is Euclidean space of dimension N , *i.e.*, $\mathbf{R}^N$ together with the usual metric and the usual coordinates. For $f : H \rightarrow U$ an immersion we define $\mathcal{O}_p(f(H))$ to be the osculating space of H at the point p , with H thought of as a submanifold of E^N . For any choice of local coordinates, $\mathcal{O}_p(f(H))$ is the linear space in E^N spanned by $\{f_{x_i}(p), f_{x_i x_k}(p)\}$. We say f is nondegenerate if the osculating space is of maximal rank at p and hence in a neighborhood of p . We use $\vdash$ both for the connection in U and in E^N , context makes clear which is meant. We shall always denote arbitrary elements in $T_p(H)$ by X or Y and use Z for a unit vector in $T_p(U)$ which is orthogonal to $T_p(H)$. Denote the second fundamental form of H in U by L and of $f(H)$ in E^N by $\bar{L}$. So $L(X, Y) = \langle Z, X \vdash \tilde{Y} \rangle$, for any extension of Y . Similarly for $\bar{L}$. Wherever necessary we use $\langle \cdot, \cdot \rangle_e$ and L_e to indicate the dependence on a specific metric. The norm of a vector in E^n is written $\|v\|$.

L takes values in $\mathbf{R}^1$ and $\bar{L}$ in the normal space to $f(H)$ at the point $f(p)$. Given a map $f : U \rightarrow E^N$ or $f : H \rightarrow E^N$ we denote by $F(f)$ the induced metric on U or H . We seek a conformal embedding. That is, given the metric g we try to find an embedding $f : U \rightarrow E^N$ and a positive function $\lambda : U \rightarrow \mathbf{R}^1$ with $F(f) = \lambda g$.

To save words later, we now note that some of our arguments are valid only if one replaces U and H by their intersections with a small ball centered at p . However, we try to state the theorems correctly.

If $f : U \rightarrow E^N$ is isometric, then

$$L(X, Y) = \langle X \vdash \bar{Y}, Z \rangle = \langle \bar{L}(f_*X, f_*Y), f_*Z \rangle$$

where the inner product is taken in E^N and f_* of course is the induced map $f_* : T_p U \rightarrow T_{f(p)} E^N$.

Definition. We write $\bar{L} > L$ if there exists $w \in T_{f(p)}(E^N)$ such that for all X and Y in $T_p H$ one has

$$L(X, Y) = \langle \bar{L}(f_*X, f_*Y), w \rangle, \quad \langle w, f_*X \rangle = 0 \text{ and } \|w\| < 1.$$

If only $\|w\| \leq 1$, then we define $\bar{L} \geq L$.

We have defined $\bar{L} > L$ with respect to the given map f and point p . Note we need only be given the map f restricted to H in order for this definition to be meaningful. So given an isometric embedding $f : H \rightarrow E^N$, the condition $\bar{L} \geq L$ is necessary for the existence of an extension $\bar{f}$ of f , $\bar{f} : U \rightarrow E^N$, $\bar{f}|_H = f$, which is an isometric embedding of U . With a nondegeneracy condition on f and suitable N we can show $\bar{L} > L$ is sufficient for such an extension [5]. Consider the case where f induces on H a metric λg conformal to g . Further assume this positive scalar function λ has an extension $\bar{\lambda}$ to U with the property that at p one has $\bar{L} > L_{\bar{\lambda}p}$. Then we claim that f can be extended to a conformal embedding of U into E^N , provided $N \geq \frac{1}{2}n(n+1) - 1$, f is nondegenerate, and everything is real analytic.

Choose local coordinates $x_1, \dots, x_{n-1}, y$ with $\partial/\partial y$ orthogonal to H with respect to the metric g and so with respect to metrics conformal to g . Also take $\langle \partial/\partial y, \partial/\partial y \rangle_p = 1$. We identify the tangent space of E^N at any point with E^N itself and write $u \cdot v$ for $\langle u, v \rangle_{E^N}$. A conformal mapping f satisfies the system

$$(1) \quad \begin{cases} f_v \cdot f_v = \lambda, \\ f_v \cdot f_{x_i} = 0, \\ f_{x_i} \cdot f_{x_j} = \lambda g_{ij}, \end{cases}$$

and hence also satisfies

$$(2) \quad \begin{cases} f_v \cdot f_v = \lambda, \\ f_v \cdot f_{x_i} = 0, \\ f_v \cdot f_{x_i x_j} = -\frac{1}{2}(\lambda g_{ij})_v, \end{cases}$$

and

$$(3) \quad \begin{cases} f_{vv} \cdot f_v = \frac{1}{2} \lambda_v, \\ f_{vv} \cdot f_{x_i} = -\frac{1}{2} \lambda_{x_i}, \\ f_{vv} \cdot f_{x_i x_j} + \frac{1}{2} \lambda_{vv} g_{ii} = -\frac{1}{2} \lambda (g_{ii})_{vv} - \lambda_v (g_{ii})_v - \frac{1}{2} \lambda_{x_i x_j} + f_{vx_i} \cdot f_{vx_j}. \end{cases}$$

Throughout this section we use $1 \leq i \leq j \leq n-1$. Conversely any solution to (3) which satisfies (2) on H is also a solution to (1). Thus, if we can find analytic functions $u \in \mathbf{R}^{n(n+1)/2-1}$ and $\mu \in \mathbf{R}^1$ defined on H and there satisfying $u \cdot u = \lambda$, $u \cdot f_{x_i} = 0$, $u \cdot f_{x_i x_j} = -\frac{1}{2} \lambda (g_{ii})_v - \frac{1}{2} \mu g_{ii}$ and further if the vectors $\{(u, 0), (f_{x_i}, 0), (f_{x_i x_j}, g_{ii})\}$ are linearly independent, then we can apply the Cauchy-Kowalewski theorem to (3) and in this way obtain a solution to (1).

Now let us explore the condition $\bar{L} > L_{\bar{\lambda}_0}$. For $N \geq \frac{1}{2}n(n+1) - 1$ and f nondegenerate we can find some analytic $w : H \rightarrow E^N$ satisfying $\|w\| < 1$, $w(q)$ orthogonal to $f(H)$ at q , and $w \cdot (f_*(\partial/\partial x_i) \vdash f_*(\partial/\partial x_i)) = \langle \mu(\partial/\partial y), \partial/\partial x_i \vdash \partial/\partial x_i \rangle_{\bar{\lambda}_0}$. Here we must take $\|\mu(\partial/\partial y)\|_{\bar{\lambda}_0} = 1$ and so $\mu = \lambda^{-1/2}$. Note that by itself $f_*(\partial/\partial x_i) \vdash f_*(\partial/\partial x_i)$ makes no sense, since it depends on the choice of an extension of $f_*(\partial/\partial x_i)$ off of $f(H)$, but $w \cdot f_*(\partial/\partial x_i) \vdash f_*(\partial/\partial x_i)$ is well defined.

Since $\partial/\partial y$ is orthogonal to H we have with respect to the metric $\bar{\lambda}g$

$$\begin{aligned} \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial x_i} \vdash \frac{\partial}{\partial x_i} \right\rangle &= \Gamma_{ii}^n \left\langle \frac{\partial}{\partial y}, \frac{\partial}{\partial y} \right\rangle \\ &= -\frac{1}{2} \bar{\lambda}^{-1} (\bar{\lambda} g_{ii})_v \left\| \frac{\partial}{\partial y} \right\|^2 \\ &= -\frac{1}{2} (\bar{\lambda} g_{ii})_v. \end{aligned}$$

Finally, since $w \cdot f_*(\partial/\partial x_i) \vdash f_*(\partial/\partial x_i) = w \cdot f_{x_i x_j}$, we obtain $w \cdot f_{x_i x_j} = -\frac{1}{2} \lambda^{-1/2} (\bar{\lambda} g_{ii})_v$.

Lemma. *If $\bar{L} > L_{\bar{\lambda}_0}$ for some extension $\bar{\lambda}$ of λ , then there exists a possibly different extension $\hat{\lambda}$ of λ , and an E^N -valued real analytic function u such that on H*

$$u \cdot u = \lambda, u \cdot f_{x_i} = 0, \text{ and } u \cdot f_{x_i x_j} = -\frac{1}{2} (\hat{\lambda} g_{ii})_v.$$

Moreover, we can arrange that $\{(u, 0), (f_{x_i}, 0), (f_{x_i x_j}, g_{ii}) \mid 1 \leq i \leq j \leq n-1\}$ be a linearly independent set of $(N+1)$ -vectors.

Proof. With w and $\bar{\lambda}$ as above, $u = \lambda^{1/2} w$ clearly satisfies the equations $u \cdot f_{x_i} = 0$ and $u \cdot f_{x_i x_j} = -\frac{1}{2} (\bar{\lambda} g_{ii})_v$.

Now consider the linear algebraic system $v \cdot f_{x_i} = 0, v \cdot f_{x_i x_j} = g_{ii}$, parametrized by points of H . Since f is nondegenerate at p these equations have a unique solution for $N = \frac{1}{2}n(n+1) - 1$, and this solution is analytic on H . For $N > \frac{1}{2}n(n+1) - 1$, we can obtain an analytic solution by requiring that $v(p) \in \mathcal{O}_p(f(H))$. For such a value of N we can also modify the choice of w by adding to it any small vector orthogonal at each point to $\mathcal{O}(f(H))$. So for $N > \frac{1}{2}n(n+1) - 1$ we assume $v \in \mathcal{O}(f(H))$ and $w \notin \mathcal{O}(f(H))$.

We now choose $\sigma : H \rightarrow \mathbf{R}$ analytic, such that $\|\lambda^{1/2}w + \sigma v\| = \lambda^{1/2}$; this is possible since $\|w\| < 1$. We obtain a quadratic equation for σ whose roots have opposite sign, and choose the positive root. It follows that $[\lambda^{1/2}w + \sigma v] \cdot v > 0$.

Take as a new extension of λ the function $\hat{\lambda}(x, y) = \lambda(x) + y(\hat{\lambda}_v(x, 0) - 2\sigma(x))$. For $u = \lambda^{1/2}w + \sigma v$ we have $u \cdot u = \lambda$, $u \cdot f_{x_i} = 0$, and $u \cdot f_{x_i x_i} = -\frac{1}{2}(\lambda g_{ii})_u$.

Let S denote the set of vectors of the lemma. We consider $N > \frac{1}{2}n(n+1) - 1$ and $N = \frac{1}{2}n(n+1) - 1$ separately. In the former case, by our choice of v and w , we have already arranged that $\mu_1 w + \mu_2 v \notin \{f_{x_i}, f_{x_i x_i}\}$ as long as $\mu_1 \neq 0$. So $u = \lambda^{1/2}w + \sigma v$ is independent of $\{f_{x_i}, f_{x_i x_i}\}$ and S has dimension $\frac{1}{2}n(n+1)$. If $N = \frac{1}{2}n(n+1) - 1$, we show $(a, \mu) \cdot b = 0$ for all $b \in S$ implies $a = 0$ and $\mu = 0$. From $f_{x_i} \cdot a = 0$ and $f_{x_i x_i} \cdot a = -\mu g_{ii}$, we conclude $a = -\mu v$. So we are done if $\mu = 0$. But otherwise $(u, 0) \cdot (a, \mu) = 0$ implies $u \cdot v = 0$, and this is impossible since $u \cdot v = [\lambda^{1/2}w + \sigma v] \cdot v > 0$.

We now consider an initial value problem for conformal embeddings (Theorem 1) and the generalization of the Cartan-Janet result (Theorem 2).

Theorem 1. *Let $f : H^{n-1} \rightarrow E^N$, $N \geq \frac{1}{2}n(n+1) - 1$, be an analytic non-degenerate mapping inducing a metric conformal to $g|_H$, i.e., $F(f) = \lambda g|_H$ where $\lambda : H \rightarrow \mathbf{R}^+$. If there exists some extension $\tilde{\lambda} : U \rightarrow \mathbf{R}^+$ of λ for which $\tilde{L} > L_{\tilde{\lambda}_v}$ at $p \in H$, then there exists an analytic conformal embedding in a U -neighborhood of p which extends f .*

Proof. We have already seen that $\tilde{L} > L_{\tilde{\lambda}_v}$ implies the existence of some analytic $u : H \rightarrow E^N$ satisfying $\|u\|^2 = \lambda$, $u \cdot f_{x_i} = 0$, and $u \cdot f_{x_i x_i} = -\frac{1}{2}(\lambda g_{ii})_u + \sigma g_{ii}$, σ also analytic. Further the vectors $\{(f_{x_i}, 0), (u, 0), (f_{x_i x_i}, g_{ii})\}$ are linearly independent. Let us take the given values of f and λ together with $f_v(x, 0) = u$, $\lambda_v(x, 0) = \tilde{\lambda}_v - 2\sigma$ as initial values and try to solve the system (3). The above indicated linear independence of certain vectors means this initial value problem can be put in normal form and the Cauchy-Kowalewski theorem may be applied to find real analytic functions $f : U \rightarrow E^N$, $\lambda : U \rightarrow \mathbf{R}^+$ satisfying (3). But the initial conditions were chosen to satisfy (2) on H and so f and λ satisfy (1) (even off of H).

Note that we have used the Cauchy-Kowalewski Theorem for the equation $Bh_{vv} = C$ where h is a k vector and C a k' vector with $k \geq k'$, i.e., a possibly underdetermined system. This is legitimate since for B of maximal rank we can find a right inverse $\hat{B}$ and solve the determined system $h_{vv} = \hat{B}C$.

We now show that conformal embeddings are always possible.

Theorem 2. *Let g be a real analytic metric on some neighborhood U of the origin in $\mathbf{R}^n$. There exists a real analytic map f of some smaller neighborhood U' into E^N , $N = \frac{1}{2}n(n+1) - 1$, which induces on U' a metric conformal to g .*

Remark. As we shall see one can even take f to be isometric on an appropriate codimension one submanifold of U' .

For the proof we rely on the following lemma:

Lemma. *With g , U , and some U' as above, there exists an analytic, non-degenerate, isometric embedding $h : U' \rightarrow E^k$, $k = \frac{1}{2}n(n+1) + n$.*

This lemma is also employed in the proof of the Cartan-Janet Theorem and indeed is equivalent to this theorem, see *e.g.*, [5]. We shall not reprove it here.

Let us now prove Theorem 2. In U pick a hypersurface H which is geodesic at the origin. In terms of our usual choice of coordinates this implies $g_{ii,y} = 0$. Hence any nondegenerate isometric embedding of H^{n-1} into some E^N has $\bar{L} > L_g$. By the lemma, we can find an isometric embedding $h : H \rightarrow E^N$, $N = \frac{1}{2}n(n+1) + 1$, which is nondegenerate and by Theorem 1 we can extend this isometric embedding to a map $f : U \rightarrow E^N$ which induces a metric conformal to the original metric.

2. Cartan-Kähler theory. In this section we give a brief introduction to the Cartan-Kähler theory of partial differential equations. More details can be found in [7] or [2]. We will assume throughout that all manifolds, differential forms, *etc.*, are real analytic.

If M is an n -dimensional differentiable manifold, we let $\Lambda_k(M)$ denote the vector space of differential k -forms on M , $\Lambda(M) = \sum_{k=0}^n \Lambda_k(M)$. The wedge product makes $\Lambda(M)$ into an algebra over the reals. A *differential ideal* is an ideal $\mathfrak{A}$ in $\Lambda(M)$ which is finitely-generated, homogeneous, and closed under exterior differentiation; here homogeneity means that

$$\mathfrak{A} = \sum_{k=0}^n \mathfrak{A}_k, \quad \text{where } \mathfrak{A}_k = \mathfrak{A} \cap \Lambda_k(M).$$

We will make the standing assumption in this paper that our differential ideals do not contain functions, *i.e.*, that $\mathfrak{A}_0 = 0$.

Let N be a p -dimensional submanifold of M with inclusion map $i : N \rightarrow M$. (We remark that in the following discussion we will often omit explicit mention of the inclusion map when convenient.) N is said to be an *integral submanifold* for $\mathfrak{A}$ if $i^*(\mathfrak{A}) = 0$, or equivalently if $i^*(\mathfrak{A}_p) = 0$. The objective of Cartan-Kähler theory is to find integral submanifolds for differential ideals.

Let $T_x(M)$ denote the tangent plane to M at x , $G_x^p(M)$ the Grassmann manifold of p -dimensional subspaces of $T_x(M)$. We let $G^p(M) = \cup \{G_x^p(M) : x \in M\}$, and give it the usual manifold structure. An element $E_x^p \in G_x^p(M)$ is called an *integral element* for $\mathfrak{A}$ if all of the differential forms of $\mathfrak{A}$ vanish when evaluated on elements of E_x^p . Notice that tangent planes to integral submanifolds are integral elements. Let $I_x^p(\mathfrak{A})$ denote the set of p -dimensional integral elements for $\mathfrak{A}$ at x , and let $I^p(\mathfrak{A}) = \cup \{I_x^p(\mathfrak{A}) : x \in M\}$, topologized as a subspace of $G^p(M)$.

Suppose that E_x^p is a p -dimensional integral element for $\mathfrak{A}$, and that $\{v_1, v_2, \dots, v_p\}$ is a basis for E_x^p . Then the *polar space* $H(E_x^p)$ is defined by

$$H(E_x^p) = \{v \in T_x(M) : \Phi(v, v_1, \dots, v_p) = 0, \text{ for all } \Phi \in \mathfrak{A}_{p+1}\}.$$

A $(p+1)$ -dimensional subspace of $T_x(M)$ containing E_x^p is an integral element

if and only if it is contained in $H(E_x^p)$. We define a function $r_{p+1} : I^p(\mathfrak{A}) \rightarrow \mathbf{R}$ by

$$r_{p+1}(E_x^p) = \dim H(E_x^p) - p - 1;$$

then E_x^p is contained in a $(p + 1)$ -dimensional integral element if and only if $r_{p+1}(E_x^p) \geq 0$.

A zero-dimensional integral element E_x^0 will be called *regular* if r_1 is constant on a neighborhood of E_x^0 in $I^0(\mathfrak{A})$. We define ordinary and regular integral elements of dimension $p \geq 1$ in the following inductive fashion: A p -dimensional integral element E_x^p is *ordinary* if it contains a $(p - 1)$ -dimensional regular integral element. E_x^p is *regular* if it is ordinary and r_{p+1} is constant on a neighborhood of E_x^p in $I^p(\mathfrak{A})$. This definition is somewhat different than Kähler's definition of regular integral element [7, p. 23], but it can be proven that an integral element which is regular in the above sense also satisfies Kähler's definition. (This fact is proven in [9, Theorem VIII.6, p. 366], and it also follows from Kähler's criterion for regular chains [7, pp. 44–47].) On the other hand, if M is connected it can be shown that our definition of regularity is equivalent to Cartan's [2, pp. 61–67], according to which an integral element E_x^p is regular if it is ordinary and $r_{p+1}(E_x^p) = r_{p+1}(\tilde{E}_x^p)$, where $\tilde{E}_x^p$ is a generic p -dimensional ordinary integral element. We will say that an integral submanifold is *regular* if its tangent planes are regular integral elements.

Cartan–Kähler Theorem. *Let N be a connected p -dimensional regular integral submanifold for $\mathfrak{A}$, and let r_{p+1} denote the constant value of $r_{p+1}(T_x(N))$ as x varies throughout N . Suppose that F is an $(n - r_{p+1})$ -dimensional submanifold of M which contains N and has the property that $(T_x F) \cap H(T_x N)$ is $(p + 1)$ -dimensional, for all $x \in N$. Then there exists a $(p + 1)$ -dimensional integral submanifold $\tilde{N}$ for $\mathfrak{A}$ such that $N \subset \tilde{N} \subset F$. If $\tilde{N}'$ is another $(p + 1)$ -dimensional integral submanifold such that $N \subset \tilde{N}' \subset F$, then $\tilde{N} = \tilde{N}'$ in a neighborhood of N .*

This theorem is proven by analytic continuation from Kähler's first existence theorem [7, p. 26]; see also [2, pp. 69–75]. An induction based upon the Cartan–Kähler Theorem shows that if E_x^q is any q -dimensional ordinary integral element, then there exists a q -dimensional integral submanifold S which contains x and satisfies the requirement that $T_x S = E_x^q$.

A system of q independent variables on M is a nowhere zero decomposable q -form Θ defined on M . We will say that a differential ideal $\mathfrak{A}$ is *in involution* with respect to the system Θ of q independent variables if at each point $x \in M$ there exists a q -dimensional ordinary integral element E_x^q on which Θ does not vanish. If $\mathfrak{A}$ is in involution with respect to Θ and $x \in M$, it follows from the Cartan–Kähler Theorem that there exists a q -dimensional integral submanifold for $\mathfrak{A}$ which contains x , and on which Θ does not vanish.

3. Conformal embeddings: Cartan's approach. Let M and $\tilde{M}$ be real analytic Riemannian manifolds with Riemannian metrics g and $\tilde{g}$ respectively. Recall

that an embedding $f : M \rightarrow \tilde{M}$ is conformal if $f^*(\tilde{g}) = \lambda g$, where λ is some smooth positive real-valued function defined on M . Suppose that $\dim M = n$, $\dim \tilde{M} = N$. In this section we will give the proof based upon Cartan-Kähler theory that conformal embeddings of M in $\tilde{M}$ exist locally, when $N \geq \frac{1}{2}n(n+1) - 1$. We will adopt the index conventions:

$$1 \leq I, J, K \leq N, \quad 1 \leq i, j, k, l \leq n, \quad n+1 \leq \lambda, \mu, \nu \leq N.$$

Let $F(M)$ denote the total space of the bundle of orthonormal frames over M ; thus

$$F(M) = \{(x, e_1, \dots, e_n) : x \in M \text{ and } (e_1, \dots, e_n) \text{ is an orthonormal basis for } T_x M\},$$

with the usual manifold structure. Let $\pi : F(M) \rightarrow M$ denote the usual projection. If $a = (a_{ij})$ is an element of the orthogonal group $O(n)$, we define a diffeomorphism L_a of $F(M)$ by

$$L_a(x, e_1, \dots, e_n) = (x, \sum_i a_{1i} e_i, \dots, \sum_i a_{ni} e_i).$$

This left action makes $F(M)$ into a principal fiber bundle over M with structural group $O(n)$. We introduce the canonical forms $\theta_1, \dots, \theta_n$ on $F(M)$, which are defined by requiring that

$$\pi_{*z}(v) = \sum_i \theta_i(v) e_i, \quad \text{when } z = (x, e_1, \dots, e_n) \in F(M) \text{ and } v \in T_z(F(M)).$$

Notice that

$$(4) \quad \pi^*g = \sum_i \theta_i \otimes \theta_i, \quad \text{and hence } \sigma^*(\sum_i \theta_i \otimes \theta_i) = g|_U,$$

whenever σ is a cross-section of the bundle of orthonormal frames defined over an open subset U of M .

According to the fundamental theorem of Riemannian geometry, there exists a unique torsion-free connection on $F(M)$, the so-called Levi-Civita connection. It is described by the unique collection of differential one-forms ω_{ij} on $F(M)$ which satisfy the structure equations:

$$(5) \quad d\theta_i = \sum_j \omega_{ij} \wedge \theta_j, \quad \omega_{ij} + \omega_{ji} = 0.$$

We mention some well-known properties of the Levi-Civita connection; compare [8] for proofs. Under the action of $O(n)$, the forms θ_i, ω_{ij} transform as follows:

$$(6) \quad L_a^* \theta_i = \sum_j a_{ij} \theta_j, \quad L_a^* \omega_{ij} = \sum_{k,l} a_{ik} a_{jl} \omega_{kl}.$$

The one-forms θ_i, ω_{ij} ($i < j$) are linearly independent and span the cotangent space at every point of $F(M)$. Finally, if we let

$$(7) \quad \Omega_{ij} = d\theta_{ij} - \sum_k \theta_{ik} \wedge \theta_{kj},$$

there exist functions R_{ijkl} defined on $F(M)$ such that

$$\Omega_{ij} = -\frac{1}{2} \sum_{k,l} R_{ijkl} \theta_k \wedge \theta_l, \quad R_{ijkl} = -R_{ijlk}.$$

The R_{ikl} 's are the components of the Riemann-Christoffel curvature tensor, and they possess the following symmetries:

$$R_{ikl} = -R_{ilk} = R_{kli}, \quad R_{ikl} + R_{jkl} + R_{kjl} = 0.$$

We let $F(\tilde{M})$ denote the total space of the bundle of orthonormal frames over $\tilde{M}$, $\pi : F(\tilde{M}) \rightarrow \tilde{M}$ the projection, $\tilde{\theta}_i$ the canonical forms on $F(\tilde{M})$, $\tilde{\omega}_{IJ}$ the Levi-Civita connection forms, etc.

We will apply the Cartan-Kähler theory to a differential ideal constructed on the product manifold

$$B = F(M) \times F(\tilde{M}) \times \mathbf{R}^+ \times \mathbf{R}^n,$$

where $\mathbf{R}^+$ denotes the positive real numbers. Let $\rho : B \rightarrow F(M)$, $\tilde{\rho} : B \rightarrow F(\tilde{M})$, $s : B \rightarrow \mathbf{R}^+$, $t : B \rightarrow \mathbf{R}^n$ denote the usual projections, t_i the i^{th} component of the $\mathbf{R}^n$ -valued function t . Using ρ and $\tilde{\rho}$ we can pull the differential forms θ_i , ω_{ij} , $\tilde{\theta}_i$, $\tilde{\omega}_{IJ}$ back to B , and we will denote the pulled-back forms by the same symbols. Let $\tilde{\mathfrak{A}}$ denote the differential ideal on B generated by

$$\tilde{\theta}_i - s\theta_i, \tilde{\theta}_\lambda, d\tilde{\theta}_i - d(s\theta_i), d\tilde{\theta}_\lambda.$$

We claim that if S is an n -dimensional integral submanifold for $\tilde{\mathfrak{A}}$ on which $\theta_1 \wedge \cdots \wedge \theta_n \neq 0$ and $z \in S$, then S defines a conformal embedding from some open neighborhood U of $(\pi \circ \rho)(z)$ into $\tilde{M}$. Indeed, by the inverse function theorem there exists a diffeomorphism h from some open neighborhood U of $(\pi \circ \rho)(z)$ onto an open neighborhood of z in S , such that $\pi \circ \rho \circ h = \text{identity}$ on U . Define $f : U \rightarrow \tilde{M}$ by $f = \tilde{\pi} \circ \tilde{\rho} \circ h$. Then by (4)

$$f^*(g) = h^*(\sum_I \theta_I \otimes \theta_I) = h^*(s^2 \sum_i \theta_i \otimes \theta_i) = s^2 g,$$

since $\rho \circ h$ is a section of $F(M)$ defined over U . Thus we find that f is a conformal immersion, and hence a conformal embedding if U is sufficiently small.

However, the differential ideal $\tilde{\mathfrak{A}}$ is not in involution with respect to the system of independent variables $\theta_1 \wedge \cdots \wedge \theta_n$ if $n > 2$. In fact, if S is an integral submanifold for $\tilde{\mathfrak{A}}$ on which $\theta_1 \wedge \cdots \wedge \theta_n \neq 0$, it follows from the equation

$$d\tilde{\theta}_i - d(s\theta_i) = \sum_j [s\tilde{\omega}_{ij} - s\omega_{ij} - \delta_{ij} ds] \wedge \theta_j = 0$$

and Cartan's Lemma that

$$\tilde{\omega}_{ij} - \omega_{ij} - (ds/s)\delta_{ij} = \sum_k f_{ijk}\theta_k,$$

where the f_{ijk} 's are smooth functions defined on S such that $f_{ijk} = f_{ikj}$. A calculation based on the identities $f_{ikj} = f_{ijk} = -f_{jik}$ now shows that

$$\tilde{\omega}_{ij} = \omega_{ij} + f_i\theta_j - f_j\theta_i, \quad \text{where} \quad ds/s = \sum_k f_k\theta_k.$$

These equations yield additional conditions which the one-dimensional integral

elements must satisfy if they are to be contained in n -dimensional integral elements on which $\theta_1 \wedge \cdots \wedge \theta_n \neq 0$. These conditions are not satisfied by the generic one-dimensional integral elements for $\tilde{\mathfrak{A}}$ if $n > 2$, and this fact can be used to show that $\tilde{\mathfrak{A}}$ is not in involution with respect to $\theta_1 \wedge \cdots \wedge \theta_n$ when $n > 2$. (See [2, pp. 130–132] for the case $n = 2$).

The above considerations lead us to introduce the differential forms

$$(8) \quad \eta = ds/s - \sum_k t_k \theta_k, \quad \varphi_{ij} = \omega_{ij} + t_i \theta_j - t_j \theta_i,$$

which are defined on B . We then have the structure equation

$$(9) \quad d(s\theta_i) = \eta \wedge (s\theta_i) + \sum_j \varphi_{ij} \wedge (s\theta_j).$$

Let $\mathfrak{A}$ be the ideal in $\Lambda(B)$ which is generated by the one-forms

$$(10) \quad \tilde{\theta}_i - s\theta_i, \tilde{\theta}_\lambda, \eta, \tilde{\omega}_{ij} - \varphi_{ij},$$

and the two-forms

$$(11) \quad d\tilde{\theta}_\lambda, d\eta, d\tilde{\omega}_{ij} - d\varphi_{ij}.$$

It is readily verified that $d(\tilde{\theta}_i - s\theta_i)$ lies in $\mathfrak{A}$, and hence $\mathfrak{A}$ is a differential ideal which contains $\tilde{\mathfrak{A}}$. We remark that $\mathfrak{A}$ can be regarded as a partial prolongation of $\tilde{\mathfrak{A}}$ ([2, p. 114]). Since $\tilde{\mathfrak{A}} \subseteq \mathfrak{A}$, if S is an n -dimensional integral submanifold for $\mathfrak{A}$ on which $\theta_1 \wedge \cdots \wedge \theta_n \neq 0$ and $z \in S$, S will define a conformal embedding from an open neighborhood of $(\pi \circ \rho)(z)$ in M to $\tilde{M}$, just as before.

There is a left action of the orthogonal group $O(n)$ on B which preserves the differential ideal $\mathfrak{A}$. Namely, if $a = (a_{ij}) \in O(n)$, we let L_a be the diffeomorphism of B which takes

$$z = ((x, e_1, \cdots, e_n), (x, \tilde{e}_1, \cdots, \tilde{e}_N), s, (t_1, \cdots, t_n))$$

to the point

$$L_a(z) = ((x, \sum_i a_{1i} e_i, \cdots, \sum_i a_{ni} e_i), (x, \sum_i a_{1i} \tilde{e}_i, \cdots, \sum_i a_{ni} \tilde{e}_i, \tilde{e}_{n+1}, \cdots, \tilde{e}_N), s, (\sum_i a_{1i} t_i, \cdots, \sum_i a_{ni} t_i)).$$

Using (6) and the corresponding equations of transformation for $\tilde{\theta}_I, \tilde{\omega}_{IJ}$, we verify that $L_a^*(\mathfrak{A}_1) \subseteq \mathfrak{A}_1$ (where $\mathfrak{A}_1 = \mathfrak{A} \cap \Lambda_1(B)$), and hence

$$(12) \quad L_a^*(\mathfrak{A}) = \mathfrak{A}, \quad \text{when } a \in O(n).$$

In order to simplify the calculation of the exterior derivatives in (11), we introduce the differential forms

$$\psi_i = dt_i + \sum_j t_j \omega_{ji} + \sum_j f_{ij} \theta_j,$$

where the functions f_{ij} will be determined later subject to the requirement that

$f_{ii} = f_{ii}$. Notice that if z is any point of B , the differential one-forms

$$\theta_i, \varphi_{ij} \quad \text{for } i < j, \tilde{\theta}_I, \tilde{\omega}_{IJ} \quad \text{for } I < J, \eta, \psi_i,$$

yield a basis for the cotangent space T_z^*B when they are evaluated at z . It follows from (5) that

$$d\eta = d(\sum_i t_i \theta_i) = \sum_i \psi_i \wedge \theta_i.$$

A more lengthy calculation based upon (5), (7), and (8) shows that

$$\begin{aligned} d\varphi_{ij} &= \sum_k \omega_{ik} \wedge \omega_{kj} + \Omega_{ij} + dt_i \wedge \theta_j - dt_j \wedge \theta_i \\ &\quad + \sum_k t_i \omega_{jk} \wedge \theta_k - \sum_k t_j \omega_{ik} \wedge \theta_k \\ &= \sum_k \varphi_{ik} \wedge \varphi_{kj} + \psi_i \wedge \theta_j - \psi_j \wedge \theta_i + \Omega_{ij} \\ &\quad + [\sum_k (t_i \theta_k - t_k \theta_i) \wedge (t_k \theta_j - t_j \theta_k) - \sum_k f_{ik} \theta_i \wedge \theta_k + \sum_k f_{jk} \theta_k \wedge \theta_i]. \end{aligned}$$

We now determine the f_{ij} 's so that the expression in brackets vanishes, *i.e.*, we set

$$f_{ij} = -t_i t_j, \quad \text{if } i \neq j, \quad f_{ii} = \frac{1}{2} \sum_k (t_k)^2 - (t_i)^2.$$

Let us use the expression $(\text{mod } \mathfrak{A}_1)$ to denote that an equation holds up to the addition of an element in the ideal generated by the one-forms (10). It is readily verified that

$$\begin{aligned} d\tilde{\theta}_\lambda &= \sum_i \tilde{\omega}_{\lambda i} \wedge \theta_i \quad (\text{mod } \mathfrak{A}_1), \\ d\tilde{\omega}_{ij} - d\varphi_{ij} &= \sum_\lambda \tilde{\omega}_{i\lambda} \wedge \tilde{\omega}_{\lambda j} + \tilde{\Omega}_{ij} - \psi_i \wedge \theta_j + \psi_j \wedge \theta_i - \Omega_{ij} \quad (\text{mod } \mathfrak{A}_1) \\ &= -\sum_\lambda \tilde{\omega}_{\lambda i} \wedge \tilde{\omega}_{\lambda j} - \psi_i \wedge \theta_j + \psi_j \wedge \theta_i \\ &\quad + \frac{1}{2} \sum_{k,l} (R_{ijkl} - s^2 \tilde{R}_{ijkl}) \theta_k \wedge \theta_l \quad (\text{mod } \mathfrak{A}_1). \end{aligned}$$

We therefore find that the ideal $\mathfrak{A}$ is generated by (10) and

$$(13) \quad \sum_i \tilde{\omega}_{\lambda i} \wedge \theta_i, \quad \sum_i \psi_i \wedge \theta_i, \\ \sum_\lambda \tilde{\omega}_{\lambda i} \wedge \tilde{\omega}_{\lambda j} + \psi_i \wedge \theta_j - \psi_j \wedge \theta_i - \frac{1}{2} \sum_{k,l} S_{ijkl} \theta_k \wedge \theta_l$$

where we have introduced the functions

$$S_{ijkl} = R_{ijkl} - s^2 \tilde{R}_{ijkl},$$

which satisfy the usual curvature symmetries.

Now let p be an integer, $0 \leq p < n$, and adopt the additional index conventions:

$$1 \leq a, b, c \leq p; \quad p+1 \leq r, s, t \leq n.$$

Suppose that E_z^p is a p -dimensional integral element for $\mathfrak{A}$ such that $\pi_* \circ \rho_*(E_z^p)$ is p -dimensional. In view of (6) we can then find an element $a \in 0(n)$ such that $\theta_{p+1} = \cdots = \theta_n = 0$ on $L_{a*}(E_z^p)$. Thus we will assume for the time being that $\theta_r = 0$ on E_z^p , and hence that E_z^p is generated by vectors $v_1, \dots, v_p$ such that

$$\theta_a(v_b) = \delta_{ab}, \quad \theta_r(v_b) = 0.$$

Then we define the quantities

$$h_{ia}^\lambda = \bar{\omega}_{\lambda i}(v_a), \quad l_{ia} = \psi_i(v_a),$$

which must satisfy certain equations, since the two-forms (13) vanish when restricted to E_z^p :

$$(14) \quad h_{ab}^\lambda = h_{ba}^\lambda, \quad l_{ab} = l_{ba},$$

$$\sum_\lambda h_{ia}^\lambda h_{jb}^\lambda - \sum_\lambda h_{ib}^\lambda h_{ja}^\lambda + l_{ia} \delta_{jb} - l_{ib} \delta_{ja} - l_{ia} \delta_{ib} + l_{jb} \delta_{ia} = S_{ijab}.$$

Finally, we let H_{ia} denote the vector $(\delta_{ia}, h_{ia}^{n+1}, \dots, h_{ia}^N)$ in $(N - n + 1)$ -dimensional euclidean space E^{N-n+1} .

Let J^p denote the set of p -dimensional integral elements E_z^p which satisfy the following conditions:

1. $\theta_1 \wedge \cdots \wedge \theta_p \neq 0$ and $\theta_{p+1} = \cdots = \theta_n = 0$ when restricted to E_z^p .
2. The vectors $\{H_{ma} : 1 \leq a \leq p, a \leq m \leq n-1\}$ are linearly independent. We then set $U^p = \{E_z^p \in I^p(\mathfrak{A}) : L_{a*}(E_z^p) \in J^p \text{ for some } a \in 0(n)\}$, and verify that U^p is an open subset of $I^p(\mathfrak{A})$.

Regularity Lemma. (a) If $0 \leq p < n$, then the function r_{p+1} is constant on U^p . (b) If $0 \leq p < n$, then J^p (and hence U^p) consists of regular p -dimensional integral elements.

Proof of (a): By (12) it suffices to show that r_{p+1} is constant on J^p . Suppose that E_z^p is an element of J^p . Then the polar space $H(E_z^p)$ consists of the vectors $v \in T_z B$ which satisfy the following "polar system" of linear equations:

$$(15) \quad s \bar{\theta}_i(v) - \theta_i(v) = 0, \quad \bar{\theta}_\lambda(v) = 0, \quad \eta(v) = 0,$$

$$(16) \quad \bar{\omega}_{i\lambda}(v) - \varphi_{i\lambda}(v) = 0,$$

$$(17) \quad \bar{\omega}_{\lambda a}(v) - \sum_i h_{ia}^\lambda \theta_i(v) = 0, \quad \psi_a(v) - \sum_i l_{ia} \theta_i(v) = 0,$$

$$(18) \quad \sum_\lambda h_{ia}^\lambda \bar{\omega}_{\lambda i}(v) - \sum_\lambda h_{ia}^\lambda \bar{\omega}_{\lambda i}(v) + l_{ia} \theta_i(v) - l_{ia} \theta_i(v)$$

$$- \delta_{ia} \psi_i(v) + \delta_{ia} \psi_i(v) - \sum_k S_{i i a k} \theta_k(v) = 0.$$

If we introduce the vectors

$$H_i(v) = (\psi_i(v), \tilde{\omega}_{n+1,i}(v), \dots, \tilde{\omega}_{N_i}(v)),$$

then (18) simplifies to become

$$(18) \quad H_{ia} \cdot H_i(v) = H_{ia} \cdot H_i(v) + l_{ia}\theta_i(v) - l_{ia}\theta_i(v) + \sum_k S_{iak}\theta_k(v),$$

where the dot denotes the ordinary dot product in E^{N-n+1} . If we specify $\theta_i(v)$, $\varphi_{i,i}(v)$, then equations (15), (16), and (17) will determine uniquely

$$(19) \quad \tilde{\theta}_I(v), \quad \eta(v), \quad \tilde{\omega}_{i,i}(v), \quad \tilde{\omega}_{\lambda_a}(v), \quad \psi_a(v);$$

in particular the vectors $H_a(v)$ are uniquely determined.

We still need to apply (18). We remark first that in the case where $1 \leq i$, $j \leq p$, (18) is an immediate consequence of (14) and (17). Hence we can replace (18) by

$$(20) \quad H_{ba} \cdot H_s(v) = H_{sa} \cdot H_b(v) + l_{sa}\theta_b(v) - l_{ba}\theta_s(v) + \sum_k S_{bsak}\theta_k(v)$$

and

$$(21) \quad H_{ra} \cdot H_s(v) = H_{sa} \cdot H_r(v) + l_{sa}\theta_r(v) - l_{ra}\theta_s(v) + \sum_k S_{rsak}\theta_k(v),$$

where to avoid duplication we will stipulate that $r < s$.

It follows from (14) that $H_{ba} = H_{ab}$, and hence (20) yields two distinct equations for $H_{ba} \cdot H_s(v)$, when $a \neq b$. The right-hand sides of these equations must agree if the system (20) is to be consistent. Fortunately, a short calculation based upon (14), (17), and the Bianchi symmetry $S_{bsak} + S_{sabk} + S_{absk} = 0$ shows that the right-hand sides do in fact agree. It therefore suffices to consider (20) in the case where $a \leq b$.

We now determine the vectors $H_{p+1}(v), \dots, H_n(v)$ in succession so that they satisfy (20) and (21). The system (20) determines the dot product of $H_{p+1}(v)$ with the $\frac{1}{2}p(p+1)$ linearly independent vectors H_{ba} ($a \leq b$); thus the components of $H_{p+1}(v)$ must satisfy a consistent linear system of rank $\frac{1}{2}p(p+1)$. Once we have chosen a particular $H_{p+1}(v)$ which satisfies this system, the dot product of $H_{p+2}(v)$ with each of the $\frac{1}{2}p(p+1) + p$ linearly independent vectors $\{H_{ma} : 1 \leq a \leq p, a \leq m \leq p+1\}$ is completely determined by (20) and (21). We continue in this fashion. In the final step we find that the dot product of $H_n(v)$ with each of the $\frac{1}{2}p(p+1) + p(n-p-1)$ linearly independent vectors $\{H_{ma} : 1 \leq a \leq p, a \leq m \leq n-1\}$ is completely determined.

Thus we find that the components $\tilde{\omega}_{\lambda_r}(v)$, $\psi_r(v)$ must satisfy a consistent system of linear equations which has rank

$$\sum_{k=1}^{n-1} [\tfrac{1}{2}p(p+1) + kp].$$

The polar system of E_z^p consists of these equations together with the equations which uniquely determine (19) in terms of $\theta_i(v)$ and $\varphi_{i,i}(v)$. Thus the dimension

of $H(E_z^p)$ depends only on N , n and p , as long as $E_z^p \in J^p$. We conclude that r_{p+1} is constant on J^p . Q.E.D.

Proof of (b): Suppose that E_z^p is an element of J^p which is generated by vectors $v_1, \dots, v_p$ such that $\theta_a(v_b) = \delta_{ab}$. If $0 \leq q \leq p$, we let E_z^q denote the q -dimensional integral element generated by $v_1, \dots, v_q$. Then E_z^q lies in J^q , and hence r_{q+1} is constant on a neighborhood of E_z^q in $I^q(\mathfrak{A})$. It follows that E_z^p is regular. Q.E.D.

Involution Theorem. *If $n \geq 2$ and $N \geq \frac{1}{2}n(n+1) - 1$, then the differential ideal $\mathfrak{A}$ is in involution with respect to the system of independent variables $\theta_1 \wedge \dots \wedge \theta_n$.*

Proof. If $z \in B$, we need to construct an n -dimensional ordinary integral element E_z^n for $\mathfrak{A}$ situated at z such that $\theta_1 \wedge \dots \wedge \theta_n \neq 0$ on E_z^n . We will do this by constructing in succession vectors $v_1, \dots, v_n \in T_z B$ which will span E_z^n . Let E_z^0 denote the unique zero-dimensional integral element at z . We then choose $v_1 \in H(E_z^0)$ so that $\theta_i(v_1) = \delta_{i1}$ and the $(N - n + 1)$ -dimensional vectors

$$H_{m1} = (\delta_{m1}, \tilde{\omega}_{n+1,m}(v_1), \dots, \tilde{\omega}_{Nn,m}(v_1)), \quad 1 \leq m \leq n-1,$$

are linearly independent. This can be done because $n-1 \leq N-n+1$, and the polar system of a zero-dimensional integral element does not involve $\tilde{\omega}_{\lambda i}(v)$. Observe that the one-dimensional integral element E_z^1 spanned by v_1 lies in J^1 .

Suppose now that we have constructed vectors $v_1, \dots, v_p \in T_z B$, $1 \leq p \leq n-1$, which span an integral element $E_z^p \in J^p$ and satisfy the requirement that $\theta_a(v_b) = \delta_{ab}$. The proof of the Regularity Lemma shows that there exists a vector $v \in H(E_z^p)$ such that $\theta_i(v) = \delta_{i,p+1}$. Moreover, suppose that V is the vector subspace of E^{N-n+1} spanned by $\{H_{ma} : 1 \leq a \leq p, a \leq m \leq n-1\}$, and that $V^\perp$ is the orthogonal complement of V in E^{N-n+1} . If we choose an arbitrary element Δ_r of $V^\perp$ for each r such that $p+1 \leq r \leq n$, there will exist a vector $v_{p+1} \in H(E_z^p)$ such that $\theta_i(v_{p+1}) = \delta_{i,p+1}$ and $H_r(v_{p+1}) = H_r(v) + \Delta_r$.

We claim that if $1 \leq p \leq n-2$, v_{p+1} can be chosen so that the vectors $\{H_{md} : 1 \leq d \leq p+1, d \leq m \leq n-1\}$ are linearly independent, where

$$H_{r,p+1} = (\delta_{r,p+1}, \omega_{n+1,r}(v_{p+1}), \dots, \omega_{Nr}(v_{p+1})).$$

Observe first that $H_{r,p+1}$ differs from $H_r(v_{p+1})$ only in the first component, and hence these vectors have the same orthogonal projection in E^{N-n} , where E^{N-n} denotes the subspace of E^{N-n+1} consisting of the vectors whose first component is zero. If $\tilde{V}^\perp$ denotes the projection of $V^\perp$ in E^{N-n} , then $\dim \tilde{V}^\perp = \dim V^\perp$, because V contains the vector H_{11} whose first component is one; hence $\dim \tilde{V}^\perp = N - n + 1 - \dim V \geq n - p + 1$. Thus it follows from the preceding paragraph that there exists a vector $v_{p+1} \in H(E_z^p)$ satisfying $\theta_i(v_{p+1}) = \delta_{i,p+1}$ such that the projections of $H_m(v_{p+1})$ in $\tilde{V}^\perp$, $p+1 \leq m \leq n-1$, are linearly independent. But then the projections of $H_{m,p+1}$ in $V^\perp$, $p+1 \leq m \leq n-1$, must be linearly independent, and our claim follows.

Hence if $1 < p < n - 2$, we obtain a vector v_{p+1} such that $v_1, \dots, v_{p+1}$ span a regular integral element $E_z^{p+1} \in J^{p+1}$, and we can continue the induction. If $p = n - 1$, we let v_n be any vector in the polar space $H(E_z^n)$ such that $\theta_i(v_n) = \delta_{in}$. Then $v_1, \dots, v_n$ span an ordinary integral element E_z^n on which $\theta_1 \wedge \dots \wedge \theta_n \neq 0$. Q.E.D.

Conformal Embedding Theorem. Let M and $\tilde{M}$ be real analytic Riemannian manifolds of dimension n and N respectively, $n \geq 2$, $N \geq \frac{1}{2}n(n+1) - 1$. If x_0 is a point of M , there exists an open neighborhood U of x_0 in M which can be conformally embedded in $\tilde{M}$.

Proof. Immediate from the preceding theorem.

Q.E.D.

We will conclude with some remarks about the initial-value problem for conformal embeddings (compare [2, pp. 204–206]), in order to clarify the relationship between the Cartan–Kähler proof of the Conformal Embedding Theorem, and the proof given in §1. Suppose that the Riemannian manifold M contains a hypersurface H which possesses a conformal embedding $f_0: H \rightarrow \tilde{M}$. If $x_0 \in H$, we seek an open neighborhood U of x_0 in M and a conformal embedding $f: U \rightarrow \tilde{M}$ such that $f|U \cap H = f_0|U \cap H$.

If $x_0 \in H$, there exists an open subset V_0 of H which contains x_0 , and on which we can construct a local adapted frame. By a local adapted frame on V_0 we mean a smooth mapping $\tilde{\sigma}: V_0 \rightarrow F(\tilde{M})$ with $\pi \circ \tilde{\sigma} = f_0$ such that if

$$\tilde{\sigma}(x) = (f_0(x), \tilde{e}_1(x), \dots, \tilde{e}_N(x)), \quad x \in V_0,$$

then $\tilde{e}_1(x), \dots, \tilde{e}_{n-1}(x)$ are tangent to $f_0(H)$. We remark that $\tilde{e}_1, \dots, \tilde{e}_N$ are vector fields “along” $f_0|V_0$ (compare [3, p. 46]). If $s: V_0 \rightarrow \mathbb{R}$ is the function defined by $f_0^*(\tilde{g}) = s^2 g$, we can define orthonormal vector fields $e_1, \dots, e_{n-1}$ on V_0 by requiring that $f_0(e_a(x)) = s(x)\tilde{e}_a(x)$, $x \in V_0$, $1 \leq a \leq p = n - 1$. We can extend these vector fields to an open subset V of M with $V \cap H = V_0$ in such a way that the extended vector fields (which will again be denoted by $e_1, \dots, e_{n-1}$) are orthonormal at every point of V . Assuming that V is orientable, we can then construct a smooth unit-length vector field e_n on V which is perpendicular to $e_1, \dots, e_{n-1}$. The vector fields $e_1, \dots, e_n$ determine a cross-section $\sigma: V \rightarrow F(M)$.

For simplicity we agree to let θ_i, ω_{ij} denote the differential forms $\sigma^*\theta_i, \sigma^*\omega_{ij}$, as well as the pull-backs of these forms to V_0 . Similarly, we will suppress the $\tilde{\sigma}^*$ and write $\tilde{\theta}_I, \tilde{\omega}_{IJ}$ for $\tilde{\sigma}^*\tilde{\theta}_I, \tilde{\sigma}^*\tilde{\omega}_{IJ}$. On V_0 we then have

$$(22) \quad \tilde{\theta}_i - s\theta_i = 0, \quad \theta_n = 0, \quad \tilde{\theta}_\lambda = 0,$$

and hence

$$0 = d(\tilde{\theta}_i - s\theta_i) = \sum_j (s\tilde{\omega}_{ij} - s\omega_{ij} - \delta_{ij} ds) \wedge \theta_j.$$

Since $\theta_1 \wedge \dots \wedge \theta_{n-1} \neq 0$ on V_0 , it follows from Cartan’s Lemma as before that

$$(23) \quad \tilde{\omega}_{ab} = \omega_{ab} + t_a \theta_b - t_b \theta_a, \quad 1 \leq a, b \leq p = n - 1,$$

where the functions $t_a : V_0 \rightarrow \mathbf{R}$ are defined by $ds/s = \sum_a t_a \theta_a$. Now choose a smooth function $t_n : V_0 \rightarrow \mathbf{R}$ and define

$$h : V_0 \rightarrow B \quad \text{by} \quad h(x) = (\sigma(x), \tilde{\sigma}(x), s(x), (t_1(x), \dots, t_n(x))).$$

In view of (22) and (23), $h(V_0)$ will be an integral submanifold for $\mathfrak{A}$ if and only if

$$(24) \quad \tilde{\omega}_{na} = h^*(\varphi_{na}) = \omega_{na} + t_n \theta_a \quad \text{on} \quad V_0.$$

Following the notation of §1, we let L denote the second fundamental form of V_0 in M , $\bar{L}$ the second fundamental form of V_0 in $\tilde{M}$; then

$$L = \sum_a \omega_{na} \otimes \theta_a, \quad \bar{L} = \sum_a (\tilde{\omega}_{na} \otimes \tilde{\theta}_a) \tilde{e}_n + \sum_{a,\lambda} (\tilde{\omega}_{\lambda a} \otimes \tilde{\theta}_a) \tilde{e}_\lambda.$$

The necessary and sufficient condition that $h(V_0)$ be an integral submanifold becomes

$$(24) \quad \langle \bar{L}, \tilde{e}_n \rangle = \sum_a h^*(\varphi_{na}) \otimes \tilde{\theta}_a = s(t_n g_0 + L),$$

where $\langle \cdot, \cdot \rangle$ denotes the Riemannian metric $\tilde{g}$ of $\tilde{M}$, and $g_0 = \sum_a \theta_a \otimes \theta_a$ is the Riemannian metric which g induces on V_0 . Using the structure equation (9), we can show that $\sum_a h^*(\varphi_{na}) \otimes \tilde{\theta}_a$ is the second fundamental form of V_0 in M , when M is given some Riemannian metric conformal to g .

After these preparations, we will prove a version of Theorem 1 of §1 within the Cartan-Kähler context. We remark that condition (25) employed below is equivalent to the condition $\bar{L} > L_{\tilde{\lambda}_0}$ of §1.

Extension Theorem. Suppose that (M, g) and $(\tilde{M}, \tilde{g})$ are Riemannian manifolds, H is a hypersurface of M , and $f_0 : H \rightarrow \tilde{M}$ is a nondegenerate conformal embedding, everything real analytic. Suppose, moreover, that we are given a real analytic normal vector field $w : H \rightarrow T(\tilde{M})$ along f_0 , and a function $\tau : H \rightarrow \mathbf{R}$ such that

$$(25) \quad \langle w, w \rangle < 1 \quad \text{and} \quad \langle \bar{L}, w \rangle = \tau g_0 + sL,$$

where g_0 is the Riemannian metric which g induces on H , and s is the function defined by $f_0^*(\tilde{g}) = s^2 g_0$. If $x_0 \in H$, there exists an open neighborhood U of x_0 in M , and a conformal embedding $f : U \rightarrow \tilde{M}$ such that $f|_{U \cap H} = f_0|_{U \cap H}$.

Proof. Since f_0 is nondegenerate, if Φ is any symmetric bilinear form on $T_x H$, there will exist a vector $v \in T_{f_0(x)} \tilde{M}$ such that $\Phi = \langle \bar{L}, v \rangle$. Hence there exists a vector field v along f_0 such that

$$\langle v, v \rangle = 1, \quad \langle \bar{L}, v \rangle = \tau g_0,$$

where $\tau : H \rightarrow \mathbf{R}^+$. We ensure that v and τ are real analytic by requiring that $v(x)$ be perpendicular to H and lie in the osculating space to f_0 at x . We then define $u : H \rightarrow T(\tilde{M})$ by $u = w + \alpha v$, where α is one of the roots of the quadratic equation

$$\alpha^2 + 2\alpha \langle v, w \rangle + \langle w, w \rangle = 1.$$

Since the roots are distinct, we can arrange that u be real analytic. We have

$$(26) \quad \langle u, u \rangle = 1, \quad \langle \bar{L}, u \rangle = (\tau + \alpha\bar{\tau})g_0 + sL.$$

Let V_0 be a contractible open neighborhood of x_0 in H . We then construct a local adapted frame $\tilde{\sigma} : V_0 \rightarrow F(\tilde{M})$ such that

$$\tilde{\sigma}(x) = (x, \tilde{e}_1(x), \dots, \tilde{e}_N(x)), \quad \text{where} \quad \tilde{e}_n(x) = u(x).$$

Using this local adapted frame and the function $t_n : V_0 \rightarrow \mathbf{R}$ defined by $st_n(x) = (\tau + \alpha\bar{\tau})(x)$, we construct a mapping $h : V_0 \rightarrow B$ as described in the preceding paragraphs. Note that (26) implies (24), and hence $h(V_0)$ will be an integral submanifold for $\mathfrak{A}$.

It remains to check that $h(V_0)$ is regular. Let E_z^{n-1} be the tangent plane to $h(V_0)$ at a given point $z = h(x)$, and choose $v_1, \dots, v_{n-1} \in E_z^{n-1}$ so that $\tilde{\theta}_a(v_b) = \delta_{ab}$. Let $h_{ab}^n = \tilde{\omega}_{na}(v_b) = \varphi_{na}(v_b)$, $h_{ab}^\lambda = \tilde{\omega}_{\lambda a}(v_b)$, and write

$$w(x) = w_n \tilde{e}_n(x) + \sum_\lambda w_\lambda \tilde{e}_\lambda(x),$$

where $w_n, w_{n+1}, \dots, w_N$ are real numbers. Then the equation $\langle u, \bar{L} \rangle = \langle w, \bar{L} \rangle + \alpha \langle v, \bar{L} \rangle$ implies that

$$h_{ab}^n = w_n h_{ab}^n + \sum_\lambda w_\lambda h_{ab}^\lambda + (\alpha\bar{\tau})(x) \delta_{ab},$$

or since $|w_n| < 1$,

$$(27) \quad h_{ab}^n = \frac{1}{1 - w_n} \left[\sum_\lambda w_\lambda h_{ab}^\lambda + (\alpha\bar{\tau})(x) \delta_{ab} \right] = c_n \delta_{ab} + \sum_\lambda c_\lambda h_{ab}^\lambda.$$

Since f_0 is nondegenerate, the $n(n-1)/2$ vectors

$$(h_{ab}^n, h_{ab}^{n+1}, \dots, h_{ab}^N), \quad 1 \leq a \leq b \leq p = n-1,$$

are linearly independent, and hence by (27) so are the vectors

$$(c_n \delta_{ab} + \sum_\lambda c_\lambda h_{ab}^\lambda, h_{ab}^{n+1}, \dots, h_{ab}^N), \quad 1 \leq a \leq b \leq p.$$

A straightforward argument now implies that the vectors

$$H_{ab} = (\delta_{ab}, h_{ab}^{n+1}, \dots, h_{ab}^N), \quad 1 \leq a \leq b \leq p,$$

are linearly independent, and hence E_z^{n-1} must lie in J^{n-1} . Using the Regularity Lemma, we conclude that E_z^{n-1} is a regular integral element, and since z is an arbitrary point of $h(V_0)$, it follows that $h(V_0)$ is regular.

If r_n is the constant value of $r_n(E_z^{n-1})$ as z varies throughout $h(V_0)$, we can construct an $(r_n - n)$ -dimensional submanifold F which contains $h(V_0)$ and satisfies the requirement that $T_z F \cap H(E_z^{n-1})$ is n -dimensional for z in some neighborhood of $h(x_0)$. Moreover, by the proof of the Involution Theorem we can arrange that $\theta_1 \wedge \dots \wedge \theta_n \neq 0$ on $T_z F \cap H(E_z^{n-1})$. The proof of the Extension Theorem can now be completed by an application of the Cartan-Kähler Theorem. Q.E.D.

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