A Characterization of the Positive Cone of $\mathbf{B}(\mathfrak{h})$

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$\mathfrak{h}$ is a complex Hilbert space with inner product $(\cdot, \cdot)$. $\mathbf{B}(\mathfrak{h})$ is the algebra of all bounded linear operators on $\mathfrak{h}$. $\mathbf{G}(\mathfrak{h})$ is the group of invertible elements of $\mathbf{B}(\mathfrak{h})$. The subset of accretive operators $\mathbf{A}(\mathfrak{h})$ is defined by

$$(1) \quad \mathbf{A}(\mathfrak{h}) = \{T \in \mathbf{B}(\mathfrak{h}) \mid \operatorname{Re} T \equiv \tfrac{1}{2}(T + T^*) \geq 0\},$$

while

$$(2) \quad \mathbf{P}(\mathfrak{h}) = \{T \in \mathbf{A}(\mathfrak{h}) \mid T = T^*\}$$

is the cone of non-negative self-adjoint operators. The principal result of this paper is the following characterization of $\mathbf{P}(\mathfrak{h})$:

Theorem 1. $T \in \mathbf{P}(\mathfrak{h})$ if and only if $T^n \in \mathbf{A}(\mathfrak{h})$, $n = 1, 2, \dots$.

This theorem seems to have first been proved by C. R. Johnson [9] in the case in which $\mathfrak{h}$ is finite dimensional. His proof is decidedly matricial in character, whereas our proof in the general case makes fundamental use of a mapping theorem for numerical ranges due to T. Kato [10]. In Section 5, we observe that the Kato theorem is in reality a mapping theorem for spectral sets.

An immediate consequence of Theorem 1 is that the only semi-groups of accretive operators are semi-groups of non-negative operators which of necessity are commutative. In Sections 2 and 3 we present the main results along with some generalizations and some discussions of open problems. Section 4 contains an application of the methods of this paper to multiplicative commutators on $\mathbf{G}(\mathfrak{h})$ for which we give an improvement of a theorem of Putnam [11].

§1. Notation and preliminaries. For $r > 0$, $\Delta_r = \{\lambda \in \mathbf{C} \mid |\lambda| \leq r\}$ is the closed disk of radius r centered at 0. $\Pi = \{\lambda \in \mathbf{C} \mid \operatorname{Re} \lambda \geq 0\}$ is the closed right half plane. For $0 < \alpha < 2$, $S_\alpha = \{\lambda \in \mathbf{C} \mid |\arg \lambda| \leq (\pi/2)\alpha\}$ is the closed wedge symmetric in the non-negative real axis centered at 0 and of opening $\pi\alpha$. If $\Sigma \subset \mathbf{C}$, Σ° is the interior of Σ ; $\operatorname{Re} \Sigma = \{t \in \mathbf{R} \mid t + is \in \Sigma, s \in \mathbf{R}\}$; $\operatorname{co} \Sigma$ is the closed convex hull of Σ ; $\partial \Sigma$ is the boundary of Σ .

For $T \in \mathbf{B}(\mathfrak{h})$, $\sigma(T)$ is the spectrum of T , $\rho(T)$ is the spectral radius of T , and $W(T)$ is the numerical range of T :

$$(3) \quad W(T) = \{(Tx, x) \mid \|x\| = 1\}.$$

The basic properties of $W(T)$ may be found in Halmos [7]. We mention a few explicitly: (i) $W(T)$ is convex, its closure $\overline{W(T)}$ is compact and contains $\sigma(T)$, and $W(T) = \{0\}$ implies $T = 0$; (ii) $W(T + S) \subset W(T) + W(S)$ and the map $W : \mathbf{B}(\mathfrak{h}) \rightarrow 2^{\mathbb{C}}$ is continuous in the sense that $W(T + S) \subset W(T) + \Delta_{\|S\|}$; (iii) $\operatorname{Re} W(T) = \overline{W(\operatorname{Re} T)}$, where $\operatorname{Re} T = \frac{1}{2}(T + T^*)$; (iv) if T is normal ($TT^* = T^*T$), then $\overline{W(T)} = \overline{\operatorname{co} \sigma(T)}$; (v) [18] $0 \notin \overline{W(S)}$ implies $\sigma(TS^{-1}) = \sigma(S^{-1}T) = \overline{W(T)/\overline{W(S)}}$; (vi) $H \in \mathbf{P}(\mathfrak{h}) \cap \mathbf{G}(\mathfrak{h})$ implies $\overline{W_H(T)} \subset \overline{W(TH)}/\overline{W(H)}$ where $W_H(T)$ is the numerical range of T relative to the inner product $(x, y)_H = (H^{-1}x, y)$ on $\mathfrak{h}$; moreover if $T \geq 0$ relative to the inner product $(x, y)_H$, then $H^{-1/2}TH^{1/2} \in \mathbf{P}(\mathfrak{h})$. We remark that $\mathbf{A}(\mathfrak{h}) = \{T \in \mathbf{B}(\mathfrak{h}) \mid W(T) \subset \Pi\} = \{T \in \mathbf{B}(\mathfrak{h}) \mid \operatorname{Re} W(T) \geq 0\}$ and $\mathbf{P}(\mathfrak{h}) = \{T \in \mathbf{B}(\mathfrak{h}) \mid W(T) \geq 0\}$.

The mapping theorem for numerical ranges mentioned in the introduction is

Lemma 1. (Kato [10]). *If f is a complex valued function holomorphic on a neighborhood Ω of Π , then $T \in \mathbf{A}(\mathfrak{h})$ implies $W(f(T)) \subset \operatorname{co} f(\Pi)$.*

Note that $\sigma(T) \subset \overline{W(T)} \subset \Pi \subset \Omega$ so that $f(T)$ is well defined as an element of $\mathbf{B}(\mathfrak{h})$ by the Dunford formula [5]:

$$(4) \quad f(T) = \frac{1}{2\pi} \int_{\Gamma} f(\lambda)(\lambda - T)^{-1} d\lambda,$$

where Γ may be taken as any rectifiable Jordan curve in Ω whose interior domain contains $\sigma(T)$. Actually it is more convenient for us to use a simple modification of Lemma 1 which we now derive. In Section 5 we shall show that a stronger conclusion holds for both Lemmas 1 and 2 and in this manner give a derivation of Lemma 2 (and Lemma 1) directly.

Lemma 2. *If g is a complex-valued function holomorphic on Π° , then $T \in \mathbf{A}(\mathfrak{h})$ and $\operatorname{Re} \sigma(T) > 0$ imply $W(g(T)) \subset \operatorname{co} g(\Pi^\circ)$.*

Proof. Observe that $g(T)$ is well defined by (4) since $\operatorname{dist}(\sigma(T), \mathbb{C} \setminus \Pi^\circ) \geq 2\epsilon > 0$. Define g_ϵ on $\Pi^\circ - \epsilon$ by $g_\epsilon(\lambda) = g(\lambda + \epsilon)$; then g_ϵ is holomorphic on a neighborhood of Π . By Lemma 1, $W(g_\epsilon(\Pi)) \subset \operatorname{co} g_\epsilon(\Pi) \subset \operatorname{co} g(\Pi^\circ)$. In (4) take Γ to lie in Π° , then

$$g_\epsilon(T) - g(T) = \frac{1}{2\pi i} \int_{\Gamma} (g(\lambda + \epsilon) - g(\lambda))(\lambda - T)^{-1} d\lambda.$$

Consequently

$$\|g_\epsilon(T) - g(T)\| \leq \frac{1}{2\pi} M_r(T) \ell(\Gamma) \max_{\lambda \in \Gamma} |g(\lambda + \epsilon) - g(\lambda)|,$$

where $M_r(T) = \max_{\lambda \in \Gamma} \|(\lambda - T)^{-1}\|$ and $\ell(\Gamma)$ is the length of Γ . Since $g_\epsilon(\lambda)$

converges uniformly to $g(\lambda)$ on Γ , $\|g_\epsilon(T) - g(T)\| \rightarrow 0$ as $\epsilon \rightarrow 0^+$ and the continuity of W yields $W(g(T)) \subset \text{co } g(\Pi^\circ)$.

In the sequel our primary concern is with numerical ranges of roots of operators in $\mathbf{A}(\mathfrak{h})$. For $T \in \mathbf{B}(\mathfrak{h})$ with $\text{Re } \sigma(T) > 0$ and for $\beta \in \mathbf{R}$, we define $T^\beta \in \mathbf{B}(\mathfrak{h})$ by (4) with $g_\beta(\lambda) = \lambda^\beta$, $g_\beta(1) = 1$, and with Γ in Π° . When β is an integer this definition agrees with the usual (algebraic) integral power of T . From [5] Chapter VII, 3.12, the following lemma is apparent.

Lemma 3. *Let $T \in \mathbf{B}(\mathfrak{h})$ with the $\text{Re } \sigma(T) > 0$. If $\alpha, \beta \in \mathbf{R}$ and $\alpha \in [-1, 1]$, $\alpha \neq 0$, then*

- (a) $T^\beta \in \mathbf{G}(\mathfrak{h})$ and commutes with all operators commuting with T ;
- (b) $\sigma(T^\alpha) \subset S_{|\alpha|}^\circ$ and $(T^\alpha)^\beta = T^{\alpha\beta}$;
- (c) $\sigma(B) \subset S_{|\alpha|}^\circ$ and $B^{1/\alpha} = T$ imply $T^\alpha = B$;
- (d) The map $T \rightarrow T^\alpha$ is continuous in the uniform topology of $\mathbf{B}(\mathfrak{h})$.

In the event that $g(\lambda) = \lambda^\alpha$, Lemma 2 may be strengthened as follows.

Proposition 1. *If $A \in \mathbf{A}(\mathfrak{h})$ with $\text{Re } \sigma(A) > 0$ and if $\alpha \in [-1, 1]$, $\alpha \neq 0$, then $W(A^\alpha) \subset S_{|\alpha|}$. When $\alpha \neq \pm 1$, $W(A^\alpha) \subset S_{|\alpha|} \setminus \{0\}$; when $W(A) \subset \Pi^\circ$, $W(A^\alpha) \subset S_{|\alpha|}^\circ$.*

Proof. The first statement is merely Lemma 2 in this special case. If $\alpha \neq \pm 1$ and if $0 \in W(A^\alpha)$, then 0 is a corner of $W(A^\alpha)$ so that $0 \in \sigma(A^\alpha)$ ([8], Theorem 1) contradicting (a) of Lemma 3. If $W(A) \subset \Pi^\circ$, then $\inf \{\text{Re } \lambda \mid \lambda \in W(A)\} \geq 2\epsilon$ with $0 < \epsilon < 1$. Set $A_\epsilon = A - \epsilon$ so that $W(A_\epsilon) \subset \Pi^\circ$. Apply Lemma 2 to A_ϵ with $g_\epsilon(\lambda) = (\lambda + \epsilon)^\alpha$ to obtain $W(g_\epsilon(A_\epsilon)) \subset \text{co } g_\epsilon(\Pi^\circ)$. But $g_\epsilon(A_\epsilon) = A^\alpha$ and $g_\epsilon(\Pi^\circ)$ is a closed convex set contained in $S_{|\alpha|}^\circ$.

§2. Main results. Theorem 1 is a fairly easy consequence of Proposition 1. Nevertheless we prefer to deduce it from a general non semigroup version of Theorem 1, namely

Theorem 2. *Let $\{T_n\}_{n=1}^\infty$ be a sequence in $\mathbf{A}(\mathfrak{h})$ satisfying $\text{Re } \sigma(T_n) > 0$, $n = 1, 2, \dots$ and let $\{r_n\}_{n=1}^\infty$ be a sequence in $\mathbf{R}^+$ converging to 0. If $\lim_{n \rightarrow \infty} T_n^{r_n} = T$ (uniform topology of $\mathbf{B}(\mathfrak{h})$), then $T \in \mathbf{P}(\mathfrak{h})$.*

Proof. There is an n_0 such that $r_n < 1$ for $n \geq n_0$. By Proposition 1 $W(T_n^{r_n}) \subset S_{r_n}$ for $n \geq n_0$. Since $T_n^{r_n} \rightarrow T$, the continuity of W and the fact that $r_n \rightarrow 0$ imply $W(T) \subset \mathbf{R}^+$. Hence $T \in \mathbf{P}(\mathfrak{h})$.

Theorem 1. *$T \in \mathbf{P}(\mathfrak{h})$ if and only if $T^n \in \mathbf{A}(\mathfrak{h})$, $n = 1, 2, \dots$.*

Proof. Since $\mathbf{P}(\mathfrak{h}) \subset \mathbf{A}(\mathfrak{h})$ and $T \in \mathbf{P}(\mathfrak{h})$ implies $T^n \in \mathbf{P}(\mathfrak{h})$, the necessity is clear. To prove the sufficiency we first observe that we may replace the hypothesis $T^n \in \mathbf{A}(\mathfrak{h})$, $n = 1, 2, \dots$ by the stronger hypothesis $W(T^n) \subset \Pi^\circ$, $n = 1, 2, \dots$. For by (ii) of §1 $W((T + \epsilon)^n) \subset \Pi + \epsilon^n \subset \Pi^\circ$ if $T^k \in \mathbf{A}(\mathfrak{h})$, $1 \leq k \leq n$ and $\epsilon > 0$. If $(T + \epsilon) \in \mathbf{P}(\mathfrak{h})$ for all $\epsilon > 0$, it follows that $T \in \mathbf{P}(\mathfrak{h})$. Hence assume $W(T^n) \subset \Pi^\circ$,

$n = 1, 2, \dots$. Thus $\sigma(T)^n = \sigma(T^n) \subset \Pi^\circ$, $n = 1, 2, \dots$. Therefore $\sigma(T) \subset \mathbf{R}^+$ and $(T^n)^{1/n} = T$ by (c) of Lemma 3. Theorem 2 may be applied to $T_n = T^n$ and $r_n = 1/n$ to give $T \in \mathbf{P}(\mathfrak{h})$.

Two immediate corollaries of Theorem 1 are given below.

Corollary 1. *Let $H \in \mathbf{P}(\mathfrak{h}) \cap \mathbf{G}(\mathfrak{h})$. If $T^n H \in \mathbf{A}(\mathfrak{h})$, $n = 1, 2, \dots$, then $H^{-1/2} T H^{1/2} \in \mathbf{P}(\mathfrak{h})$.*

Proof. By (vi) of §1 $W_H(T^n) \subset \Pi$, $n = 1, 2, \dots$. Hence Theorem 1 implies $T \geq 0$ relative to the $(x, y)_H = (H^{-1}x, y)$ inner product. Again by (vi), it follows that $H^{-1/2} T H^{1/2} \in \mathbf{P}(\mathfrak{h})$.

Corollary 2. *The only multiplicative semi-groups in $\mathbf{A}(\mathfrak{h})$ are the (necessarily commutative) semi-groups in $\mathbf{P}(\mathfrak{h})$.*

It is interesting to consider what happens when the condition $H = H^*$ in Corollary 1 is removed. A result of this type is contained in Proposition 2. Theorem 3 below is actually a corollary of Proposition 2 and of Theorem 2 so that it implies Theorem 1.

Proposition 2. *If $T^n D \in \mathbf{A}(\mathfrak{h})$, $n = 0, 1, 2, \dots$, then $0 \in \overline{W(D)}$ or $\sigma(T) \subset \mathbf{R}$.*

Proof. Assume $0 \notin \overline{W(D)}$. Because $\operatorname{Re} W(D) \geq 0$ and $\overline{W(D)}$ is convex, there exists $0 < \alpha < 1$ such that one of the two sets $e^{\pm i(\pi/2)\alpha} \overline{W(D)}$ is contained in $S_{1-\alpha}$. Consequently by (v) $\sigma(T^n) = \sigma(T)^n \subset \overline{W(T^n D)}/\overline{W(D)}$, $n = 1, 2, \dots$, so that $\sigma(T^n) \subset K$, where K omits an open non-void sector with vertex at zero and negative real axis as edge. But the hypotheses of the theorem are unchanged if T is replaced by $T + r$, $r \geq 0$, so that $\sigma((T + r)^n) \subset K$, $r \geq 0$ and $n = 1, 2, \dots$. Thus $\lambda \in \sigma(T)$ implies $(\lambda + r)^n \in K$, $r \geq 0$ and $n = 1, 2, \dots$. Clearly this is possible only if $\lambda \in \mathbf{R}$.

Theorem 3. *Let $\overline{W(D)} \subset \Pi^\circ$ and $T^n D \in \mathbf{A}(\mathfrak{h})$, $n = 1, 2, \dots$. If $TD = DT$, then $T \in \mathbf{P}(\mathfrak{h})$.*

Proof. By Proposition 2, $\sigma(T) \subset \mathbf{R}$. Moreover $\overline{W(D)} \subset \Pi^\circ$ implies the existence of $0 \leq \beta < 1$ such that $\overline{W(D)} \subset S_\beta^\circ$ so that by (v), $\sigma(T) \subset S_{1+\beta}$. Hence $\sigma(T) \subset \mathbf{R}^+$. As in the proof of Theorem 1 we may replace T by $T_\epsilon = T + \epsilon$, $\epsilon > 0$ and $\sigma(T_\epsilon) \subset \mathbf{R}^+$. Since $T_\epsilon D = DT_\epsilon$, $T_\epsilon D^{1/n} = D^{1/n} T_\epsilon$ and $\sigma(D^{1/n}) \subset S_{1/n}^\circ$ by (b) of Lemma 3. Furthermore [13] $\sigma(T_\epsilon D^{1/n}) \subset \sigma(T_\epsilon) \sigma(D^{1/n}) \subset S_{1/n}^\circ$ and $(T_\epsilon D^{1/n})^n = T_\epsilon^n D$ so that by (c) of Lemma 3 $(T_\epsilon^n D)^{1/n} = T_\epsilon D^{1/n}$. Because $\lim_{n \rightarrow \infty} D^{1/n} = 1$, Theorem 2 may be applied with $T_n = T_\epsilon^n D$ and $r_n = 1/n$ to give $T_\epsilon \in \mathbf{P}(\mathfrak{h})$. Therefore $T \in \mathbf{P}(\mathfrak{h})$.

A slightly different version of Theorem 3 is given by

Corollary 3. *Suppose that $T^n B \in \mathbf{A}(\mathfrak{h})$, $n = 1, 2, \dots$, and that $\overline{W(B)} \subset \Pi^\circ$. If $TV = VT$, where $B = V|B| = V(B^*B)^{1/2}$ is the polar decomposition of B , then $|B|^{-1/2} T |B|^{1/2} \in \mathbf{P}(\mathfrak{h})$.*

Proof. By (vi) of §1, $T^n B \in \mathbf{A}(\mathfrak{h})$ implies $W_{|B|}(T^n V) \subset \Pi$, $n = 1, 2, \dots$ and

$\overline{W(B)} \subset \Pi^\circ$ implies $\overline{W_{|B|}(V)} \subset \Pi^\circ$. As T commutes with V we conclude from Theorem 3 that $T \geq 0$ relative to the $|B|$ inner product. Therefore, again by (vi), $|B|^{-1/2} T |B|^{1/2} \in \mathbf{P}(\mathfrak{h})$.

§3. Extensions and remarks. We first remark on possible extensions of Theorem 1 to Banach algebras other than $\mathbf{B}(\mathfrak{h})$ and then comment on some questions raised by the results of §2.

(1). For a general unital Banach algebra $\mathfrak{A}$ over the complex field, the numerical range $V(a)$ of an element $a \in \mathbf{A}$ (see [1]) is given by

$$V(a) = \{f(a) \mid f \in \mathfrak{A}^*, f(1) = 1 = \|f\|\}.$$

Here $\mathfrak{A}^*$ is the dual of $\mathfrak{A}$. The positive hermitian (self-adjoint) elements $P(\mathfrak{A})$ of $\mathfrak{A}$ are those for which $V(a) \geq 0$. The desired characterization of $a \in \mathbf{P}(\mathfrak{A})$ would be $\operatorname{Re} V(a^n) \geq 0$, $n = 1, 2, \dots$. Unfortunately in such a general setting this characterization is false. In [12] the method of Bollobas [2] is used to construct a commutative unital Banach algebra $\mathfrak{A}$ which is generated by an element a with $V(a) = [-1, 1]$ so that $1 + a \in \mathbf{P}(\mathfrak{A})$. In such Bollobas algebras every entire function $\hat{f}(z)$ with $\hat{f}(1) = 1$ and $|\hat{f}(z)| \leq \exp |\operatorname{Re} z|$ determines an $f \in \mathfrak{A}^*$ with $f(1) = 1 = \|f\|$. $\hat{f}(z) = 5/9 + 4/9 \cosh z - 2/3 \sinh z$ is such a function and it is found that $f((1+a)^3) = -1/3$. It is also shown in [12] that the sufficiency part of the characterization is false in $\mathbf{B}(\mathbf{B}(\mathfrak{h}))$.

We do not know whether the conjectured characterization is valid in more restrictive Banach algebras (C^* algebras aside) such as hermitian algebras. We remark that there appears to be a dismal lack of mapping theorems for numerical ranges in general Banach algebras.

If $\mathfrak{A}$ is a C^* -algebra, then $\mathfrak{A}$ has a faithful $*$ -representation as a closed self-adjoint subalgebra of $\mathbf{B}(\mathfrak{h})$ for some Hilbert space $\mathfrak{h}$. Moreover, for $a \in \mathfrak{A}$, $V(a)$ coincides with $\overline{W(T(a))}$, where $T(a)$ is the representation of a in $\mathbf{B}(\mathfrak{h})$, and $T(\mathbf{P}(\mathfrak{A}))$ coincides with $\mathbf{P}(\mathfrak{h}) \cap T(\mathfrak{A})$. Consequently Theorem 1 may be trivially transplanted to this situation. An interesting result is obtained when $\mathfrak{h}$ is a separable Hilbert space, $\mathcal{K}(\mathfrak{h})$ is the closed ideal of compact operators in $\mathbf{B}(\mathfrak{h})$, and $\mathfrak{A} = \mathbf{B}(\mathfrak{h})/\mathcal{K}(\mathfrak{h})$ is the Calkin algebra of $\mathbf{B}(\mathfrak{h})$ [3]. $\mathfrak{A}$ is a C^* -algebra [3]. Stampfli and Williams [6], [16] introduced the notion of an essential numerical range $W_*(T)$ for $T \in \mathbf{B}(\mathfrak{h})$:

$$W_*(T) = \bigcap_{K \in \mathcal{K}} \overline{W(T + K)}.$$

They showed that $W_*(T) = V(p(T))$ where p is the canonical homomorphism of $\mathbf{B}(\mathfrak{h})$ into $\mathfrak{A}$. Since $\mathfrak{A}$ is a C^* algebra, we have

Theorem 4. *Let $T \in \mathbf{B}(\mathfrak{h})$; then $W_*(T^n) \subset \Pi$, $n = 1, 2, \dots$ if and only if $\operatorname{Im} T \equiv (1/2i)(T - T^*) \in \mathcal{K}(\mathfrak{h})$ and $W_*(T) \geq 0$.*

(2). We observe that Theorem 2 implies and is implied by

Theorem 2'. Let $\{T_n\}_{n=1}^\infty$ be a sequence in $\mathbf{B}(\mathfrak{h})$, $\{r_n\}_{n=1}^\infty$ a sequence in $\mathbf{R}^+$ which converges to 0, and $\{\theta_n\}_{n=1}^\infty$ a sequence in $\mathbf{R}$ such that $(2/\pi) \sup_n |\theta_n| = \alpha < 1$. If $\sigma(T_n) \subset S_{1-\alpha}^\circ$, $n = 1, 2, \dots$, and if $W(T_n) \subset e^{i\theta_n}\Pi$, then $\lim_{n \rightarrow \infty} T_n^{r_n} = T$ implies $T \in \mathbf{P}(\mathfrak{h})$.

(3). Proposition 2 is "sharp" in that there exist T and D such that $T^n D \in \mathbf{A}(\mathfrak{h})$, $n = 0, 1, \dots$, $0 \in \overline{W(D)}$ and $\sigma(T)$ is not real. For, if $\mathfrak{h} = \mathbf{C}^2$, then

$$D = i \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad T = i \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

satisfy the above conditions. If the assumption $\overline{W(D)} \subset \Pi^\circ$ of Theorem 3 is replaced by the weaker assumption $D \in \mathbf{A}(\mathfrak{h})$ and $0 \notin \overline{W(D)}$, then the conclusion $T \in \mathbf{P}(\mathfrak{h})$ is not valid. This follows from the example:

$$\mathfrak{h} = \mathbf{C}^2, D = \begin{bmatrix} i & 0 \\ 0 & 1 \end{bmatrix}, \quad \text{and} \quad T = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

However the following slight generalization of Theorem 3 does hold.

Theorem 3'. Let $T^n D \in \mathbf{A}(\mathfrak{h})$, $n = 0, 1, 2, \dots$. If $TD = DT$, then either $0 \in \overline{W(D)}$ or $T = T^*$.

(4). If the commutativity assumption is dropped in Theorem 3, we do not know whether the conclusion remains valid. Nevertheless it is remarkable that the following theorem holds:

Theorem 3''. If $T^n B \in \mathbf{A}(\mathfrak{h})$, $n = 0, 1, 2, \dots$ and if T is convexoid $\overline{W(T)} = \overline{\text{co } \sigma(T)}$, then $\overline{W(B)} \subset \Pi^\circ$ implies $T \in \mathbf{P}(\mathfrak{h})$.

Note that if T is normal or even hyponormal, T is convexoid. The proof of the theorem follows from Proposition 2 and from the fact that $\sigma(T) \geq 0$ since $\overline{W(B)} \subset \Pi^\circ$ (see proof of Theorem 3). Comparing Theorem 3'' with Corollary 3, we observe that T convexoid, $TV = VT$, and $T^n B \in \mathbf{A}(\mathfrak{h})$, $n = 0, 1, 2, \dots$ must imply $TB = BT$.

(5). Consider the following situation:

$W(T^n) \subset e^{i\theta}\Pi$, $n = 1, 2, \dots$ for some $\theta \in (-\pi, \pi]$. If $|\theta| < \pi/2$, it follows from Theorem 3 that $T \in \mathbf{P}(\mathfrak{h})$. If $|\theta| \geq \pi/2$ it is readily seen that $\sigma(T) \subset \mathbf{R}$ and in fact $\sigma(T) = \{0\}$ for $|\theta| > \pi/2$. For $|\theta| = \pi/2$, Theorem 3' yields $T = T^*$. When $|\theta| > \pi/2$, $\sigma(T) = \{0\}$ and $0 \in \partial W(T^n)$, $n = 1, 2, \dots$. A result of Sinclair and Crabb [15] may be invoked to give $T = 0$.

§4. Multiplicative commutators. In [11] Putman studies the multiplicative commutator

$$(5) \quad T = VB V^{-1} B^{-1}$$

of a pair of invertible operators V and B where

$$(6) \quad TV = VT \quad \text{and} \quad V^{-1} = V^*.$$

His results depend essentially upon the induced spectral properties of T when the sets $W(T^n B)$, $n = 1, 2, \dots$ are suitably constrained. We state his principal result as Proposition 3 which we deduce from Proposition 2. As a matter of fact a considerably stronger conclusion than that of Proposition 3 is true under the same hypotheses. This is the content of Theorem 5 below, which turns out to be a direct consequence of the preceding proposition and of the Sz.-Nagy theorem [17] on bounded representations of the additive group of integers in $B(\mathfrak{h})$. We are indebted to the referee for this observation. In [4] Theorem 5 is obtained as a special instance of a more general multiplicative commutator result which, however, is proved by methods completely different from those in this paper.

Proposition 3. (Putnam [11]). *If (5) and (6) hold, then $0 \notin \overline{W(B)}$ or $\sigma(T) = \{1\}$.*

Proof. From (5) and (6) it follows easily that

$$(7) \quad T^n B = V^n B V^{*n}, \quad n \in \mathbf{Z}.$$

Hence

$$(8) \quad \|T^n\| \leq \|B\| \|B^{-1}\|, \quad n \in \mathbf{Z},$$

and in view of the unitary invariance of the numerical range

$$(9) \quad W(T^n B) \subset W(B), \quad n \in \mathbf{Z}.$$

Now assume that $0 \notin \overline{W(B)}$. By the convexity of $\overline{W(B)}$ there exists $\theta \in [0, 2\pi)$ such that $W(e^{i\theta} B) \subset \Pi^\circ$. As T , (8) and (9) are unchanged when B is replaced by $e^{i\theta} B$, we may and do assume $\overline{W(B)} \subset \Pi^\circ$ so that $K = \overline{W(B)}/\overline{W(B)}$ is a compact set with the property that $K \cap \mathbf{R} \subset \mathbf{R}^+$. By (v) §1, $\sigma(T)^n \subset K$, $n \in \mathbf{Z}$. On the other hand $T^n B \in \mathbf{A}(\mathfrak{h})$, $n \in \mathbf{Z}$, by (9) so that Proposition 2 is applicable to give $\sigma(T) \subset \mathbf{R}$. Therefore $\sigma(T)^n \subset \mathbf{R} \cap K \subset \mathbf{R}^+$, $n \in \mathbf{Z}$, from which we conclude that $\sigma(T) = \{1\}$. This proves Proposition 3.

Theorem 5. *If (5) and (6) hold, then $0 \notin \overline{W(B)}$ or $T = I$ (i.e., $VB = BV$)*

Proof. In view of (8) the theorem of Sz.-Nagy cited above implies T is similar to a unitary U . Hence if $0 \notin \overline{W(B)}$, $\sigma(U) = \sigma(T) = \{1\}$ by Proposition 3. Therefore $U = I$ and, because of the similarity between T and U , $T = I$.

It may be of interest to remark that when $\dim \mathfrak{h}$ is finite, Theorem 5 may be made to follow from Proposition 3 in an elementary fashion, i.e. without the use of the Nagy theorem. To this end we observe (regardless of $\dim \mathfrak{h}$) that (8) implies

$$(10) \quad \ker(I - T) \cap (I - T)\mathfrak{h} = \{0\}.$$

For otherwise there would exist $u, v \in \mathfrak{h}$ with $\|u\| = 1$, $(u, v) = 0$, $Tu = u$, and

$Tv = v + u$. Thus, for $n \in \mathbb{Z}$, $T^n v = v + nu$ so that $\|T^n v\|^2 = \|v\|^2 + n^2$ which contradicts (8). If $\dim \mathfrak{h}$ is finite and $0 \notin W(B) (= \overline{W(B)})$, Proposition 3 then implies $I - T$ is nilpotent and consequently, because of (10), $I - T = 0$.

§5. Mappings of spectral sets. If $T \in \mathcal{B}(\mathfrak{h})$ and if Σ is a closed subset of $\mathbb{C}$ such that $\sigma(T) \subset \Sigma$, then Σ is said to be spectral for T if, whenever q is a rational complex-valued function with poles outside Σ ,

$$\|q(T)\| \leq \sup_{\lambda \in \Sigma} |q(\lambda)|.$$

Spectral sets were introduced by von Neumann and have been studied extensively. The properties we use (see [14]) are (a) Σ spectral for T and $\Sigma' \supset \Sigma$ then Σ' is spectral for T ; (b) Δ_1 is spectral for T if and only if $\|T\| \leq 1$; moreover, if f is holomorphic in a neighborhood of Δ_1 , then $\|T\| \leq 1$ implies $\|f(T)\| \leq \sup_{\lambda \in \Delta_1} |f(\lambda)|$; (c) Π is spectral for T if and only if $T \in \mathcal{A}(\mathfrak{h})$. (d) If Σ is closed convex and spectral for T , then $W(T) \subset \Sigma$.

We shall show

Theorem 6. *If f is holomorphic in a neighborhood of Π , then $T \in \mathcal{A}(\mathfrak{h})$ implies $\text{co } f(\Pi)$ is spectral for $f(T)$.*

Theorem 7. *If g is holomorphic on Π° , then $T \in \mathcal{A}(\mathfrak{h})$ and $\text{Re } \sigma(T) > 0$ imply $\text{co } g(\Pi^\circ)$ is spectral for $g(T)$.*

Note that because of (d) Lemmas 1 and 2 are consequences of Theorems 6 and 7 respectively. The proofs are based on a continuity principle for convex spectral sets which may be of independent interest.

If $\Sigma_n, \Sigma, n = 1, 2, \dots$, are closed convex sets such that $\Sigma_n \supset \Sigma$, we say that Σ_n tends to Σ , $\Sigma_n \rightarrow \Sigma$, whenever for each $\epsilon > 0$ and each compact set K , there exists a positive integer $n_0(\epsilon, K)$ such that $n \geq n_0(\epsilon, K)$ implies $\Sigma_n \cap K \subset (\Sigma \cap K) + \Delta_\epsilon$.

Proposition 4. *Let Σ_n, Σ be closed convex subsets of $\mathbb{C}$ such that $\Sigma_n \supset \Sigma$ and $\Sigma_n \rightarrow \Sigma$ and $T_n \in \mathcal{B}(\mathfrak{h})$ with Σ_n spectral for T_n . If $T_n \rightarrow T$ (in the uniform topology), then Σ is spectral for T .*

We shall need only a weakened form of this proposition, namely the case when $\Sigma_n = \Sigma, n = 1, 2, \dots$. The case in which $T_n = T, n = 1, 2, \dots$ is contained in [8].

Proof. Let q be a rational function with poles outside Σ . Then there is a $\delta > 0$ such that q is holomorphic on $\Sigma + \Delta_\delta$. Since the poles of q are finite in number, it follows from $\Sigma_n \rightarrow \Sigma$ that there is an n_0 such that for $n \geq n_0$ q is holomorphic on Σ_n . Since Σ_n is spectral for T_n and is closed and convex, $\sigma(T_n) \subset \overline{W(T_n)} \subset \Sigma_n$. Hence $q(T_n)$ is well defined for $n \geq n_0$. Moreover $\Sigma_n \rightarrow \Sigma$ implies $\overline{W(T)} \subset \Sigma$ since Σ is convex and $\overline{W(T)}$ is compact. Consequently $\sigma(T) \subset \Sigma$

and $q(T)$ is defined. Moreover there is an $n_1 \geq n_0$ such that $n \geq n_1$ implies $\overline{W(T_n)} \subset \overline{W(T)} + \Delta_\delta \subset \Sigma + \Delta_\delta$. Let Γ be a rectifiable Jordan curve in $\Sigma + \Delta_\delta$ which contains $\overline{W(T)} + \Delta_\delta$ in its interior domain. Hence, for $n \geq n_1$ and $\lambda \in \Gamma$, $\text{dist}(\lambda, W(T_n)) \geq \delta$ and $\text{dist}(\lambda, W(T)) \geq \delta$. Consequently for $n \geq n_1$

$$\begin{aligned} q(T) - q(T_n) &= \frac{1}{2\pi i} \int_{\Gamma} q(\lambda) \{(\lambda - T)^{-1} - (\lambda - T_n)^{-1}\} d\lambda \\ &= \frac{1}{2\pi i} \int_{\Gamma} q(\lambda) (\lambda - T)^{-1} (T - T_n) (\lambda - T_n)^{-1} d\lambda. \end{aligned}$$

Therefore

$$\|q(T) - q(T_n)\| \leq \frac{1}{2\pi} \max_{\lambda \in \Gamma} |q(\lambda)| \ell(\Gamma) \delta^{-2} \|T - T_n\|, \quad n \geq n_1,$$

where we have used the fact that $\|(\lambda - T)^{-1}\| \leq \text{dist}(\lambda, W(T))^{-1}$. Thus $q(T_n) \rightarrow q(T)$.

Let

$$M_n = \sup_{\lambda \in \Sigma_n} |q(\lambda)| \quad \text{and} \quad M = \sup_{\lambda \in \Sigma} |q(\lambda)|.$$

Then $M \leq M_n$ and $\|q(T_n)\| \leq M_n$ since Σ_n is spectral for T_n . If Σ is compact, so are the Σ_n for sufficiently large n . Then $M_n = |q(\lambda_n)|$ for some $\lambda_n \in \Sigma_n$ and compactness shows that $\lim M_n \leq M$ and, thus $\lim M_n = M$. Hence $\|q(T)\| = \lim \|q(T_n)\| \leq M$. If Σ is not compact, and if $q(\infty) = \infty$, there is nothing to prove. So assume $q(\infty)$ is finite. We may then transfer considerations to the Riemann sphere and again use the compactness argument above to give $\|q(T)\| \leq M$.

Theorems 6 and 7 imply each other. A proof of Theorem 7 may be given which parallels the proof of Kato's theorem [10] except that Proposition 4 rather than the continuity of W is used. We shall deduce Theorem 6 from Theorem 7 and then prove the latter.

Let f be holomorphic in a neighborhood of Π and let $W(T) \subset \Pi$. For integral n , set $A_n = (1 + nT)(n + T)^{-1}$. We note that $W(A_n) \subset \Pi + 1/n$ so that $\text{Re } \sigma(A_n) > 0$. Moreover it is readily seen that $A_n \rightarrow T$ and $f(A_n) \rightarrow f(T)$ as $n \rightarrow \infty$. Since A_n and f satisfy the hypotheses of Theorem 7, $\text{co } f(\Pi^\circ)$ is a spectral set for $f(A_n)$. Hence $\text{co } f(\Pi)$ is spectral for $f(A_n)$. Recourse to Proposition 4 yields $\text{co } f(\Pi)$ is spectral for $f(T)$.

To prove Theorem 7 we first use the von Neumann property (b) from which it follows that h holomorphic in Δ_1° , $\|A\| \leq 1$, and $\sigma(A) \subset \Delta_1^\circ$ together imply $h(\Delta_1^\circ)$ (and hence $\text{co } h(\Delta_1^\circ)$) is spectral for $h(A)$. If f is holomorphic in Π° , then $h(z) = f((1+z)/(1-z))$ is holomorphic in Δ_1° . Since $W(T) \subset \Pi$ and $\text{Re } \sigma(T) > 0$, we see that $A = (T - 1)(T + 1)^{-1}$ satisfies $\|A\| \leq 1$, $\sigma(A) \subset \Delta_1^\circ$, and $h(A) = f(T)$. But $\text{co } h(\Delta_1^\circ) = \text{co } f(\Pi^\circ)$ is spectral for $h(A)$ and therefore for $f(T)$.

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