Products of Spaces with m -Dense Subsets

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Introduction. Let X be an arbitrary topological space. In [3] Professor A. Wilansky introduced the notion of an m dense subset of X where m is a fixed infinite cardinal. We recall the definition here. (If A is a set, then $c(A)$ will denote the cardinal of A .) A pair (D, I) is an m -density pair for X if the following are satisfied: (a) $D \subset X$ and $c(D) \leq m$, (b) I is a totally ordered set with $c(I) \leq m$, (c) each point in X is the limit of a net in D which is directed by I . If X has an m -density pair, then X is said to have an m -dense subset. In [3] Professor Wilansky conjectured that if $\{X_a : a \in A\}$ is a family of topological spaces each of which has an m -dense subset and if $X = \prod \{X_a : a \in A\}$, then X has an m -dense subset if $c(A) < 2^m$ and has, in general, no m -dense subset if $c(A) = 2^m$. Assuming the generalized continuum hypothesis (G.C.H.), we show (Theorem 2.6) that if m is a regular cardinal and if each of the factors satisfies a condition (H), then the product space has an m -dense subset if $c(A) < 2^m$. (Condition (H) is satisfied by Hausdorff spaces and topological groups.) In Example 2.8 we show that if $c(A) = 2^m$, then the product space need not have an m -dense subset. Finally, we make some remarks concerning the conjecture when m is a singular cardinal.

As will be seen, our methods involve little topology. However, they draw heavily from the theory of ordinal and cardinal numbers. For this reason, we will begin with a summary of various results about ordinal numbers which we require. The reader should consult the elegant monograph of Takeuti and Zaring [2] for further details.

§1. Ordinal and cardinal numbers. If I and J are partially ordered sets, a *similarity transformation* of I into J is a one-one, order preserving function from I into J . If there is a similarity transformation of I onto J , then I and J are said to be *similar*. A subset K of I is said to be *cofinal* in I if for each $i \in I$, there is a $k \in K$ with $i \leq k$. We write $\text{cof}(I, J)$ if J is similar to a cofinal subset of I .

Proposition 1.1. *Let W be a well ordered set. Then there is a unique ordinal number α such that W is similar to α .*

A proof of this proposition may be found in [2]. (See Theorem 7.51, p. 47.) The unique ordinal α guaranteed by the proposition will be denoted by $\text{ord}(W)$.

If α is an ordinal, then $\text{cf}(\alpha)$ will denote the least ordinal β such that $\text{cof}(\alpha, \beta)$. (The ordinal $\text{cf}(\alpha)$ exists since $\text{cof}(\alpha, \alpha)$ holds, and hence the set of ordinals β with $\text{cof}(\alpha, \beta)$ is not empty.) It is clear that $\text{cf}(\alpha)$ is a cardinal number. (See [2], Theorem 10.62, p. 85.) A cardinal number n is called *regular* if $\text{cf}(n) = n$ and *singular* if $\text{cf}(n) < n$. For example, $\aleph_\alpha$ is regular whenever α is a non-limit ordinal ([2], Theorem 11.13, p. 92), whereas $\aleph_\omega$ is clearly singular. It is also clear that for any ordinal α , $\text{cf}(\alpha)$ is a regular cardinal. The following is proved in [2] (Theorem 11.28, p. 100).

Proposition 1.2. *Let m and n be cardinals. If $n < \text{cf}(m)$, then $m^n = m$. If $\text{cf}(m) \leq n \leq m$, then $m^n = 2^m$.*

The following proposition is easily proved with Zorn's lemma. (See [1], exercise p. 68.)

Proposition 1.3. *Let I be a totally ordered set. Then I contains a well ordered, cofinal subset.*

§2. Spaces with m -dense subsets.

Lemma 2.1. *Let (D, I) be an m -density pair for X , and assume that $\text{cof}(I, J)$ holds where J is a totally ordered set. Then (D, J) is an m -density pair for X .*

Proof. Let f be a similarity transformation of J onto a cofinal subset of I . Then we clearly have $c(J) \leq c(I) \leq m$. Furthermore, if $x \in X$, there is a net φ in D directed over I (i.e., φ is a function from I into D) which converges to x . But then $\varphi \circ f$ is a net in D directed through J which converges to x . The proof is complete.

The m -density spectrum of the topological space X which we denote by $\text{Spec}(X; m)$ is the set of all regular cardinals n with the property that there is a set $D \subset X$ such that (D, n) is an m -density pair. (Obviously, $n \in \text{Spec}(X; m)$ implies that $n \leq m$.) It is clear that $\text{Spec}(X; m)$ is a topological invariant. Furthermore, we have the following.

Proposition 2.2. *Let X and Y be topological spaces, and let f be a continuous function from X onto Y . Then $\text{Spec}(X; m) \subset \text{Spec}(Y, m)$.*

Proof. Let $n \in \text{Spec}(X; m)$, and let (D, n) be an m -density pair for X . Then $(f[D], n)$ is an m -density pair for Y . Indeed, since $c(f[D]) \leq m$ is obvious, all we must show is that every point of Y is the limit of a net in $f[D]$ directed over n . Hence take $y \in Y$. Since f is onto, there is a point $x \in X$ with $f(x) = y$. Let φ be a net in D which is directed over n and which converges to x . Since f is continuous, the net $f \circ \varphi$ converges to $f(x) = y$. The proof is complete.

Corollary 2.3. *Let $\{X_a : a \in A\}$ be a family of topological spaces, and let $X = \coprod \{X_a : a \in A\}$. Then $\text{Spec}(X; m) \subset \bigcap \{\text{Spec}(X_a; m) : a \in A\}$.*

Proof. Let π_a be the projection map of X onto X_a . Since π_a is continuous, it follows from Proposition 2.2 that $\text{Spec}(X; m) \subset \text{Spec}(X_a; m)$. The proof is complete.

The following proposition relates the m -density spectrum to the notion of m -density.

Proposition 2.4. *Let X be a topological space. Then X has an m -dense subset if and only if $\text{Spec}(X; m) \neq \emptyset$.*

Proof. ($\Leftarrow$) This is immediate from the definitions.

($\Rightarrow$) Let (D, I) be an m -density pair for X . By Proposition 1.3, there is a cofinal, well ordered subset W of I . Let $n = \text{cf}(\text{ord}(W))$. Then n is a regular cardinal. Furthermore, it is clear that $\text{cof}(I, n)$ holds. Hence by Lemma 2.1, (D, n) is a m -density pair for X . That is, $n \in \text{Spec}(X; m)$. The proof is complete.

A topological space X has *property (H)* if given $x, y \in X$, either the neighborhood system at x is identical to the neighborhood system at y or x and y have disjoint neighborhoods. Clearly the class of spaces which satisfy property (H) contains the class of all Hausdorff spaces. It also contains the class of all linear topological spaces and the class of all topological groups. It is not hard to verify that the product of a family of spaces which satisfy property (H) also satisfies property (H). If X satisfies property (H), let X^* denote the set of equivalence classes of X modulo the equivalence relation E defined by xEy if x and y have identical neighborhood systems. Let $H(X)$ denote the cardinal of X^* . We then have the following.

Proposition 2.5. *Assume the G.C.H. Let $\{X_a : a \in A\}$ be a family of topological spaces each of which satisfies property (H), and let $X = \prod \{X_a : a \in A\}$. Then the following hold.*

(1) *If $H(X) \leq m$, then $\text{Spec}(X; m) = \bigcap \{\text{Spec}(X_a; m) : a \in A\} = \{n : n \text{ a regular cardinal and } n \leq m\}$.*

(2) *Assume that $c(A) \leq \text{cf}(m)$. If $m < H(X)$, then $\text{Spec}(X; m) = \{n : n \text{ a regular cardinal and } c(A) \leq n\} \cap \bigcap \{\text{Spec}(X_a; m) : a \in A\}$.*

Proof. (1) If $H(X) \leq m$, then by the axiom of choice there is a set $D \subset X$ such that D intersects each element of X^* in exactly one point. Hence $c(D) \leq m$. It is clear that (D, n) is an m -density pair for each regular cardinal $n \leq m$. The proof is complete.

(2) Let $n \in \text{Spec}(X; m)$. We will show that $c(A) \leq n$. Hence assume that $n < c(A)$. Let (D, n) be an m -density pair for X . Since D^n is the set of nets in D directed by n and since X is a space with property (H), we certainly have that $H(X) \leq c(D^n) \leq m^n$. Since $n < c(A) \leq \text{cf}(m)$, it follows from Proposition 1.2 that $m^n = m$. Hence we obtain $H(X) \leq m$ contrary to assumption. Thus $c(A) \leq n$ as claimed.

Now let $n \in \bigcap \{\text{Spec}(X_a; m) : a \in A\}$ and assume that $c(A) \leq n$. We must show that $n \in \text{Spec}(X; m)$. Without loss of generality, we may assume that

$A = c(A)$ (that is, that A is a cardinal number). For each $a \in A$, let (D_a, n) be an m -density pair for X_a .

Fix a point $x_0 \in X$. For each $a \in A$, define Δ_a to be the set of all points x in X such that $x(\alpha) \in D_a$ for $\alpha \leq a$ and $x(\alpha) = x_0(\alpha)$ for $a < \alpha$. (Recall that A is a cardinal so that the definition makes sense.) Finally, define $D = \bigcup \{\Delta_a : a \in A\}$.

We will show that (D, n) is an m -density pair for X . First note that for each $a \in A$, $c(\Delta_a) \leq m^a$. But A is a cardinal so that $a < A \leq \text{cf}(m)$. Hence by Proposition 1.2, $m^a = m$. Hence $c(\Delta_a) \leq m$ for all $a \in A$. But then, $c(D) \leq A \cdot m \leq m \cdot m = m$.

Now let $x \in X$ be arbitrary. For each $a \in A$, let φ_a be a net in D_a which is directed over n and which converges to $x(a)$. We now define a net φ in D directed over n as follows. For each $\alpha \in n$, $\varphi(\alpha)$ is the point $x_a \in X$ defined for each $a \in A$ by

$$x_a(a) = \begin{cases} \varphi_a(\alpha), & \text{if } \alpha \leq a, \\ x_0(\alpha), & \text{if } a < \alpha. \end{cases}$$

Since convergence in X is coordinate-wise convergence and since $A \leq n$, it follows that the net φ so defined converges to x . Hence (D, n) is an m -density pair for X . That is, $n \in \text{Spec}(X; m)$. The proof is complete.

Proposition 2.6. *Assume the G.C.H. Let m be a regular cardinal, and let X be a topological space which satisfies property (H). If X has an m -dense subset, then $m \in \text{Spec}(X; m)$.*

Proof. Since X has an m -dense subset, $\text{Spec}(X; m) \neq \emptyset$ by Proposition 2.4. Let (D, n) be an m -density pair with n a regular cardinal. If $n = m$, we are done. Hence assume that $n < m$. Then D^n is the set of all nets in D directed over n . Let D^* be a subset of X which intersects each element of D^n in exactly one point. Then we have, $c(D^*) \leq c(D^n) \leq m^n$. But since m is regular (that is, $\text{cf}(m) = m$) and since $n < m$, it follows from Proposition 1.2 that $m^n = m$. Thus $c(D^*) \leq m$. It is now clear that (D^*, m) is a density pair for X . That is, $m \in \text{Spec}(X; m)$. The proof is complete.

We are now in a position to prove the following theorem which gives a partial solution to Wilansky's conjecture.

Theorem 2.7. *Assume the G.C.H., and assume that m is a regular cardinal. Let $\{X_a : a \in A\}$ be a family of topological spaces each of which has property (H) and each of which has an m -dense subset. Let $X = \prod \{X_a : a \in A\}$. If $c(A) < 2^m$, then X has an m -dense subset.*

Proof. By the G.C.H. $c(A) \leq m = \text{cf}(m)$, and by Proposition 2.6, $m \in \text{Spec}(X_a; m)$ for each $a \in A$. But then by Proposition 2.5, it follows that $m \in \text{Spec}(X; m)$. Hence by Proposition 2.4, X has an m -dense subset.

We will now give an example to show that Wilansky's conjecture holds for $c(A) = 2^m$.

Example 2.8. Let $A = 2^m$; and, for each $a \in A$, let $X_a = m$ with the discrete topology. It is clear that (X_a, m) is an m -density pair for each $a \in A$. We will show that $X = \prod \{X_a : a \in A\}$ has no m -dense subset. Indeed, assume that X has an m -dense subset. Then $\text{Spec}(X; m) \neq \emptyset$ by Proposition 2.2. Let (D, n) be an m -density pair where n is a regular cardinal. Since X is Hausdorff, it is clear that $c(X) \leq c(D^n) \leq m^n \leq m^m \leq 2^m$. But $c(X) = m^{(2^m)} \geq 2^{(2^m)} > 2^m$. This is a contradiction. Hence X has no m -dense subset.

For any cardinal m , if X is a topological space with $c(X) \leq m$, then (X, m) is trivially an m -density pair for X and $\text{Spec}(X; m)$ is the set of all regular cardinals n with $n \leq m$. Furthermore, if X is a space with property (H) and if X has an m -dense subset, then $H(X) \leq 2^m$. (This is easily verified since if (D, n) is an m -density pair for X , then $H(X) \leq c(D^n) \leq m^n \leq m^m \leq 2^m$.) The question arises whether there is an example of a space X with an m -dense subset for which $c(X) = 2^m$. We will now show that there are indeed such spaces if the G.C.H. holds.

Example 2.9. Let $X = 2^m$ with the product topology. (Of course $2 = \{0, 1\}$ has the discrete topology.) Then X has an m -dense subset. Indeed, for each $\alpha < m$ let D_α denote the set consisting of all points $x \in X$ with the property that $x(\beta) = 0$ for all $\beta \geq \alpha$. Let $D = \bigcup \{D_\alpha : \alpha < m\}$. For $\alpha < m$ it follows that $c(D_\alpha) = 2^\alpha \leq m$ by the G.C.H. Hence $c(D) \leq m \cdot m = m$. We will now show that (D, m) is an m -density pair. To this end, take $x \in X$; and define a net φ in D which is directed by m as follows. For $\alpha < m$, $\varphi(\alpha)$ is the point x_α in D defined for each $\beta < m$ by

$$x_\alpha(\beta) = \begin{cases} x(\beta), & \text{if } \beta < \alpha, \\ 0, & \text{if } \alpha \leq \beta. \end{cases}$$

It is clear that the net φ converges to x . Hence (D, m) is an m -density pair for X as claimed.

The next example shows that the hypothesis that $c(A) \leq \text{cf}(m)$ in Proposition 2.5 is necessary. Clearly m must be a singular cardinal in such an example.

Example 2.10. Let $\aleph_\omega$ have the discrete topology, and let $X = \aleph_\omega^q$ with the product topology where $q = \aleph_p$ for some $p \leq \omega$. It is clear that $\text{Spec}(\aleph_\omega; \aleph_\omega) = \{\aleph_n : n \in \omega\}$. (Of course, $\text{cf}(\aleph_\omega) = \omega = \aleph_0$ so that $\aleph_\omega$ is a singular cardinal and $\text{cf}(\aleph_\omega) \leq q \leq \aleph_\omega$.) We will show that $\text{Spec}(X; \aleph_\omega) = \{\aleph_0\}$.

We first verify that $\aleph_0 \in \text{Spec}(X; \aleph_\omega)$. For each $n \in \omega$, let A_n be the subset of X consisting of all those points x such that for each $\alpha < q$, $x(\alpha) \in \aleph_n$ for $\alpha < \aleph_n$ and $x(\alpha) = 0$ for $\aleph_n \leq \alpha$. Then, $c(A_n) \leq \aleph_n^{\aleph_n} = \aleph_{n+1} < \aleph_\omega$. (We use the G.C.H.) Define $D = \bigcup \{A_n : n \in \omega\}$. Then $c(D) \leq \sum_{n \in \omega} c(A_n) \leq \sum_{n \in \omega} \aleph_{n+1} = \aleph_\omega$. We claim that $(D, \aleph_0)$ is an $\aleph_\omega$ -density pair for X . Indeed, fix $x \in X$. Define a net φ in D directed over $\aleph_0$ as follows. For each $n \in \aleph_0$, φ_n is the element of D defined for $\alpha < q$ by

$$\varphi_n(\alpha) = \begin{cases} x(\alpha), & \text{if } \alpha < \aleph_n \text{ and } x(\alpha) \in \aleph_n, \\ 0, & \text{otherwise.} \end{cases}$$

It is not hard to verify that $\varphi_n \rightarrow x$.

We will now verify that $\aleph_n \notin \text{Spec}(X; \aleph_\omega)$ for $n \in \omega - \{0\}$. Indeed, assume that $\aleph_n \in \text{Spec}(X; \aleph_\omega)$ with $n \neq 0$; and let $(D, \aleph_n)$ be an $\aleph_\omega$ -density pair. Define a set $D_\omega = \{x \in \aleph_\omega^\omega : \text{there is } y \in D \text{ with } y|_\omega = x\}$. (Of course, $y|_\omega$ denotes the restriction of y to ω .) If $x \in \aleph_\omega^\omega$, take $x^* \in X$ with $x^*|_\omega = x$. Let φ be a net in D which is directed over $\aleph_n$ and which converges to x^* . Then $\varphi|_\omega$ converges to x in $\aleph_\omega^\omega$. Since $\aleph_0 < \aleph_n$, there is an $\alpha < \aleph_n$ for which $\varphi_\alpha|_\omega = x$. That is, $x \in D_\omega$. Hence we have shown that $D_\omega = \aleph_\omega^\omega$. But then we have, using Proposition 1.2, that $c(D) \geq c(D_\omega) = c(\aleph_\omega^\omega) = 2^{\aleph_\omega} > \aleph_\omega$. This contradicts the assumption that $(D, \aleph_n)$ was an $\aleph_\omega$ -density pair. The proof is complete.

It is not hard to see that the arguments used in the above example can be applied to show that if m is any singular cardinal, then m^n is a space with $\text{Spec}(m^n; m) = \{\text{cf}(m)\}$ whenever $\text{cf}(m) \leq n \leq m$.

Finally, although we cannot prove Wilansky's conjecture in general, the following is an equivalent formulation of it for the class of spaces with property (H), which appears more approachable.

Theorem 2.11. *The following are equivalent.*

(a) *Wilansky's conjecture holds for the class of topological spaces with property (H).*

(b) *If m is a singular cardinal and if X is a space with property (H), then either $\text{Spec}(X; m) = \emptyset$ or $\{\text{cf}(m)\} \in \text{Spec}(X; m)$.*

Proof. (a) $\Rightarrow$ (b): If (b) is false, then there is a singular cardinal m and a space X with $\{\text{cf}(m)\} \notin \text{Spec}(X; m)$ and with $\text{Spec}(X; m) \neq \emptyset$. Let $Y = m^m$ as above. Then $\text{Spec}(Y; m) = \{\text{cf}(m)\}$. Hence by Corollary 2.3, $\text{Spec}(X \times Y; m) = \emptyset$. Thus Wilansky's conjecture fails.

(b) $\Rightarrow$ (a): Let $\{X_\alpha : \alpha < m\}$ be a family of spaces with property (H) each of which has an m -dense subset. (We may assume without loss of generality that $A = m$.) By assumption (b), for each $\alpha < m$, there is an m -density pair $(D_\alpha, \text{cf}(m))$. Let f be a similarity transformation of $\text{cf}(m)$ onto a cofinal subset of m . Let $D_\alpha = \{d_{\alpha, \beta} : \beta < m\}$ be an enumeration of D_α . (If $c(D_\alpha) < m$, let repetition occur.) For each $\beta < \text{cf}(m)$, define Δ_β to be the set of all x in $X = \prod \{X_\alpha : \alpha < m\}$ with the property that for each $\alpha < f(\beta)$, $x(\alpha) \in \{d_{\alpha, \gamma} : \gamma < f(\beta)\}$ and for $f(\beta) \leq \alpha$, $x(\alpha) = \delta_{\alpha, 0}$. Then $c(\Delta_\beta) = c(f(\beta)^{f(\beta)}) = c(2^{f(\beta)}) < m$. Let $D = \cup \{\Delta_\beta : \beta < \text{cf}(m)\}$. Then $c(D) \leq \sum_{\beta \in \text{cf}(m)} c(2^{f(\beta)}) \leq \text{cf}(m) \cdot m \leq m \cdot m = m$.

We will now show that $(D, \text{cf}(m))$ is an m -density pair for X . Hence take $x \in X$. Let $\varphi^{(\alpha)}$ be a net in D_α directed over $\text{cf}(m)$ which converges to $x(\alpha)$. Define a net φ in D directed over $\text{cf}(m)$ as follows. For each $\beta < \text{cf}(m)$, φ_β is the element of D defined for each $\alpha < m$ by

$$\varphi_\beta(\alpha) = \begin{cases} \varphi_\beta^{(\alpha)}, & \text{if } \alpha < f(\beta) \text{ and } \varphi_\beta^{(\alpha)} = d_{\alpha, \gamma} \text{ for some } \gamma < f(\beta), \\ \delta_{\alpha, 0}, & \text{otherwise.} \end{cases}$$

It is now easy to verify that $\varphi_\beta \rightarrow x$ (since convergence in X is pointwise). The proof is complete.

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