Pairs of Positive Solutions of Nonlinear Elliptic Partial Differential Equations

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Introduction. The existence of positive solutions of nonlinear elliptic partial differential equations has been the object of a great deal of study. (See *e.g.*, [1]–[28].) In this paper we treat a family of nonlinear elliptic eigenvalue problems for which positive solutions occur in pairs.

To describe our result more precisely, let $\Omega \subset \mathbf{R}^n$ be a smooth bounded domain and

$$(0.1) \quad \begin{cases} Lu \equiv - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} (a_{ij}(x) u_{x_j}) + c(x)u = \lambda f(x, u), & x \in \Omega, \\ u = 0, & x \in \partial\Omega. \end{cases}$$

Here $\lambda \in \mathbf{R}^+ = \{r \in \mathbf{R} \mid r \geq 0\}$, the coefficients a_{ij} have Hölder continuous first derivatives, c is Hölder continuous and nonnegative, and $-L$ is uniformly elliptic in Ω . The function f satisfies

- (f₁) f is locally Lipschitz continuous on $\bar{\Omega} \times \mathbf{R}^+$ with $f(x, 0) = 0$.
- (f₂) There exists $\bar{z} > 0$ such that $f(x, z) < 0$ if $z \geq \bar{z}$.
- (f₃) If $F(x, z) = \int_0^z f(x, s) ds$, there exists $(a, y) \in \Omega \times [0, \bar{z})$ such that $F(a, y) > 0$.
- (f₄) $f(x, z) = o(z)$ at $z = 0$ uniformly in $x \in \bar{\Omega}$.

These conditions more than suffice for our purposes and it will be indicated later how they may be relaxed. A simple example of an f satisfying (f₁)–(f₄) is, *e.g.*, $f(z) = z^2 - z^3$.

A solution of (0.1) is a pair (λ, u) satisfying (0.1) in the classical sense and u is a solution of (0.1) _{λ} if (λ, u) is a solution of (0.1). (f₁) implies that (0.1) possesses the line of trivial solutions $\{(\lambda, 0) \mid \lambda \in \mathbf{R}^+\}$.

We will show there exists a $\lambda > 0$ such that for all $\lambda > \lambda$, (0.1) _{λ} possesses at least two distinct solutions $\bar{u}(\lambda)$, $\underline{u}(\lambda) > 0$ in Ω . Such a lower bound, λ , is essential since for $\lambda > 0$ and sufficiently small, $u = 0$ is the unique solution of

$(0.1)_\lambda$ (Lemma 1.5). For a smaller class of f 's, a stronger statement is obtained concerning the structure of the set of positive solutions of (0.1), namely it possesses a component $\mathcal{C}$ such that for all $\lambda > \bar{\lambda}$, there are distinct points $(\lambda, \bar{u}(\lambda))$, $(\lambda, \underline{u}(\lambda)) \in \mathcal{C}$ with $\bar{u}, \underline{u}$ as above.

Our results are obtained by a combination of analytical and topological arguments. To give a rough sketch of what we do, first (f_2) is used to obtain *a priori* bounds for solutions of (0.1). These bounds together with (f_1) , (f_3) , and a variational argument imply the existence of a positive solution of $(0.1)_\lambda$ provided that λ is sufficiently large. A perturbation argument then enables us to find a *subsolution* (see e.g., [2] or [3] for this terminology) of $(0.1)_\lambda$ which is not a solution. A corresponding *supersolution* is easily constructed using (f_2) . This information in conjunction with (f_3) and a topological degree computation then implies the existence of a second solution of $(0.1)_\lambda$.

The basic ingredients of our argument, i.e., variational techniques, constructive and nonconstructive positivity arguments using sub- and supersolutions, and topological degree methods are indeed the existence machinery generally used to treat equations such as (0.1) in [1–28]. However unlike these papers, we seem to need all of them in the same problem.

The result in the literature which is closest to our work is an interesting recent paper by Amann [4]. He finds bounds for the number of fixed points a mapping on an order interval in an ordered Banach space possesses in terms of the number of appropriately ordered “sub-” and “superfunctions” one has in the order interval. The abstract results are applied in particular to nonlinear elliptic equations (Theorem E of [4]). Once we have obtained a sub- and supersolution for $(0.1)_\lambda$, it is possible to revise the remainder of our argument to put it into the framework of Amann’s result (in particular to find another sub- and supersolution). In so doing we avoid using a degree computation for our first result. However, these computations are essential for the more delicate theorem on the existence and structure of $\mathcal{C}$.

The existence theorems will be proved in §1. It is important for these proofs that the spectrum of L lies in $\mathbf{R}^+ - \{0\}$ (which is the case under the hypotheses on its coefficients). In §2 we briefly study what happens when L has nonpositive eigenvalues. Here, using the results of §1, the existence of at least one positive solution can still be shown.

§1. The existence results. In this section the structure of the set of solutions of (0.1) will be studied and in particular the existence of pairs of positive solutions of $(0.1)_\lambda$ for λ sufficiently large will be established. The first step in this process is to obtain *a priori* bounds for solutions of (0.1). Suppose u is a solution of $(0.1)_\lambda$ with $u \geq 0$ and the maximum of u occurs at $x \in \Omega$. Then from (0.1) (after a rotation of coordinates if necessary),

$$(1.1) \quad 0 \leq Lu(x) = \lambda f(x, u(x)),$$

so by (f_2) , $0 \leq u(x) < \bar{z}$.

In a standard fashion *a priori* bounds can now be obtained for the derivatives of u . For this purpose some function spaces are needed. Using multiindex notation, let

$$||u||_k = \sum_{|\sigma| \leq k} \max_{x \in \Omega} |D^\sigma u(x)|$$

with $||u||_0 = ||u||$; for $\alpha \in (0, 1)$, let

$$||u||_{k,\alpha} = ||u||_k + \max_{\substack{|\sigma|=k \\ x,y \in \Omega}} \frac{|D^\sigma u(x) - D^\sigma u(y)|}{|x - y|^\alpha}$$

and for $p \geq 1$, let

$$|u|_{k,p} = \left(\sum_{|\sigma| \leq k} \int_{\Omega} |D^\sigma u(x)|^p dx \right)^{1/p}.$$

The closure of infinitely differentiable functions with compact support in Ω with respect to $|\cdot|_{k,p}$ will be denoted by $W_0^{k,p}$. Clearly $W_0^{0,p} = L^p(\Omega)$ and the norm of this space will be denoted by $|\cdot|_p$.

If $u \geq 0$ is a solution of $(0.1)_\lambda$, by the L^p estimates for uniformly elliptic equations [29] and (1.1) for any $p > 1$,

$$(1.2) \quad |u|_{2,p} \leq c_1 \lambda |f(\cdot, u)|_p \leq c_2 \lambda \max_{x \in \Omega, 0 \leq z \leq \bar{z}} |f(x, z)| = c_3 \lambda.$$

In (1.2) and what follows c_1, c_2 , etc. will denote various positive constants. By a general imbedding theorem [34], for $p > n$,

$$(1.3) \quad ||u||_{1,1-n/p} \leq c_4 |u|_{2,p},$$

so combining (1.2) and (1.3) gives

$$(1.4) \quad ||u||_{1,1-n/p} \leq c_5 \lambda.$$

The Schauder estimates for uniformly elliptic equations [24] and $(0.1)_\lambda$ then give $||\cdot||_{2,\alpha}$ bounds for u .

As is well known, once bounds such as (1.4) have been obtained for $(0.1)_\lambda$, an iteration scheme can be constructed converging to a unique maximal non-negative solution $\hat{u}(\lambda)$ [31 II, 2]. However, it may be the case that $\hat{u}(\lambda) \equiv 0$; indeed this is so for small λ .

Lemma 1.5. *Suppose f satisfies (f_1) – (f_2) . Then for all $0 \leq \lambda$ sufficiently small, $u \equiv 0$ is the unique nonnegative solution of $(0.1)_\lambda$.*

Proof. If not, (0.1) possesses solutions (λ_m, u_m) with $u_m \geq 0$ and $u_m \not\equiv 0$, $\lambda_m > 0$ and $\lambda_m \rightarrow 0$ as $m \rightarrow \infty$. From (1.4) $u_m \rightarrow 0$ in $||\cdot||_1$. Let $v_m = u_m / ||u_m||_1$. Then by (1.2)–(1.3),

$$(1.6) \quad ||v_m||_1 \leq c_6 \lambda_m \frac{|f(\cdot, u_m)|_p}{||u_m||_1} \leq \frac{c_7 \lambda_m ||f(\cdot, u_m)||}{||u_m||_1}.$$

Since $f(x, 0) = 0$ and f is Lipschitz continuous at 0, there exists a constant $K > 0$ such that for all m sufficiently large,

$$(1.7) \quad \|f(x, u_m(x))\| \leq K \|u_m\| \leq K \|u_m\|_1.$$

Hence (1.6) and (1.7) imply $\|v_m\|_1 \rightarrow 0$ as $m \rightarrow \infty$. But $\|v_m\|_1 = 1$, a contradiction.

Next a variational argument will be employed to show that $(0.1)_\lambda$ possesses a positive solution for all large λ . For technical reasons, f has to be modified. Let

$$(1.8) \quad \begin{cases} \bar{f}(x, z) = f(x, z), & 0 \leq z \leq \bar{z}, \\ & = f(x, \bar{z}), & z \geq \bar{z}, \\ & = 0, & z < 0. \end{cases}$$

Then $\bar{f}$ is Lipschitz continuous in $\bar{\Omega} \times \mathbf{R}$. Let $\bar{F}(x, z) = \int_0^z \bar{f}(x, s) ds$ and note that $\bar{F}(x, z) = F(x, z)$ for $z \in [0, \bar{z}]$. Let $\bar{E} = W_0^{1,2}$ and define a functional on $\bar{E}$ by

$$(1.9) \quad I(\lambda, \varphi) = \frac{1}{2} \int_{\Omega} \left[\sum_{i,j=1}^n a_{ij}(x) \varphi_{x_i}(x) \varphi_{x_j}(x) + c(x) \varphi^2(x) \right] dx \\ - \lambda \int_{\Omega} \bar{F}(x, \varphi(x)) dx \equiv I_1(\varphi) - \lambda I_2(\varphi).$$

As is well known, I_1 is weakly lower semicontinuous on $\bar{E}$ [38]. The functional I_2 is weakly continuous on $\bar{E}$. To verify this, let $(\varphi_m) \subset \bar{E}$ with (φ_m) converging weakly in $\bar{E}$ to φ . By standard imbedding theorems, E is compactly imbedded in $L^p(\Omega)$ for $1 \leq p < 2n/(n-2)$ [34]. (The cases $n = 1, 2$ are simpler to treat and their proof will be omitted.) Therefore $\varphi_m \rightarrow \varphi$ in $L^{2n\theta/(n-2)}(\Omega)$ for any $\theta \in [\frac{1}{2}, 1]$. If g is continuous on $\bar{\Omega} \times \mathbf{R}$, the map $\varphi(x) \rightarrow g(x, \varphi(x))$ is continuous from $L^p(\Omega)$ to $L^q(\Omega)$ if $|g(x, z)| \leq c_1 + c_2 |z|^{p/q}$ [32]. Since $|\bar{F}(x, z)| \leq c_3 + |z|$, $\bar{F}$ is continuous from $L^{2n\theta/(n-2)}(\Omega)$ to $L^1(\Omega)$ and therefore $\bar{F}(x, \varphi_m(x)) \rightarrow \bar{F}(x, \varphi(x))$ in $L^1(\Omega)$, i.e., $I_2(\varphi_m) \rightarrow I_2(\varphi)$. Hence I_2 is weakly continuous.

The hypotheses on the coefficients of L imply that $I_1(\varphi) \geq (\beta/2) |\varphi|_{1,2}^2$ for all $\varphi \in \bar{E}$, β being the ellipticity constant of the principal part of L . Since $0 \leq |I_2(\varphi)| \leq c_1 + c_2 |\varphi|_{1,2}$ there exists $R = R(\lambda) > 0$ such that $I(\lambda, \varphi) \geq 1$ if $|\varphi|_{1,2} \geq R(\lambda)$. Let $\bar{B}_\lambda = \{\varphi \in \bar{E} \mid |\varphi|_{1,2} \leq R(\lambda)\}$. We pose the variational problem: minimize $I(\lambda, \varphi)$ on $\bar{B}_\lambda$. The above remarks imply that $I(\lambda, \varphi)$ is weakly lower semicontinuous on the ball $\bar{B}_\lambda$ and therefore by a standard theorem in the calculus of variations [38], the minimum of $I(\lambda, \varphi)$ is achieved at $u(\lambda) \in \bar{B}_\lambda$. Moreover since $I(\lambda, 0) = 0$ while $I(\lambda, \partial \bar{B}_\lambda) \geq 1$, $u(\lambda)$ lies in the interior of $\bar{B}_\lambda$.

Up to this point no restrictions have been placed on the size of λ so it may occur—and indeed does for λ sufficiently small—that $u(\lambda) \equiv 0$. Here (f_3) plays its role; by (f_3) there exists $\varphi \in \bar{E}$ such that $I_2(\varphi) > 0$. This follows since there is a $\delta > 0$ such that $\bar{F}(x, s) \geq \gamma > 0$ if $|x - a| \leq \delta$, $|y - s| \leq \delta$. Define $\varphi(x) = y$ if $|x - a| \leq \delta/2$ and let φ go to 0 linearly along rays in the shell $\delta/2 \leq |x - a| \leq \delta/2 + \eta$ where $\eta > 0$ is so small that

$$\int_{\delta/2 \leq |x-a| \leq \delta/2+\eta} \bar{F}(x, \varphi(x)) dx < \frac{1}{2} \int_{|x-a| \leq \delta/2} \bar{F}(x, \varphi(x)) dx.$$

Therefore $I_2(\varphi) > 0$. For λ sufficiently large, $\lambda I_2(\varphi) > I_1(\varphi)$. Consequently for such values of λ , $I(\lambda, u(\lambda)) < 0$ so $u(\lambda) \neq 0$.

It is now a straightforward matter to show that $I(\lambda, \varphi)$ is Fréchet differentiable in φ and its Fréchet derivative at φ is given by

$$(1.10) \quad I'(\lambda, \varphi)v = \int_{\Omega} \left[\sum_{i,j=1}^n a_{ij}(x) \varphi_{x_i} v_{x_j} + c(x) \varphi v - \lambda \bar{f}(x, \varphi) v \right] dx$$

for $v \in \tilde{E}$. Since $u(\lambda)$ is an interior minimum for $I(\lambda, \varphi)$ on $\tilde{B}_{\lambda}$, $I'(\lambda, u(\lambda))v = 0$ for all $v \in \tilde{E}$. Standard bootstrap arguments using the L^p and Schauder estimates imply that $u(\lambda)$ is a classical solution of

$$(1.11) \quad Lu = \lambda \bar{f}(x, u), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega.$$

If u is any solution of (1.11), letting $A = \{x \in \Omega \mid u < 0\}$, (1.8) implies $Lu = 0$ in A and $u = 0$ on ∂A . By the maximum principle, $u \equiv 0$ in A so $A = \emptyset$. Therefore $u \geq 0$ in Ω . Also as in (1.1), $\|u\| < \bar{z}$ so $\bar{f}(x, u) = f(x, u)$ and $u(\lambda)$ satisfies $(0.1)_{\lambda}$.

Next let $P^+ = \{\varphi \in C^1(\Omega) \mid \varphi > 0 \text{ in } \Omega, \partial\varphi/\partial\nu < 0 \text{ on } \partial\Omega\}$ where $\nu = \nu(x)$ denotes the outward pointing normal to $\partial\Omega$. We will show $u(\lambda) \in P^+$. Since f satisfies (f_1) , $f(x, z) = g(x, z)z$ where g is well defined and continuous for $z \neq 0$. Define $g(x, 0) \equiv 0$. (f_1) further implies that $g(x, u(\lambda, x))$ is bounded. Let $g^+(x) = \max(g(x, u(\lambda, x)), 0) \geq 0$ and $g^-(x) = g(x, u(\lambda, x)) - g^+(x) \leq 0$. Equation $(0.1)_{\lambda}$ can be written

$$(1.12) \quad (L - \lambda g^-)u = \lambda g^+ u \geq 0, \quad x \in \Omega,$$

since $u \geq 0$ in Ω . $L - \lambda g^-$ has bounded coefficients, Ω is smooth and $u \neq 0$ in Ω ; therefore the strong maximum principle [35] implies that $u(\lambda) \in P^+$.

Lastly observe that if $I(\lambda, \cdot)$ has a negative minimum for $\lambda = \mu$, it has a negative minimum for all $\lambda \geq \mu$. Therefore $(0.1)_{\lambda}$ possesses a solution $u(\lambda) \in P^+$ for all $\lambda > \bar{\lambda} = \inf \{\lambda \in \mathbf{R}^+ \mid I(\lambda, \cdot) \text{ has a negative minimum on } \tilde{B}_{\lambda}\}$.

Combining these results, we have shown

Theorem 1.13. *Suppose f satisfies (f_1) – (f_3) . Then for all $\lambda > \bar{\lambda} > 0$, $(0.1)_{\lambda}$ possesses a solution $u(\lambda) \in P^+$. Moreover $u(\lambda)$ minimizes $I(\lambda, \cdot)$ on $\tilde{E}$.*

Remark 1.14. If $c = c(x, u) \leq 0$, the above arguments also work modulo some minor changes. Observe also that the condition (f_2) was only used to get *a priori* bounds for (0.1) . It can be replaced by other conditions which imply such bounds. For example, as has been noted by others (e.g., [2]), if f is Hölder continuous in u and $|f(x, z)| \leq c_1 + c_2 |z|^{\sigma}$ where $\sigma \in [0, 1)$, then by the L^p estimates and (1.3), if u satisfies $(0.1)_{\lambda}$,

$$(1.15) \quad \|u\| \leq c_3 \|u\|_{2,p} \leq c_4 \lambda \|f(\cdot, u)\|_p \leq c_5 \lambda (1 + \|u\|^{\sigma})$$

which readily implies $\|u\| < c_6$, a constant depending on λ . However if f is only Hölder continuous, the proof that $u(\lambda) \in P^+$ breaks down and we can only conclude $u(\lambda) \geq 0$ in Ω .

By also assuming (f_4) , a slightly stronger version of Theorem 1.13 obtains.

Corollary 1.16. *If f satisfies (f_1) – (f_4) , then for all $\lambda \geq \lambda > 0$, $(0.1)_\lambda$ possesses a solution $u(\lambda) \in P^+$ which minimizes $I(\lambda, \cdot)$ on $\tilde{E}$.*

Proof. All we need show is that $(0.1)_\lambda$ has a solution in P^+ . Let $\lambda_m > \lambda$ and $\lambda_m \rightarrow \lambda$ as $m \rightarrow \infty$. From (1.4) and the corresponding bounds for the second derivatives of $u(\lambda_m)$, a subsequence of $(\lambda_m, u(\lambda_m))$ converges in $\mathbf{R} \times C^2(\Omega)$ to $(\lambda, u(\lambda))$ with $u(\lambda) \geq 0$ and satisfying (0.1). If $u(\lambda) \equiv 0$, $u(\lambda_m) \rightarrow 0$ in $\|\cdot\|$ along the subsequence. But then by (1.2), (1.3) and (f_4) , for m sufficiently large,

$$(1.17) \quad \|u_m\| \leq c_6 \lambda_m \|f(x, u_m)\| \leq \frac{1}{2} \|u_m\|$$

which is impossible since $u_m \in P^+$. Hence $u(\lambda) \not\equiv 0$ and as in the proof of Theorem 1.13, $u(\lambda) \in P^+$. Note also that $I(\lambda, u(\lambda)) = 0$ and by the definition of λ , $u(\lambda)$ minimizes $I(\lambda, \cdot)$ on $\tilde{E}$.

Remark 1.18. The remarks following (1.11) and the argument of the above lemma show that any nontrivial solution of (1.11) lies in P^+ and that $u = 0$ is an isolated solution of $(0.1)_\lambda$ in $\|\cdot\|$ (and *a fortiori* in $\|\cdot\|_1$). Moreover this is true uniformly in λ on bounded λ intervals.

To obtain a second solution of $(0.1)_\lambda$ in P^+ , some preliminaries are needed. Corollary 1.16 and the remarks following (1.4) imply that $(0.1)_\lambda$ possesses a unique maximal solution $\hat{u}(\lambda) \in P^+$ for all $\lambda \geq \lambda$ and $\hat{u}(\lambda)$ may be obtained via a monotone iteration scheme. Since it is important for what follows, the construction of $\hat{u}(\lambda)$ will be indicated next. By (f_1) there exists $M > 0$ such that $|f(x, z_1) - f(x, z_2)| \leq M |z_1 - z_2|$ for all $x \in \bar{\Omega}$ and $0 \leq z_1, z_2 \leq \bar{z}$. Let $u_0(\lambda) \equiv \bar{z}$ and $u_{m+1}(\lambda)$ satisfy

$$(1.19) \quad (L + \lambda M)u_{m+1} = \lambda(Mu_m + f(x, u_m)), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega,$$

for $m \geq 0$. These equations can be recursively solved using the Schauder theory and the functions u_m form a monotone decreasing sequence converging to $\hat{u}(\lambda) \geq u(\lambda)$ in Ω .

Remark 1.20. By using (1.19) and (1.2)–(1.3), it is easily shown that $u_1(\lambda)$ is a continuous function of λ in $\|\cdot\|_1$. We also note that $u_0(\lambda)$ is a *super-solution* for $(0.1)_\lambda$, i.e., $Lu_0 = cu_0 \geq 0 \geq \lambda f(x, u_0)$ and $u_0 \geq 0$ on $\partial\Omega$.

Let $\Gamma^+ = \{(x, z) \in \bar{\Omega} \times \mathbf{R}^+ \mid f(x, z) \geq 0\}$, $\Gamma^- = \{(x, z) \in \bar{\Omega} \times \mathbf{R}^+ \mid f(x, z) \leq 0\}$, let $\rho^+(x, z)$ denote the (Euclidean) distance from (x, z) to Γ^+ and $\rho^-(x, z)$ the distance from (x, z) to Γ^- , and set $\theta(x, z) = (1 - t\rho^-(x, z))(1 - t\rho^+(x, z))^{-1}$ where $t > 0$ is a constant. Then θ is Lipschitz continuous and by making t sufficiently small, $\theta \leq 1$ on $\Gamma^+ \cap (\bar{\Omega} \times [0, \bar{z}])$, $\theta \geq 1$ on $\Gamma^- \cap (\bar{\Omega} \times [0, \bar{z}])$, and θ is as close to 1 as desired on $\bar{\Omega} \times [0, \bar{z}]$. Choosing $\lambda > \lambda$ and replacing f

by θf in Theorem 1.13, by that result the modified functional $I_\theta(\lambda, \cdot)$ is minimized by $w(\lambda) \in P^+$ provided that t (which we now fix) is sufficiently small. Moreover $w(\lambda)$ satisfies

$$(1.21) \quad Lw = \lambda \theta(x, w)f(x, w), \quad x \in \Omega; \quad w = 0, \quad x \in \partial\Omega,$$

and $0 \leq w < \bar{z}$. The choice of θ implies that

$$(1.22) \quad Lw \leq \lambda f(x, w), \quad x \in \Omega,$$

i.e., $w(\lambda)$ is a *subsolution* for $(0.1)_\lambda$. (This is also true if $w(\lambda)$ is replaced by $\hat{w}(\lambda)$, the corresponding maximal solution of (1.21).) Taking $\bar{u}_0(\lambda) = w(\lambda)$ and iterating as in (1.19), the iterates $\bar{u}_m(\lambda)$ form a monotone increasing sequence converging to a solution $\bar{u}(\lambda)$ of $(0.1)_\lambda$ with $\hat{u}(\lambda) \geq \bar{u}(\lambda)$ [31 II, 2]. In addition the choice of θ and the maximum principle imply that $u_1(\lambda) > \bar{u}(\lambda) > w(\lambda)$ in Ω and $\partial u_1 / \partial \nu < \partial \bar{u} / \partial \nu < \partial w / \partial \nu$ on $\partial\Omega$.

We suspect that in general $\hat{u}(\lambda) = \bar{u}(\lambda) = u(\lambda)$. In any event, this possibility cannot be excluded. With the above preliminaries in hand, a topological argument will be employed to find a second solution of $(0.1)_\lambda$ distinct from $\hat{u}(\lambda)$. For technical reasons we replace f by $\bar{f}$ in $(0.1)_\lambda$ and treat $(0.1)_\lambda$ in the form

$$(1.23) \quad (L + \lambda M)u = \lambda(Mu + \bar{f}(x, u)), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega.$$

Remark 1.18 shows the only nontrivial solutions (1.23) possess lie in P^+ and for any such solution $\bar{f}(x, u) = f(x, u)$.

Let $E = \{\varphi \in C^1(\Omega) \mid \varphi = 0 \text{ on } \partial\Omega\}$. (1.23) can be converted to an equivalent operator equation in $\mathbf{R} \times E$ in a standard fashion. For $(\lambda, u) \in \mathbf{R} \times E$, let $v = T(\lambda, u)$ satisfy

$$(1.24) \quad (L + \lambda M)v = \lambda(Mu + \bar{f}(x, u)), \quad x \in \Omega; \quad v = 0, \quad x \in \partial\Omega.$$

The Schauder theory implies there exists a unique such $v \in C^2(\Omega)$ and by the L^p estimates and the boundedness of $\bar{f}$,

$$(1.25) \quad |v|_{2,p} \leq c_1 |Mu + \bar{f}(x, u)|_p \leq c_2(|u|_p + 1)$$

c_2 depending on λ . Combining (1.25) with (1.3) yields

$$(1.26) \quad \|v\|_{1,1-n/p} \leq c_3(|u|_p + 1)$$

from which it readily follows that T is a compact map on $\mathbf{R} \times E$.

Let $\Phi(\lambda, u) = u - T(\lambda, u)$. By the above remarks, any zero of $\Phi(\lambda, \cdot)$ is a solution of (1.23) in P^+ and conversely. Moreover by (1.1) and (1.25) if u satisfies (1.23),

$$(1.27) \quad \|u\|_1 \leq c_4(\bar{z} + 1) < c_4(\bar{z} + 1) + 1 \equiv q(\lambda)$$

with $q(\lambda)$ a continuous function of λ . Let

$$\alpha_\lambda = \left\{ \varphi \in E \mid \|\varphi\|_1 < q(\lambda), w(\lambda) < \varphi < u_1(\lambda) \text{ in } \Omega, \frac{\partial w}{\partial \nu} > \frac{\partial \varphi}{\partial \nu} > \frac{\partial u_1}{\partial \nu} \text{ on } \partial\Omega \right\}$$

where $u_1(\lambda)$ was defined by (1.19). The maximum principle shows that if $\varphi \in \bar{\alpha}_\lambda$, $w(\lambda) < T(\lambda, w(\lambda)) \leq T(\lambda, \varphi) \leq T(\lambda, u_1(\lambda)) < u_1(\lambda)$ in Ω with reversed inequalities for $\partial/\partial\nu$ on $\partial\Omega$. Moreover by (1.25) and (1.27), $\|T(\lambda, \varphi)\|_1 < q(\lambda)$. Hence $T(\lambda, \cdot) : \bar{\alpha}_\lambda \rightarrow \alpha_\lambda$.

α_λ is a bounded, open, convex set and by our above remarks, $0 \notin \Phi(\lambda, \partial\alpha_\lambda)$. Therefore the Leray-Schauder degree of $\Phi(\lambda, \cdot)$ relative to the set α_λ and point 0 is defined and will be denoted by $d(\Phi(\lambda, \cdot), \alpha_\lambda, 0)$. See e.g., [33, 37] for the properties of topological degree. Since $T : \bar{\alpha}_\lambda \rightarrow \alpha_\lambda$, $d(\Phi(\lambda, \cdot), \alpha_\lambda, 0) = 1$. To verify this, let $a \in \alpha_\lambda$, $S_t(\lambda, u) = tT(\lambda, u) + (1-t)a$, and $\Psi_t(\lambda, u) = u - S_t(\lambda, u)$ for $t \in [0, 1]$. Then $S_t : \bar{\alpha}_\lambda \rightarrow \alpha_\lambda$ so $0 \notin \Psi_t(\lambda, \partial\alpha_\lambda)$ for $t \in [0, 1]$ and therefore by the homotopy invariance of degree,

$$(1.28) \quad d(\Psi_t(\lambda, \cdot), \alpha_\lambda, 0) \equiv \text{constant} = c_1, \quad t \in [0, 1].$$

Since $\Psi_1(\lambda, u) = u - a$, the definition of degree implies $c_1 = 1$.

Let $B_\rho(\psi) = \{\varphi \in E \mid \|\varphi - \psi\|_1 < \rho\}$ and $B_\rho \equiv B_\rho(0)$. By remark 1.18, if $\rho = \rho(\lambda)$ is sufficiently small, $u = 0$ is the unique solution of $(0.1)_\lambda$ in $\bar{B}_{\rho(\lambda)}$ and ρ can be chosen independently of λ on bounded λ intervals. Fix $\lambda > \lambda$, choose such a ρ for $[0, \lambda]$, and let

$$\mathcal{B}(\lambda) = \{(\mu, \varphi) \in \mathbf{R} \times P^+ \mid \mu \in [0, \lambda], \rho < \|\varphi\|_1 < q(\lambda)\}.$$

Since $q(\lambda)$ is continuous, $\mathcal{B}(\lambda)$ is a bounded open set in $[0, \lambda] \times E$ with no zeroes of Φ on $\partial\mathcal{B}(\lambda)$ (relative to $[0, \lambda] \times E$). By the homotopy invariance of degree,

$$(1.29) \quad d(\Phi(\mu, \cdot), \mathcal{B}_\mu(\lambda), 0) \equiv \text{const.} = c_2, \quad \mu \in [0, \lambda],$$

where $\mathcal{B}_\mu(\lambda) = \{(\mu, \varphi) \in \mathcal{B}(\lambda)\}$. Since $\Phi(0, \cdot)$ has no zeroes in $\mathcal{B}_0(\lambda)$, $c_2 = 0$. By the additivity of degree and (1.28),

$$(1.30) \quad d(\Phi(\lambda, \cdot), \mathcal{B}_\lambda(\lambda), 0) = d(\Phi(\lambda, \cdot), \alpha_\lambda, 0) \\ + d(\Phi(\lambda, \cdot), \mathcal{B}_\lambda(\lambda) - \bar{\alpha}_\lambda, 0) = 1 + d(\Phi(\lambda, \cdot), \mathcal{B}_\lambda(\lambda) - \bar{\alpha}_\lambda, 0).$$

Hence (1.29) and (1.30) show

$$(1.31) \quad d(\Phi(\lambda, \cdot), \mathcal{B}_\lambda(\lambda) - \bar{\alpha}_\lambda, 0) = -1$$

and therefore there exists $u(\lambda) \in \mathcal{B}_\lambda(\lambda) - \bar{\alpha}_\lambda \subset P^+$ satisfying $(0.1)_\lambda$. Thus we have shown

Theorem 1.32. *Let f satisfy $(f_1)-(f_4)$. Then for all $\lambda > \lambda$, $(0.1)_\lambda$ possesses at least 2 distinct solutions $\bar{u}(\lambda)$, $u(\lambda) \in P^+$.*

Remark 1.32. As was noted in the introduction, Theorem 1.32 can be obtained from Theorem E of [4]. To do so, another sub- and supersolution of (1.23) is needed and an appropriate choice to make is $\pm \epsilon v_1$, where $v_1 > 0$ in Ω is the eigenfunction of L corresponding to its smallest eigenvalue and normalized so that $\|v_1\|_1 = 1$ and $\epsilon > 0$ is sufficiently small.

Corollary 1.34. Suppose f satisfies (f_1) , (f_4) , $f(x, z) \geq 0$ and $\neq 0$ in $\bar{\Omega} \times [0, \bar{z}]$, and $f(x, z) \leq 0$ in $\bar{\Omega} \times [\bar{z}, \infty)$. Then (0.1) possesses an unbounded component $\mathcal{C}$ of solutions in $\mathbf{R}^+ \times P^+$ such that for each $\lambda > \lambda$, there are two distinct points $(\lambda, \bar{u}(\lambda))$, $(\lambda, \underline{u}(\lambda)) \in \mathcal{C}$ with $0 \leq \bar{u}, \underline{u} \leq \bar{z}$.

Proof. A component of solutions of (0.1) is a maximal closed connected set of solutions. Since $f \geq 0$ and $\neq 0$ in $\bar{\Omega} \times [0, \bar{z}]$, f satisfies (f_3) . Let (λ, u) be a solution of (0.1) with $u \geq 0$ in Ω and let $A = \{x \in \Omega \mid u > \bar{z}\}$. Then $Lu \leq 0$ in A with $u = \bar{z}$ on ∂A . Hence by the maximum principle $A = \emptyset$ and $0 \leq u \leq \bar{z}$ in Ω . Letting $\bar{f}(x, z) = f(x, z)$ for $0 \leq z \leq \bar{z}$ and $\bar{f} \equiv 0$ otherwise, the arguments of Theorems 1.13 and 1.32 go through unchanged to give a distinct pair of solution of $(0.1)_\lambda$ in P^+ for all $\lambda > \lambda$.

To obtain $\mathcal{C}$, let $\mu = \inf \{\lambda \in \mathbf{R}^+ \mid (0.1)_\lambda \text{ has a solution in } P^+\}$. Lemma 1.5 implies $\mu > 0$ and the argument of Corollary 1.16 shows $(0.1)_\mu$ has a solution in P^+ . Let $\hat{u}(\mu)$ denote the unique maximal solution of $(0.1)_\mu$, let $\mathcal{S}$ denote the set of nontrivial solutions of (0.1), and $\mathcal{C}$ the component of $\mathcal{S}$ to which $(\mu, \hat{u}(\mu))$ belongs. Previous arguments imply $\mathcal{C} \subset \mathbf{R}^+ \times P^+$. We will show for all $\lambda > \mu$ (and *a fortiori* for all $\lambda > \lambda$) (0.1) has a distinct pair of solutions $(\lambda, \bar{u}(\lambda))$, $(\lambda, \underline{u}(\lambda)) \in \mathcal{C}$.

Since $0 \leq \hat{u}(\mu) \leq \bar{z}$, $f(x, \hat{u}(\mu, x)) \geq 0$ and from (0.1),

$$(1.35) \quad L\hat{u}(\mu) = \mu f(x, \hat{u}) \leq \lambda f(x, \hat{u}) = \lambda \bar{f}(x, \hat{u})$$

for all $\lambda > \mu$. Thus $\hat{u}(\mu)$ is a subsolution of $(0.1)_\lambda$ for all $\lambda > \mu$. Replace $w(\lambda)$ by $\hat{u}(\mu)$ in the definition of $\hat{\alpha}_\lambda$ in Theorem 1.32, denote the resulting set by $\hat{\alpha}_\lambda$, and let $\hat{\alpha} = \bigcup_{\lambda > \mu} \{\lambda\} \times \hat{\alpha}_\lambda$. Then $\hat{\alpha}$ is an open subset of $\mathbf{R} \times E$ and as in (1.28),

$$(1.36) \quad d(\Phi(\lambda, \cdot), \hat{\alpha}_\lambda, 0) = 1 \quad \text{for all } \lambda > \mu.$$

Let $\mathcal{O}$ be a bounded open set in $\mathbf{R} \times E$ containing $(\mu, \hat{u}(\mu))$. We will show $\partial \mathcal{O} \cap \mathcal{S} \cap \hat{\alpha} \neq \emptyset$. Since this is true for all such $\mathcal{O}$, by an elementary argument from point set topology [36], $\mathcal{C}$ contains an unbounded subcontinuum $\hat{\mathcal{C}}$ lying in $\hat{\alpha} \cup \{(\mu, \hat{u}(\mu))\}$ and meeting $(\mu, \hat{u}(\mu))$. To find $(\lambda^*, u^*) \in \partial \mathcal{O} \cap \mathcal{S} \cap \hat{\alpha}$, note first that the bounds for solutions of (1.23) and maximality of $\hat{u}(\mu)$ imply that for any $\epsilon > 0$, there exists a $\delta > 0$ such that if $0 < \lambda - \mu < \delta$ and $(\lambda, u) \in \hat{\alpha} \cap \mathcal{S}$, then $u \in B_\epsilon(\hat{u}(\mu))$. Therefore by (1.36) and the properties of degree,

$$(1.37) \quad d(\Phi(\lambda, \cdot), \hat{\alpha}_\lambda, 0) = d(\Phi(\lambda, \cdot), \hat{\alpha}_\lambda \cap B_\epsilon(\hat{u}(\mu)), 0) = 1$$

for $0 < \lambda - \mu < \delta$. Choose $\epsilon, a > 0$ such that $\mathcal{O} \supset \{a\} \times B_\epsilon(\hat{u}(\mu))$ where $\mu < a < \mu + \delta$. Further choose $b > \mu$ and so large that $(\{b\} \times \hat{\alpha}_b) \cap \mathcal{O} = \emptyset$. By (1.36), (1.37), and the properties of degree [30, 36] there exists a subcontinuum $\mathcal{J}$ of $\mathcal{S}$ in $\hat{\alpha}$ which meets $\{a\} \times (\hat{\alpha}_a \cap B_\epsilon(\hat{u}(\mu)))$ and $\{b\} \times \hat{\alpha}_b$. Hence $\mathcal{J}$ must intersect $\partial \mathcal{O}$ at (λ^*, u^*) .

Next consider $\mathcal{C} - \hat{\mathcal{C}}$. By the definition of μ , $\mathcal{C} - \hat{\mathcal{C}}$ lies in $\{(\lambda, u) \in \mathbf{R}^+ \times P^+ \mid \lambda \geq \mu, \rho(\lambda) < \|u\|_1 < q(\lambda)\} - \hat{\alpha}$ where $\rho(\lambda)$ is chosen so that 0 is the

unique zero of $\Phi(\lambda, \cdot)$ in $\bar{B}_{\rho(\lambda)}$. If $\mathfrak{C} - \hat{\mathfrak{C}}$ were bounded in this set, we could find a bounded open set $\mathfrak{N}$ such that $\mathfrak{C} \subset \mathfrak{N} \cup \hat{\mathfrak{C}} \equiv \mathfrak{N}$ and $\mathfrak{S} \cap \partial\mathfrak{N} = \emptyset$. Letting $\mathfrak{N}_\lambda = \{u \in E \mid (\lambda, u) \in \mathfrak{N}\}$,

$$(1.38) \quad d(\Phi(\lambda, \cdot), \mathfrak{N}_\lambda, 0) \equiv \text{const.} = c_1, \quad \lambda \in \mathbf{R}.$$

The definition of μ implies $\Phi(\lambda, \cdot) \not\equiv 0$ for $\lambda < \mu$ and therefore $c_1 = 0$. But for large λ , $\mathfrak{N}_\lambda = \hat{\mathfrak{C}}_\lambda$, contradicting (1.36) and the corollary is proved.

Remark 1.39. Suppose $u(\lambda)$ is as in Theorem 1.13 and f is continuously differentiable in $\Omega \times \mathbf{R}^+$. Since $u(\lambda)$ minimizes $(0.1)_\lambda$,

$$I''(\lambda, u(\lambda))(v, v) \equiv \int_\Omega \left[\sum_{i,j=1}^n a_{ij}(x) v_{x_i} v_{x_j} + c(x) v^2 - \lambda f_u(x, u(\lambda, x)) v^2 \right] dx \geq 0$$

for all $v \in \tilde{E}$. This implies the spectrum of $\tilde{L}(u(\lambda)) = L - \lambda f_u(\cdot, u(\lambda))$ is non-negative. If 0 is not an eigenvalue of $\tilde{L}(u(\lambda))$, the implicit function theorem implies (0.1) possesses a continuously differentiable curve of solutions $(s, U(s))$ for s near λ with $U(\lambda) = u(\lambda)$. For fixed s the index of this curve is ± 1 so for this case a much simpler degree argument can be used to get a second solution of $(0.1)_\lambda$. However there is no guarantee that this case occurs. If 0 is an eigenvalue of $\tilde{L}(u(\lambda))$, it is simple and the corresponding eigenfunction v_1 is of one sign in Ω . It can be made unique by requiring that $v_1 \in P^+$ and $\|v_1\|_1 = 1$. Suppose further that f satisfies the hypotheses of Corollary 1.34 so $f(\cdot, u(\lambda)) \geq 0$ in Ω and $\not\equiv 0$. Considering $\tilde{L}(u(\lambda))$ as a mapping from $C^2(\Omega)$ to $C(\Omega)$, $f(\cdot, u(\lambda))$ does not lie in the range of $\tilde{L}(u(\lambda)) = \{\varphi \in C^1(\Omega) \mid \int_\Omega \varphi v_1 dx = 0\}$. It then follows from Theorem 3.2 of [8] that if Z is any complement of $\text{span } \{v_1\}$ in $C^2(\Omega)$, there is a C^1 curve of solutions of (0.1) of the form $(\lambda + \tau(s), u(\lambda) + sv_1 + z(s))$ where $z(s) \in Z$ and $\tau(0) = 0 = \tau'(0)$, $z(0) = 0 = z'(0)$. Thus under these conditions $(\lambda, u(\lambda))$ lies on a curve of solutions of (0.1) for each $\lambda \geq \lambda$. However we do not know how to continue these curves or piece them together in any global fashion.

Remark 1.40. Using arguments similar to those of Corollary 1.34, it can also be shown in the context of Theorem 1.32 that for all $\lambda > \lambda$ there are solution pairs which lie on the same component of $\mathfrak{S}$. However we were not able to show the corresponding components were unbounded in $\mathbf{R} \times E$, the difficulty being that $w(\lambda)$ need not be a continuous function of λ .

Note that under the hypotheses of Corollary 1.34, $\hat{u}(\lambda)$ is a pointwise monotone increasing function of λ .

The condition that there exists a line such as $z = \bar{z}$ along which f vanishes is not necessary. More general zero sets can be permitted for f . The hypotheses of Corollary 1.32 can be satisfied in particular in the "autonomous" case which we treat next.

Corollary 1.41. Suppose $g(z)$ satisfies (f_1) , (f_4) , $g > 0$ in a deleted neighborhood of $z = 0$, and $g(\hat{z}) = 0$ for some $\hat{z} > 0$. Then $\mathfrak{S}$ possesses a component $\mathfrak{C}$

as in Corollary 1.34 with $0 \leq \bar{u}(\lambda)$, $u(\lambda) \leq z_1$ where z_1 is the smallest positive zero of g .

Proof. Set $f(z) = g(z)$ for $z \in [0, z_1]$ and $f \equiv 0$ for $z > z_1$. The result then follows immediately from Corollary 1.34.

Remark 1.42. Under the hypotheses of Corollary 1.41, (f_2) is unnecessary and, in particular, functions such as $g(z) = z^2(1 - z)^2$ are permitted.

If in Corollary 1.41, $\hat{z} \neq z_1$, by truncating g at $\hat{z}$ and, e.g., extending it in a Lipschitz continuous fashion to be negative for $z > \hat{z}$, Theorem 1.32 becomes applicable. The same procedure may be carried out at any zero of g . However we may not obtain any additional solutions by repeatedly invoking Theorem 1.32 in this fashion. For example suppose $\underline{z} < \bar{z}$ are a pair of adjacent zeroes of g with $g < 0$ in $[\underline{z}, \bar{z}]$ and g is truncated as above at $\bar{z}$. Let u be a solution of $(0.1)_\lambda$ with $0 \leq u \leq \bar{z}$. The corresponding maximal solution $\hat{u}$ is obtained by the iteration scheme (1.19) and the maximum principle implies $\hat{u} < u_0 \equiv \bar{z}$ in Ω . Since (1.1) shows the maximum of u cannot lie in $(\underline{z}, \bar{z})$, $0 \leq u \leq \underline{z}$. Thus working with $\bar{z}$ adds nothing new. On the other hand if, e.g., $g > 0$ in $(\underline{z}, \bar{z})$ and for some $\bar{\lambda} \geq \lambda$, $I(\bar{\lambda}, \cdot)$ possesses a smaller negative minimum on $\{\varphi \in \tilde{E} \mid 0 \leq \varphi \leq \bar{z}\}$ than on $\{\varphi \in \tilde{E} \mid 0 \leq \varphi \leq \underline{z}\}$, an analysis of our arguments shows $(0.1)_\lambda$ will possess a pair of solutions distinct from those on $\mathcal{C}$ for all $\lambda > \bar{\lambda}$. We do not know if this can also be shown via Theorem E of [4].

§2. Varying the spectrum of L . The assumptions made on its coefficients in §1 together with the boundary conditions imply the spectrum of L is contained in the positive reals. In this section we briefly study the effect of varying the spectrum of L on the results of §1. Of interest in particular is what happens when L has nonpositive eigenvalues. Let

$$(2.1) \quad L_t \equiv - \sum_{i,j=1}^n \frac{\partial}{\partial x_i} \left(a_{ij}(x) \frac{\partial}{\partial x_j} \right) + tc(x) \equiv \hat{L} + tc, \quad t \in \mathbf{R},$$

where the a_{ij} are as in §1, c is Hölder continuous and is such that the smallest eigenvalue of L_1 is negative. Consider

$$(2.2)_t \quad L_t u = \lambda f(x, u), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega.$$

A solution of $(2.2)_t$ is a pair (λ, u) . We will study how the solution set of $(2.2)_t$ changes as t ranges over $[0, 1]$.

Let $\mu_1(t)$ denote the smallest eigenvalue of L_t under the boundary conditions. The variational characterization of $\mu_1(t)$ [31 I] implies that $\mu_1(t)$ is a continuous function of t with $\mu_1(0) > 0$, $\mu_1(1) < 0$ and therefore there exists $\underline{t} \in (0, 1)$ with $\mu_1(\underline{t}) = 0$. Moreover for $t > \underline{t}$ its variational characterization implies $\mu_1(t)$ is monotonic decreasing for $t > \underline{t}$ so $\underline{t}$ is the unique zero of $\mu_1(t)$. The structure of the solution set for $(2.2)_t$ depends strongly on whether $t < \underline{t}$ or $t \geq \underline{t}$.

Theorem 2.3. *If f satisfies (f_1) – (f_4) , there exists $\bar{\lambda} > 0$ such that for each*

$t \in [0, \underline{t}]$ and all $\lambda > \bar{\lambda}$, $(2.2)_t$ possesses at least 2 distinct solutions $(\lambda, \bar{u}(\lambda, t))$, $(\lambda, \underline{u}(\lambda, t)) \in \mathbf{R}^+ \times P^+$ while for $t \in [\underline{t}, 1]$, $(2.2)_t$ has at least one solution $(\lambda, \bar{u}(\lambda, t)) \in \mathbf{R}^+ \times P^+$.

Proof. Consider

$$(2.4) \quad \hat{L}u + \max(c, 0)u = \lambda f(x, u), \quad x \in \Omega; \quad u = 0, \quad x \in \partial\Omega.$$

By Theorem 1.32, there is a $\lambda > 0$ such that the conclusions of Theorem 1.32 hold for (2.4) for all $\lambda > \lambda$. Let $(\lambda, \hat{w}(\lambda)) \in \mathbf{R}^+ \times P^+$ be any solution of (2.4). Then

$$(2.5) \quad \hat{L}\hat{w} + t\hat{w} \leq \hat{L}\hat{w} + \max(c, 0)\hat{w} = \lambda f(x, \hat{w})$$

and $\hat{w}$ is a subsolution for $(2.2)_t$ for all $t \in [0, 1]$. Since $f(x, \bar{z}) < 0$, arguing as in (1.1), there exists $\hat{\lambda} > 0$ such that $\bar{z}$ is a supersolution for $(2.2)_t$ for all $\lambda \geq \hat{\lambda}$ and $t \in [0, 1]$. Hence by earlier remarks $(2.2)_t$ possesses a solution $(\lambda, \bar{u}(\lambda)) \in \mathbf{R}^+ \times P^+$ for all $t \in [0, 1]$ and all $\lambda > \bar{\lambda} = \max(\lambda, \hat{\lambda})$.

As in Remark 2.18, for each $t \in [0, \underline{t}]$ and fixed λ , $(\lambda, 0)$ is an isolated solution of $(2.2)_t$. Therefore by converting $(2.2)_t$ to an equivalent equation $\Phi_t(\lambda, u) = 0$ in $\mathbf{R} \times E$ as in §1 and using the degree calculation of Theorem 1.32 for $t = 0$ together with the above $\hat{w}(\lambda)$, a simple homotopy argument gives a second solution $(\lambda, \underline{u}(\lambda)) \in \mathbf{R}^+ \times P^+$ of $(2.2)_t$ distinct from $(\lambda, \bar{u}(\lambda))$.

Remark 2.6. Fix $\lambda > \bar{\lambda}$ and consider the solutions (t, u) of $(2.2)_\lambda$. Since $L_{\underline{t}}$ has a null space, the solutions $(t, 0)$ of $(2.2)_\lambda$ may not be isolated for t near $\underline{t}$. In fact if, e.g., $c < 0$ in $\bar{\Omega}$, then for all $\lambda > \bar{\lambda}$, $(\underline{t}, 0)$ is a bifurcation point for $(2.2)_\lambda$. This follows from results of [22] since $\underline{t}$ is a simple eigenvalue of

$$(2.7) \quad \hat{L}v = -scv, \quad x \in \Omega; \quad v = 0, \quad x \in \partial\Omega.$$

Moreover by modifying the argument of Theorem 2.12 of [22], $(\underline{t}, 0)$ lies on an unbounded component $\mathcal{D}$ of solutions of $(2.2)_\lambda$ in $(\mathbf{R} \times P^+) \cup \{(\underline{t}, 0)\}$. If in addition $f > 0$ near $z = 0$, it is not difficult to show there exists $\bar{t} > -\infty$ such that $\mathcal{D} \subset [\bar{t}, \infty) \times P^+$ and for each t in some interval $[\underline{t} - \delta, \underline{t}]$, $(2.2)_t$ has at least two distinct solutions on $\mathcal{D}$. We suspect a more subtle argument would show $\underline{t} - \delta < 0$, thereby giving an independent proof of Theorem 1.32 for this case.

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