Initial-value Problems for Pseudo-parabolic Partial Differential Equations

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1. Introduction. We shall be concerned with an equation of the form,
(1.1)
$$(A - I)u_t + Bu = f,$$

for an unknown function $u(x, t)$ defined in the whole space $R^m \times R$, where A and B are second-order differential operators with coefficients being functions of $x \in R^m$ and $t \in R$. In what follows A will be assumed to be uniformly elliptic in $R^m \times R$ and the coefficient of u_t to be strictly negative. For earlier works on equations of this type, see the references in [29] where mixed problems in a cylindrical domain have been studied. Subsequently, various generalizations of the results in [29] have been obtained, [18], [28] and certain mildly non-linear equations of this kind have also been considered, [7], [12], [28].

Although many physical problems are governed by equations of the form in (1.1), the most significant application is to the two temperature theory for heat conduction, [6]. Based on this theory, various questions can be raised about the solution of (1.1). Here, we only generalize the results in [26] and [27] by considering the initial-value problem of finding a solution u of (1.1) in $R^m \times R$, which assumes prescribed values at a given instant, say $t = 0$.

It turns out that physical theory [6] requires that the coefficient of u_t in (1.1) be strictly negative. This together with the ellipticity of A ensures the existence of the principal fundamental solution of the equation $(A - I)u = 0$. In §2, basic properties of the principal fundamental solution have been studied. Some of these results are new and some old. In either case, these results have been stated in the form suitable for the present applications. Results in §2 are then applied in §3 to obtain an $W^{2,p}$ -estimate for the solutions of the equation $(A - I)u = v$ in the whole space. In §4, the initial value problem is reduced to solving an integro-differential equation which is then solved by successive approximations via the estimate obtained in §3. Finally, the regularity of the solution is studied in §5 under further restrictions on the initial data and on the coefficients in (1.1).

In what follows, we shall write $x = (x_1, \dots, x_m)$, $dx = dx_1 \cdots dx_m$, $D_x^\alpha = (\partial/\partial x_1)^{\alpha_1} \cdots (\partial/\partial x_m)^{\alpha_m}$, $\alpha = (\alpha_1, \dots, \alpha_m)$ with α_i being non-negative integers and $|\alpha| = \sum \alpha_i$. As usual, $L^p(R^m)$, $1 < p < \infty$, stands for the Banach space of functions defined on R^m having Lebesgue integrable p^{th} power over R^m and $W^{k,p}(R^m)$, $1 < p < \infty$, is the Banach space of functions defined on R^m , which together with their distribution derivatives up to order k belong to $L^p(R^m)$. The norm on $W^{k,p}(R^m)$ is defined by

$$\|f\|_{k,p} = \left[\sum_{0 \leq |\alpha| \leq k} \int |D_x^\alpha f|^p dx \right]^{1/p} \quad \text{for all } f \text{ in } W^{k,p}(R^m).$$

For later reference we state two theorems in a way convenient for applications. The first one concerns the property of mean-zero of the second derivatives of a Levi function, see [26] or [20] for a proof; the second is the Calderon-Zygmund estimate for singular integral operators [3], [4].

Theorem 1.1. *Let $B(y, 1)$ be the unit ball in R^m centered at y and let $\theta \equiv (x - y)/|x - y|$. If $\varphi(y, \theta)$ is defined on $R^m \times \partial B(0, 1)$ and if φ together with its first derivatives with respect to θ is continuous and bounded, then for $i = 1, 2, \dots, m$*

$$(1.2) \quad \frac{\partial}{\partial x_i} \left(\frac{\varphi(y, \theta)}{r^{m-1}} \right) = \frac{f_i(y, \theta)}{r^m}, \quad r = |x - y|,$$

such that f_i satisfies the relation

$$\int_{\partial B(y, 1)} f_i(y, \theta) ds = 0, \quad i = 1, 2, \dots, m,$$

where $\partial B(y, 1)$ denotes the surface of the ball $B(y, 1)$ and ds the element of surface measure.

Theorem 1.2. *Let $N(y, y - x)$ be defined for all (x, y) in $R^m \times R^m$ with $x \neq y$. Suppose that*

- (i) $N(y, z)$ is homogeneous of degree $-m$ in z ,
- (ii) for every fixed y , $N(y, z)$ is integrable over the unit sphere $|z| = 1$ and its integral is zero,
- (iii) $\sup \{|N(y, z)|; y \in R^m, |z| = 1\} \equiv \|N\|$ is finite.

Then, for every f in $L^p(R^m)$, $1 < p < \infty$, the sequence of functions,

$$g_\epsilon(x) \equiv \int_{|y-x|>\epsilon} N(y, y-x)f(y) dy,$$

converges as $\epsilon \rightarrow 0$ in $L^p(R^m)$ to a function $g(x)$ such that

$$\|g\|_{0,p} \leq A_{m,p} \|N\| \|f\|_{0,p},$$

where $\|N\|$ is the number defined in (iii) and $A_{m,p}$ is a constant depending only on m and p .

2. Basic properties of principal fundamental solutions. Briefly, principal

fundamental solutions are fundamental solutions in the large which, together with their first two derivatives, decay exponentially as the distance between the two variable points increases indefinitely. This behavior will enable us to derive certain L^p -estimates for the solutions of certain non-homogeneous elliptic equations and their first and second derivatives.

Let A be the elliptic differential operator given by

$$(2.1) \quad Au \equiv \sum_{i,j=1}^m a_{ij}(x)u_{ij} + \sum_{i=1}^m a_i(x)u_i + a_0(x)u,$$

where u_{ij} denotes the partial derivative of u with respect to x_i and x_j and where the functions a_{ij} , a_i and a_0 are all defined on R^m . Let $A(x)$ denote the determinant of the matrix (a_{ij}) , $S(x)$ the positive square root of $A(x)$ and $(A_{ij}(x))$ the inverse of $(a_{ij}(x))$.

Definition. For all integers $m \geq 2$, x, y in R^m , $x \neq y$, the *original Levi function* for the operator A is defined by the formula,

$$(2.2) \quad L(y; y-x) \equiv \begin{cases} ((m-2)\omega_m S(y))^{-1} \rho(y; y-x)^{2-m}, & m > 2, \\ (2\pi S(y))^{-1} \log(1/\rho(y; y-x)), & m = 2, \end{cases}$$

where ω_m denotes the surface area of a unit hypersphere in R^m , and where for x, y, z in R^m

$$(2.3) \quad \rho(z; y-x) \equiv \left\{ \sum_{i,j=1}^m A_{ij}(z)(y_i - x_i)(y_j - x_j) \right\}^{1/2}.$$

Let Δ be the diagonal of $R^m \times R^m$, i.e., the set of points (x, y) in $R^m \times R^m$ with $x = y$. Let A_x denote the operator A operating at the point x . A function $N(x, y)$ defined on $R^m \times R^m - \Delta$ is a fundamental solution of the equation $Au = 0$ if it is twice differentiable in x there and if

(i) $A_x N(x, y) = 0$ in $R^m \times R^m - \Delta$,

(ii) $N(x, y) = (1 + o(1))L(y; y-x)$ uniformly in x as $|x - y| \rightarrow 0$.

A principal fundamental solution $G(x, y)$ of $Au = 0$ is a fundamental solution of $Au = 0$ such that for $0 \leq |\alpha| \leq 1$,

$$(2.4) \quad |D_x^\alpha G(x, y)| \leq \text{const. } e^{-ar} \text{ for all } r \equiv |x - y| > R_0,$$

where a and R_0 are some constants and all the constants in (2.4) are independent of x and y .

Theorem [Giraud]. Suppose that the coefficients of the operator A are all bounded and uniformly Hölder continuous in R^m with exponent λ , $0 < \lambda \leq 1$. If for some positive constants $\lambda(A)$ and $\Lambda(A)$

$$\lambda(A) |\xi|^2 \leq \sum_{i,j=1}^m a_{ij}(x) \xi_i \xi_j \leq \Lambda(A) |\xi|^2 \text{ for all } \xi \text{ in } R^m,$$

and if the function $a_0(x)$ is bounded above by a strictly negative constant, then the equation $Au = 0$ possesses a unique principal fundamental solution.

Since our proof of the main theorem of this section depends on the construction of such a principal fundamental solution, we now give a summary of Giraud's construction [14]. The first step is to construct the principal fundamental solution of the modified equation $Au - k^2u = 0$ for a sufficiently large positive constant k . To do this one uses the decisive fact that the function,

$$(2.5) \quad H_k(y; y - x) \equiv \frac{e^{-k\rho}(2\pi\rho)^{(1-m)/2}k^{(m-3)/2}}{S(y) \cdot 2 \cdot \Gamma((m-1)/2)} \int_0^\infty t^{(m-3)/2} \left(1 + \frac{t}{2k\rho}\right)^{(m-3)/2} e^{-t} dt,$$

$\rho \equiv \rho(y; y - x)$, satisfies the equation

$$(2.6) \quad \sum a_{i,j}(y) \partial^2 H_k(y; y - x) / \partial x_i \partial x_j - k^2 H_k = 0 \text{ in } R^m \times R^m - \Delta.$$

In fact, the following lemma shows that it is the principal fundamental solution of equation (2.6).

Lemma 2.1. *Suppose that the functions, $a_{i,j}(x)$, satisfy all the hypotheses in Giraud's theorem and that H_k and L are the functions defined in (2.5) and (2.2), respectively. Let $H_k(x, y) \equiv H_k(y, y - x)$, $r \equiv |x - y|$. If $m \geq 3$, then (i) there exists a positive constant a less than k such that for $0 \leq |\alpha| \leq 2$,*

$$(2.7) \quad |D_x^\alpha H_k(x, y)| \leq \text{const. } e^{-ar} r^{2-m-|\alpha|} \text{ in } R^m \times R^m - \Delta;$$

and (ii) for each $R_0 > 0$,

$$(2.8) \quad |D_x^\alpha (H_k - L)| \leq \text{const. } r^{3-m-|\alpha|}, \quad 0 \leq |\alpha| \leq 2, \quad 0 < r \leq R_0.$$

On the other hand, if $m = 2$, then (iii) there is a positive constant a less than k such that in $R^m \times R^m - \Delta$

$$(2.9) \quad |H_k(x, y)| \leq \text{const. } e^{-ar} |\log r|,$$

$$(2.10) \quad |D_x^\alpha H_k(x, y)| \leq \text{const. } e^{-ar} r^{-|\alpha|}, \quad 1 \leq |\alpha| \leq 2;$$

and (iv) for each $R_0 > 0$, $1 \leq |\alpha| \leq 2$,

$$(2.11) \quad |H_k - L| \leq \text{const.}, \quad |D_x^\alpha (H_k - L)| \leq \text{const. } r^{\beta-|\alpha|}$$

whenever $0 < r \leq R_0$, where β is a number independent of k such that $0 < \beta < 1$.

The proof of this lemma is based on formulas (2.2) and (2.5). If $m \geq 3$ and is an odd integer, one expands the factor $(1 + t/2k\rho)^{(m-3)/2}$ into a series of $t/2k\rho$ and then evaluates the integral; while if m is an even integer one applies Cauchy's integral theorem to replace the integral in (2.5) by a properly chosen contour integral. The details of these verifications, although elementary, are rather lengthy and involved, but everything has been carried out in [26]. We make no attempt to reproduce it here. Related results are also given in [15], where Giraud showed that $H_k = (1 + o(1))L$ uniformly in x as $r \equiv |x - y| \rightarrow 0$.

Now, for all integers $n \geq 1$, we define

$$(2.12) \quad \begin{aligned} Z_k^{(1)}(x, y) &\equiv A_x H_k(y; y - x) - k^2 H_k(y; y - x), \\ Z_k^{(n)}(x, y) &\equiv \int Z_k(x, \xi) Z_k^{(n-1)}(\xi, y) d\xi, \end{aligned}$$

where the integral, here as well as in what follows, is extended over R^m . For simplicity, we shall write $Z_k^{(1)}$ as Z_k . The following lemma summarizes the crucial properties of these iterated kernels.

Lemma [Giraud]. *If the operator A satisfies all the hypotheses in Giraud's theorem, then*

(i) *there exists a positive constant a less than k such that*

$$(2.13) \quad |Z_k^{(n)}(x, y)| \leq \begin{cases} \text{const. } r^{n\lambda-m} e^{-ar} & \text{if } n\lambda < m, \\ \text{const. } |\log r| & \text{if } n\lambda = m \text{ and } r < \frac{1}{2}, \\ \text{const. } e^{-ar} & \text{if } n\lambda = m \text{ and } r \geq \frac{1}{2}, \\ \text{const. } e^{-ar} (1 + r^{n\lambda-m}) & \text{if } n\lambda > m, \end{cases}$$

where λ is the Hölder exponent of the coefficients of the operator A .

(ii) *for k sufficiently large, there is a constant δ such that*

$$(2.14) \quad \int |Z_k(x, y)| dx, \quad \int |Z_k(x, y)| dy < \delta < 1,$$

for all y in R^m and for all x in R^m respectively; and

(iii) *for any two positive integers p and q with $p + q = n$*

$$(2.15) \quad Z_k^{(n)}(x, y) = \int Z_k^{(p)}(x, \xi) Z_k^{(q)}(\xi, y) d\xi.$$

Now, let us consider the integral equation,

$$(2.16) \quad G_k(x, y) = H_k(x, y) + \int G_k(x, \xi) Z_k(\xi, y) d\xi$$

for determining the unknown function $G_k(x, y)$. It has a unique solution given by the formula,

$$(2.17) \quad G_k(x, y) = H_k(x, y) + \sum_{n=1}^{s-1} \int H_k(x, \xi) Z_k^{(n)}(\xi, y) d\xi + W_k(x, y),$$

where $W_k(x, y)$ is the solution of the integral equation

$$(2.18) \quad W_k(x, y) = \int H_k(x, \xi) Z_k^{(s)}(\xi, y) d\xi + \int W_k(x, \xi) Z_k(\xi, y) d\xi.$$

Indeed, if $s \geq m/\lambda$, then $Z_k^{(s)}$ is continuous and bounded on $R^m \times R^m$. Consequently, equation (2.18) has a unique continuous bounded solution given by

$$(2.19) \quad W_k(x, y) = \sum_{n=s}^{\infty} \int H_k(x, \xi) Z_k^{(n)}(\xi, y) d\xi.$$

If the constant k is so large that (2.14) is satisfied, then the function G_k so constructed is the principal fundamental solution of the equation $Au - k^2 u$

= 0, [14]. That is,

$$A_x G(x, y) - k^2 G(x, y) = 0 \text{ in } R^m \times R^m - \Delta.$$

The next step is to construct the principal fundamental solution $G_{k-\Delta k}$ for the equation $Au - (k - \Delta k)^2 u = 0$ with $k > \Delta k > 0$. This construction is similar to that of G_k and we have

$$(2.20) \quad G_{k-\Delta k}(x, y) = G_k(x, y) + \sum_{n=1}^{\infty} \int G_k(x, \xi) Z_{k-\Delta k}^{(n)}(\xi, y) d\xi,$$

provided Δk is so chosen that $k > \Delta k > 0$ and that

$$(2.21) \quad \Delta k(2k - \Delta k) \int |G_k(x, y)| dx < \delta < 1,$$

where $Z_{k-\Delta k}^{(1)} \equiv AG_k - (k - \Delta k)^2 G_k$ and $Z_{k-\Delta k}^{(n)}$ are the corresponding iterated kernels.

Based on the hypotheses on the coefficients of A , especially $a_0 < 0$, it was proved in [14] that there exists a number θ such that the inequality,

$$\Delta k(2h - \Delta k) \int |G_h(x, y)| dx < \delta < 1,$$

is satisfied for all h , $0 < h < k$, provided $\Delta k < \theta$. Thus, we can reduce k to zero in a finite number of steps with $0 < \Delta k < \theta$ in each step and construct the principal fundamental solution at every step. The principal fundamental solution $G(x, y)$ is the one constructed at the last step. We now state the main theorem of this section.

Theorem 2.1. *Suppose that the coefficients of the elliptic differential operator A are all bounded and uniformly Hölder continuous in R^m with exponent λ , $0 < \lambda \leq 1$. Suppose also that $a_0(x) < 0$ in R^m . Let G be the principal fundamental solution of $Au = 0$ and let L be the function defined in (2.2). Then there exists positive constants a and R_0 such that*

$$(2.22) \quad |D_x^\alpha(x, y)| \leq \text{const. } e^{-ar} \text{ for all } r \geq R_0, 0 \leq |\alpha| \leq 2;$$

$$(2.23) \quad |D_x^\alpha(G(x, y) - L(x, y))| \leq \text{const. } r^{2+\lambda-m-|\alpha|}, \quad 0 \leq \alpha \leq 2, 0 < r \leq R_0,$$

if $m \geq 3$; and if $m = 2$, then for $0 < r \leq R_0$ and $\gamma = \min(\beta, \lambda)$

$$(2.24) \quad |G - L| \leq \text{const.}, |D_x^\alpha(G - L)| \leq \text{const. } r^{\gamma-|\alpha|}, 1 \leq |\alpha| \leq 2,$$

where β is the same constant in (2.11).

We emphasize that all the constants appearing in the statements depend only on $\lambda(A)$, $\Lambda(A)$ and the bounds and the Hölder constant of the coefficients of A . Hence, they are all independent of x and y . For clarity we split the proof into several lemmas

Lemma 2.2. *Suppose that the coefficients of the operator A satisfy the hy-*

potheses stated in Theorem 2.1. Let G_k and H_k be the functions defined, respectively in (2.17), (2.19) and (2.5). If $m \geq 3$, then

(i) there exists a positive constant $a < k$ such that

$$(2.25) \quad |D_x^\alpha G_k(x, y)| \leq \text{const. } e^{-ar} r^{2-m-|\alpha|}, \quad 0 \leq |\alpha| \leq 2; \text{ and}$$

(ii) for a constant $R_0 > 0$, $0 < r \leq R_0$

$$(2.26) \quad |D_x^\alpha (G_k - H_k)| \leq \text{const. } r^{2+\lambda-m-|\alpha|}, \quad 0 \leq |\alpha| \leq 2.$$

On the other hand, if $m = 2$, then

(iii) there exists a positive a less than k such that

$$(2.27) \quad |G_k(x, y)| \leq \text{const. } e^{-ar} |\log r|.$$

$$(2.28) \quad |D_x^\alpha G_k(x, y)| \leq \text{const. } e^{-ar} r^{-|\alpha|}, \quad 1 \leq |\alpha| \leq 2;$$

and

(iv) for every constant $R_0 > 0$, $0 < r \leq R_0$,

$$(2.29) \quad |G_k - H_k| \leq \text{const.}, |D_x^\alpha (G_k - H_k)| \leq \text{const. } r^{\lambda-|\alpha|}, \quad 1 < |\alpha| \leq 2.$$

Lemma 2.3. (i) If, for some constant $b > 0$,

$$F_1(x, y) = e^{-b|x-y|} |x - y|^{\alpha-m}, \quad F_2(x, y) = e^{-b|x-y|} |x - y|^{\beta-m},$$

with $\alpha \geq 0$, $\beta > 0$ being some constants such that $\alpha + \beta < m$, then

$$(2.30) \quad \int F_1(X, \xi) F_2(\xi, y) d\xi \leq \text{const. } e^{-b|x-y|} |x - y|^{\alpha+\beta-m}.$$

(ii) Suppose that the operator A satisfies all the hypotheses in Theorem 2.1. If H_k and Z_k are respectively the functions defined in (2.5) and (2.12) and if $\partial_{x_i x_j} \equiv \partial^2 / \partial x_i \partial x_j$, $i, j = 1, 2, \dots, m$, then

$$(2.31) \quad \partial_{x_i x_j} \int H_k(x, \xi) Z_k(\xi, y) d\xi \\ = \int \partial_{x_i x_j} H_k(x, \xi) Z_k(\xi, y) d\xi - \frac{1}{m} A_{ij}(x) Z_k(x, y),$$

$$(2.32) \quad \left| \partial_{x_i x_j} \int H_k(x, \xi) Z_k(\xi, y) d\xi \right| \leq \text{const. } e^{-a|x-y|} |x - y|^{\lambda-m},$$

where a is a positive constant less than k .

Proof of Lemma 2.3. Part (i) of this lemma was proved by Giraud [14, pp. 288–290] for $\alpha > 0$. A proof for the case $\alpha = 0$ is given in [20, pp. 64–65]. As for part (ii) we first show that formula (2.31) is true if H_k is replaced by the function L , i.e.,

$$(2.31)' \quad \partial_{x_i x_j} \int L(x, \xi) Z_k(\xi, y) d\xi = \int \partial_{x_i x_j} L(x, \xi) Z_k(\xi, y) d\xi - \frac{1}{m} A_{ij}(x) Z_k(x, y).$$

To derive this formula we write $R^m = T \cup C(T)$, where T is an open ball centered at x and not containing y , while $C(T)$ is the complement of T . Thus, for $|\alpha| = 2$,

$$(2.33) \quad D_x^\alpha \int L(x, \xi) Z_k(\xi, y) d\xi = \left\{ D_x^\alpha \int_T + D_x^\alpha \int_{C(T)} \right\} L(x, \xi) Z_k(\xi, y) d\xi.$$

As a function of ξ , $Z_k(\xi, y)$ is Hölder continuous in T with exponent λ . It follows [22, pp. 28–32] that

$$(2.34) \quad \partial_{x_i x_i} \int L(x, \xi) Z_k(\xi, y) d\xi = \int_T \partial_{x_i x_i} L(x, \xi) Z_k(\xi, y) d\xi - \frac{1}{m} A_{ii}(x) Z_k(x, y).$$

On the other hand, since $x \notin T$, $L(x, \xi)$ has no singularity when $\xi \in C(T)$ and since $|Z_k| \leq \text{const.} (e^{-a r} r^{m-\lambda})$ for some constant $a > 0$, we also have, for $|\alpha| = 2$

$$(2.35) \quad D_x^\alpha \int_{C(T)} L(x, \xi) Z_k(\xi, y) d\xi = \int_{C(T)} D_x^\alpha L(x, \xi) Z_k(\xi, y) d\xi.$$

Formula (2.31)' now follows directly from (2.34) and (2.35).

By (2.8) in Lemma 2.1, the function $H_k - L$ and its first and second derivatives are weakly singular, we have, $|\alpha| = 2$,

$$(2.36) \quad D_x^\alpha \int (H_k - L)(x, \xi) Z_k(\xi, y) d\xi = \int D_x^\alpha (H_k - L)(x, \xi) Z_k(\xi, y) d\xi.$$

By combining (2.31)' and (2.36) we obtained the desired formula (2.32).

Finally, inequality (2.32) is a simple consequence of part (i) of this lemma and the estimates on Z_k and H_k . In proving (2.32), we must choose $\beta > \lambda$, and $\lambda < 1$, where β is the same constant in Lemma 2.1.

Proof of Lemma 2.2. Our proof is based on Lemma 2.3 and the explicit expressions in (2.17) and (2.19) for the function G_k . In fact, we have

$$(2.37) \quad G_k(x, y) = H_k(x, y) + \sum_{n=1}^{\infty} \int H_k(x, \xi) Z_k^{(n)}(\xi, y) d\xi.$$

It is known [14] that for $|\alpha| = 1$

$$(2.38) \quad D_x^\alpha G_k(x, y) = D_x^\alpha H_k(x, y) + \sum_{n=1}^{\infty} \int D_x^\alpha H_k(x, \xi) Z_k^{(n)}(\xi, y) d\xi.$$

However, for $|\alpha| = 2$ we note that for $n \geq 2$

$$\int H_k(x, \xi) Z_k^{(n)}(\xi, y) d\xi = \int \left\{ \int H_k(x, \eta) Z_k(\eta, \xi) d\eta \right\} Z_k^{(n-1)}(\xi, y) d\xi.$$

Moreover, it follows from (2.2) and the estimates in (2.8), (2.13) and the part (i) in Lemma 2.3 that the function $\int H_k(x, \eta) Z_k(\eta, \xi) d\eta$ and its first and second derivatives are weakly singular. Hence

$$D_x^\alpha \int H_k(x, \xi) Z_k^{(n)}(\xi, y) d\xi = \int D_x^\alpha \left\{ \int H_k(x, \eta) Z_k(\eta, \xi) d\eta \right\} Z_k^{(n-1)}(\xi, y) d\xi$$

for $|\alpha| \leq 2$. Thus, we conclude from (2.14) and estimate (2.32) that for $|\alpha| = 2$

$$(2.39) \quad D_x^\alpha G_k(x, y) = D_x^\alpha H_k(x, y) + D_x^\alpha \int H_k(x, \xi) Z_k(\xi, y) d\xi \\ + \sum_{n=2}^{\infty} \int D_x^\alpha \left\{ \int H_k(x, \eta) Z_k(\eta, \xi) d\eta \right\} Z_k^{(n-1)}(\xi, y) d\xi.$$

In fact, it is not difficult to check that after deleting the first finite number of terms the above infinite series is dominated by a geometric series with constant terms.

Inequality (2.25) was proved by Giraud in [14] for $0 \leq |\alpha| \leq 1$. For $|\alpha| = 2$, the same method used there may still be used to derive inequality (2.25) as far as the exponential behavior is concerned. The singularity behavior in (2.25) follows from the fact that $D_x^\alpha H_k(x, y)$ is the dominating term in (2.39) and from the fact that $\sum Z_k^{(n)}(x, y)$ is a continuous bounded function in $R^m \times R^m$ when $n \geq s \geq m/\lambda$. This proves inequality (2.25).

To prove inequality (2.26) we note that the dominating term in the series representation for $D_x^\alpha(G_k - H_k)$ is

$$D_x^\alpha \int H_k(x, \xi) Z_k(\xi, y) d\xi, \quad 0 \leq |\alpha| \leq 2.$$

The remaining terms in the series representation for $D_x^\alpha(G_k - H_k)$ either have singularities weaker than that of the dominating term or are simply continuous and bounded functions. Thus, the singularity behavior of $D_x^\alpha(G_k - H_k)$ is the same as that of $D_x^\alpha \int H_k(x, \xi) Z_k(\xi, y) d\xi$. The singularity behavior of this dominating term is easy to establish when $0 \leq |\alpha| \leq 1$, while it obeys inequality (2.32) when $|\alpha| = 2$.

Part (iii) and (iv) can be proved in a similar way with some modifications, since they are the results corresponding to part (i) and (ii) for the case $m = 2$.

Proof of Theorem 2.1. By the method for constructing $G(x, y)$, we have, by choosing Δk as integral fraction of k ,

$$(2.40) \quad G = (G - G_{\Delta k}) + (G_{\Delta k} - G_{2\Delta k}) + \cdots + (G_k - H_k) + H_k,$$

where each of the functions appearing above is the principal fundamental solution of elliptic equations with the same variable coefficients except, of course, H_k . The similarity in the behavior of all these functions and in the construction of each of them from the previous ones makes it sufficient to prove the theorem for G_k . For example, one can immediately extend the results to the function $G_{k-\Delta k}$ with the aid of equation (2.20) and inequality (2.21). Thus, we need only to prove every statement in Theorem 2.1 for the function G_k .

Since $G_k - L = (G_k - H_k) + (H_k - L)$, the required estimate (2.23) follows

from inequalities (2.8) and (2.26). Inequality (2.22) is a direct consequence of the inequality (2.25) under the assumption $r > R_0$. Inequality (2.24) follows from part (iv) of Lemma 2.1 and part (iv) of Lemma 2.2. This completes the proof of the theorem.

3. $W^{2,p}$ -estimates. In §2 we considered the principal fundamental solution of a second order elliptic equation. Now we use the principal fundamental solution to derive $W^{2,p}$ -estimates for the solutions of the corresponding non-homogeneous equation. Such estimates will enable us to solve the initial value problem described in §1.

Throughout the remainder of this work we shall assume that the operator A satisfies all the hypotheses in Giraud's theorem. The symbol $D_{ij}G(\cdot, y)$ denotes the second derivative of a function G of two variables, with respect to the i th and j th coordinates of the first variable. Similar meaning applies to the first derivatives.

Theorem 3.1. *Let A be the elliptic differential operator given by equation (2.1) such that all the coefficients of A are bounded and uniformly Hölder continuous in R^m with exponent λ , $0 < \lambda \leq 1$, and $a_0(x) < 0$. Let G be the principal fundamental solution of the equation $Au = 0$. Then for every v in $L^p(R^m)$, $1 < p < \infty$,*

$$(3.1) \quad \begin{cases} u \equiv \int G(x, y)v(y) dy & \text{belongs to } W^{2,p}(R^m), \\ Au = -v & \text{holds in } L^p(R^m) \text{ and} \\ \left\| \int G(\cdot, y)v(y) dy \right\|_{2,p} \leq \text{const. } \|v\|_{0,p}. \end{cases}$$

We split the proof into lemmas. The first one is to establish the property of mean zero of the second derivatives of the original Levi function.

Lemma 3.1. *Let $L(y, y - x)$ be the function defined in (2.2) and let $\theta \equiv (x - y)/|x - y|$. Then we can write*

$$(3.2) \quad \partial^2 L(y; y - x) / \partial x_i \partial x_j = r^{-m} f_{ij}(y, \theta)$$

such that the functions f_{ij} satisfy the relations

$$(3.3) \quad \int_{\partial B(y, 1)} f_{ij}(y, \theta) ds = 0, \quad i, j = 1, 2, \dots, m,$$

where $\partial B(y, 1)$ stands for the boundary of $B(y, 1)$ and ds the surface element.

Proof. Since ρ/r is a function of y and θ only, we have, for all integers $m \geq 2$, $x \neq y$

$$\begin{aligned} \partial L(y; y - x) / \partial x_i &= \text{const. } (S(y))^{-1} \rho^{1-m} \partial \rho / \partial x_i \\ &= \text{const. } (S(y))^{-1} r^{1-m} (\rho/r)^{-m} \sum_l A_{il}(y) \theta_l \end{aligned}$$

$$\equiv \varphi_i(y, \theta) r^{1-m}.$$

From the definition of $\varphi_i(y, \theta)$ it is clear that it has continuous derivatives with respect to the components of θ . Consequently, Theorem 1.1 ensures that

$$\partial(\varphi_i(y, \theta) r^{1-m}) / \partial x_i = f_{ii}(y, \theta) r^{-m}$$

and that f_{ii} , $i, j = 1, 2, \dots, m$, has property (3.3).

Lemma 3.2. *Let G be the principal fundamental solution of the equation $Au = 0$. Then for every v in $L^p(R^m)$ the function $\int D_{ii}G(\cdot, y)v(y)dy$ belongs to $L^p(R^m)$ and*

$$(3.4) \quad \left\| \int D_{ii}G(\cdot, y)v(y) dy \right\|_{0,p} \leq \text{const.} \|v\|_{0,p}.$$

Proof. Take R_0 to be the same constant as that in Theorem 2.1 and write

$$\int D_{ii}G(x, y)v(y) dy \equiv g_1(x) + g_2(x) + g_3(x) \quad \text{with}$$

$$g_1(x) \equiv \int_{B(x; R_0)} D_{ii}L(y; y-x)v(y) dy,$$

$$g_2(x) \equiv \int_{B(x; R_0)} D_{ii}(G-L)(x, y)v(y) dy \quad \text{and}$$

$$g_3(x) \equiv \int_{R^m - B(x; R_0)} D_{ii}G(x, y)v(y) dy.$$

Clearly, to prove (3.4) it suffices to show that

$$(3.5) \quad \|g_i\|_{0,p} \leq \text{const.} \|v\|_{0,p} \quad \text{for } i = 1, 2, 3.$$

Now, Theorem 2.1 ensures that

$$(3.6) \quad \|g_2\|_{0,p} \leq \text{const.} \left\| \int_{B(x, R_0)} r^{\lambda-m} |v(y)| dy \right\|_{0,p}.$$

Consequently, the desired estimate (3.5) for g_2 follows by an application of Young's inequality to the right-hand side of inequality (3.6). Similarly, for $i = 3$, (3.5) is an immediate consequence of Young's inequality and the estimate (2.22) in Theorem 2.1.

To establish (3.5) for g_1 we put

$$N(y, y-x) = \psi(y-x) \partial^2 L(y, y-x) / \partial x_i \partial x_i,$$

where $\psi(y-x) = 1$ for $0 \leq |y-x| \leq R_0$ and zero otherwise. Then

$$g_1(x) = \int_{R^m} N(y, y-x)v(y) dy.$$

Since the singular kernel $N(y, y - x)$ satisfies all the conditions in Theorem 1.2, g_1 is, indeed, subject to the estimates (3.5). The proof is now complete.

Lemma 3.3. *Let G be the principal fundamental solution of $Au = 0$. Then for every v in $L^p(R^m)$, $1 < p < \infty$, the function $\int G(x, y)v(y)dy$ belongs to $W^{2,p}(R^m)$ and its derivatives are given by the formulas:*

$$(3.7) \quad D_i \int G(x, y)v(y) dy = \int D_i G(x, y)v(y) dy,$$

$$(3.8) \quad D_{ii} \int G(x, y)v(y) dy = \int D_{ii} G(x, y)v(y) dy - \frac{1}{m} A_{ii}(x)v(x).$$

Proof. Since $\partial G/\partial x_i$ is weakly singular and is subject to the estimate in (2.22), the truth of (3.7) is evident. As for (3.8), the same reasoning as that for deriving formula (2.31) insures that for all ψ_n in $C_0^\infty(R^m)$ and for all x in R^m

$$(3.9) \quad D_{ii} \int G(x, y)\psi_n(y) dy = \int D_{ii} G(x, y)\psi_n(y) dy - \frac{1}{m} A_{ii}(x)\psi_n(x).$$

For a given v in $L^p(R^m)$ we choose a sequence $\{\psi_n\}$ in $C_0^\infty(R^m)$ such that $\|v - \psi_n\|_{0,p} \rightarrow 0$ as $n \rightarrow \infty$. Then, it follows from Lemma 3.2 and the boundedness of $A_{ii}(x)$ that

$$(3.10) \quad \text{R.S. in (3.9)} \rightarrow \text{R.S. in (3.8) strongly in } L^p(R^m).$$

On the other hand, if we write $(f, g) = \int fg$, then for every φ in $C_0^\infty(R^m)$,

$$(3.11) \quad \left| \left(D_{ii} \int G(\cdot, y)(v - \psi_n)(y)dy, \varphi \right) \right| \\ = \left| \left(\int D_i G(\cdot, y)(v - \psi_n)(y) dy, D_i \varphi \right) \right| \\ \leq \left\| \int D_i G(\cdot, y)(v - \psi_n)(y) dy \right\|_{0,p} \|D_i \varphi\|_{0,q}.$$

In fact, (3.7) insures that all the expressions above are meaningful and the last inequality is that of Hölder. Since Theorem (2.1) implies that

$$\left\| \int D_i G(\cdot, y)(v - \psi_n)(y) dy \right\|_{0,p} \leq \text{const.} \|v - \psi_n\|_{0,p},$$

we conclude from (3.11) that

$$(3.12) \quad \text{L.S. in (3.9)} \rightarrow \text{L.S. in (3.8) weakly in } L^p(R^m).$$

By combining (3.9), (3.10) and (3.12) we conclude that (3.8) holds in the distribution sense. As is well-known [23], this implies that the right-hand side in (3.8) is also the strong derivative of $D_i \int G(x, y)v(y)dy$. The proof of Lemma 3.3 is now complete.

Proof of Theorem 3.1. The first statement in (3.1) is the first conclusion in Lemma 3.3 and the second statement in (3.1) can be easily verified by using formulas (3.7) and (3.8). To prove the inequality in (3.1) we first note that both $G(x, y)$ and $\partial G(x, y)/\partial x_i$ are weakly singular at $x = y$ and decay exponentially as $r \equiv |x - y| \rightarrow \infty$. Hence

$$\left\| \int G(\cdot, y)v(y) dy \right\|_{0,p}, \quad \left\| D_i \int G(\cdot, y)v(y) dy \right\|_{0,p} \leq \text{const.} \|v\|_{0,p}.$$

Moreover, it follows from formula (3.8) and inequality (3.4) that

$$\left\| D_{i,i} \int G(\cdot, y)v(y) dy \right\|_{0,p} \leq \text{const.} \|v\|_{0,p}.$$

By combining the three inequalities above, we obtain the inequality in (3.1).

To solve the initial value problem we shall allow the operator A to depend on a parameter t , $t \in R$, and restrict its dependence on t in such a way that the corresponding principal fundamental solution $G(x, y; t)$ exists for all t in R and that the following theorem holds.

Theorem 3.1.' Suppose that the coefficients of the operator A depend on a parameter t , $t \in R$, and that the hypotheses in Theorem 3.1 hold uniformly in t , i.e., $\lambda(A)$, $\Lambda(A)$ and the upper bounds as well as the Hölder constant and Hölder exponent of the coefficients of A can be chosen to be independent of t . Also, $a_0(x, t)$ is less than a constant < 0 . If $G(x, y; t)$ is the principal fundamental solution of the equation $Au = 0$, then for every v in $L^p(R^m)$, $1 < p < \infty$, the function

$$u(x, t) \equiv \int G(x, y; t)v(y) dy,$$

as a function of x , belongs to $W^{2,p}(R^m)$ for all t in R and it satisfies the equation $Au = -v$ in $L^p(R^m)$ for all t in R . Moreover, for all v in $L^p(R^m)$ and for all $|t| \leq T$,

$$(3.13) \quad \left\| \int G(x, y; t)v(y) dy \right\|_{2,p} \leq \text{const.} \|v\|_{0,p},$$

where the constant can be chosen to be independent of t .

The theorem can be proved by precisely the same reasonings as that for Theorem 3.1. However, it must be checked that all the estimation constants used in proving Theorem 3.1 can be chosen to be independent of t . To avoid such a long repetition of what has been done in §§2 and 3, we satisfy ourselves with a few crucial observations when the coefficients of A depend on a parameter t .

Concerning Lemma 2.1. The functions H_k and L depend only on the functions $a_{i,i}(x, t)$. Consequently, the constants in Lemma 2.1 depend only on

$$|a_{i,i}|_0, k, \quad \lambda(A) \quad \text{and} \quad \Lambda(A)$$

where $|\cdot|_0$ stands for supremum norm. Since the constant was so chosen that

(2.14) is satisfied, it depends only on

$$|a_{ij}|_0, |a_i|_0, |a_0^{-1}|_0.$$

Since each of these numbers can be chosen to be independent of t for all $|t| \leq T$, so are the constants in Lemma 2.1 for $H_k(x, y; t)$.

Concerning Lemma 2.2. The important estimates used in the proof are those for H_k and Z_k . Estimates for H_k have just been checked and we only need to consider Z_k . Since

$$Z_k(x, y; t) = \sum (a_{ij}(x, t) - a_{ij}(y, t))H_{k,ij}(x, y; t) \\ + \sum a_i(x, t)H_{k,i}(x, y; t) + a_0(x, t)H_k(x, y; t),$$

the constants in Giraud's Lemma depend only on

$$\lambda(A), \Lambda(A), M, |a_{ij}|_0, |a_i|_0, |a_0|_0,$$

where M is the Hölder constant of $a_{ij}(x, t)$ with respect to x . From this and the representation for G_k ,

$$G_k(x, y; t) = H_k(x, y; t) + \sum_{n=1}^{\infty} \int H_k(x, \xi; t) Z_k^{(n)}(\xi, y; t) d\xi,$$

we can conclude that all the constants in Lemma 2.2 for $G_k(x, y; t)$ can be chosen to be independent of t .

Concerning $G_{k-\Delta k}$ and G . The formula

$$G_{k-\Delta k}(x, y; t) = G_k(x, y; t) + \sum_{n=1}^{\infty} \int G_k(x, \xi; t) Z_{k-\Delta k}^{(n)}(\xi, y; t) d\xi,$$

where $Z_{k-\Delta k}(x, y; t) \equiv (A - (k - \Delta k)^2)G_k(x, y; t) = \Delta k(2k - \Delta k)G_k(x, y; t)$, and $Z_{k-\Delta k}^{(n)}$ are iterates of $Z_{k-\Delta k}^{(1)}$ for $n \geq 2$, the behavior of $G_k(x, y; t)$ and the inequality,

$$\Delta k(2k - \Delta k) \int |G_k(x, y; t)| dx < \delta < 1 \quad \text{for all values of } k,$$

ensure that similar estimates as that for $G_k(x, y; t)$ hold for $G_{k-\Delta k}(x, y; t)$ with constant independent of t . Then the expression in (2.40) ensures that the estimates in Theorem 2.1 hold for $G(x, y; t)$ with constants independent of t .

Concerning Lemmas 3.1 and 3.2. Since the operator A is assumed to be uniformly elliptic on $R^m \times R$, the second x -derivatives of $L(y, y - x; t)$, which are defined in (2.2) with $a_{ij}(x, t)$ being functions of x and t , still satisfy all the conditions in Lemma 3.1 and Theorem 1.2. Consequently, the constant in (3.4) can be chosen to be independent of t .

In the following applications, it is Theorem 3.1' that will be used.

4. The initial value problem. Let A and B be the differential operators:

$$(4.1) \quad Au \equiv \sum_{i,j=1}^n a_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n a_i(x, t) \frac{\partial u}{\partial x_i} + a_0(x, t)u,$$

$$(4.2) \quad Bu \equiv \sum_{i,j=1}^n b_{ij}(x, t) \frac{\partial^2 u}{\partial x_i \partial x_j} + \sum_{i=1}^n b_i(x, t) \frac{\partial u}{\partial x_i} + b_0(x, t)u,$$

satisfying the following three assumptions.

(A₁). The coefficients of the operator A are uniformly continuous and bounded on R^m and on every compact subset of R . For some constant λ , $0 < \lambda \leq 1$, all these coefficients are Hölder continuous in x uniformly on $R^m \times R$ with exponent λ . For some constant $g > 0$, the coefficient $a_0(x, t)$ is strictly less than $1 - g^2$ for all (x, t) in $R^m \times R$. Note that the operator, previously denoted by A , is now replaced by $A - I$.

(A₂). There exist positive constants $\lambda(A)$ and $\Lambda(A)$ such that for all real vectors $\xi \equiv (\xi_1, \dots, \xi_m)$ and all (x, t) in $R^m \times R$

$$(4.3) \quad \lambda(A) |\xi|^2 \leq \sum_{i,j=1}^n a_{ij}(x, t) \xi_i \xi_j \leq \Lambda(A) |\xi|^2.$$

(A₃). The coefficient of the operator B are bounded measurable functions of x for all t in R and are continuous in t uniformly in x except on a set of measure zero in R^m .

Let $u_0 \in W^{2,p}(R^m)$ and $f : R \rightarrow L^p(R^m)$ be two given functions with $1 < p < \infty$. Consider the problem of finding a function $u(x, t)$ such that as a function of x , $u(x, t)$ belongs to $W^{2,p}(R^m)$, $1 < p < \infty$, for all t in R and that it satisfies the equation

$$(4.4) \quad Lu \equiv (A - I) \frac{\partial u}{\partial t} + Bu = f$$

in $L^p(R^m)$ for each t in R and

$$(4.5) \quad \lim_{t \rightarrow 0} u(x, t) = u_0(x) \quad \text{in } W^{2,p}(R^m),$$

where $\partial u / \partial t$ is understood to be the limit as $h \rightarrow 0$ of the difference quotient $h^{-1}(u(t+h) - u(t))$ in $W^{2,p}(R^m)$. Our objective is to establish

Theorem 4.1. Suppose that the differential operators A and B defined in (4.1) and (4.2) satisfy the assumptions (A₁)–(A₃). If the function $f : R \rightarrow L^p(R^m)$ is continuous and bounded, then the Cauchy problem described by equations (4.4) and (4.5) has a unique solution.

To this end, we first observe that the assumptions (A₁) and (A₂) ensure the existence of the principal fundamental solution $G(x, y; t)$ of the equation $Au - u = 0$ and guarantee the validity of Theorem 3.1' for the function $G(x, y; t)$. Consequently, for every $f : R \rightarrow L^p(R^m)$, the function (as a function of x)

$$(4.6) \quad (A - I)^{-1}f(x, t) \equiv \int G(x, y; t)f(y, t) dy$$

belongs to $W^{2,p}(R^m)$ for all t in R .

Lemma 4.1. *Under the hypotheses in Theorem 4.1, the function,*

$$(A - I)^{-1}f : R \rightarrow W^{2,p}(R^m),$$

is continuous.

Proof. With the aid of the notation in (4.6) we can write

$$\begin{aligned} ((A - I)^{-1}f)(x, t + h) - ((A - I)^{-1}f)(x, t) &= \int G(x, y; t + h)[f(y, t + h) - f(y, t)] dy \\ &\quad + \int [G(x, y; t + h) - G(x, y; t)]f(y, t) dy \\ &= I_1 + I_2. \end{aligned}$$

Since Theorem 3.1' states that as a linear operator mapping $L^p(R^m)$ into $W^{2,p}(R^m)$, $(A - I)^{-1}$ has its norm bounded by a constant independent of t , we have

$$\|I_1\|_{2,p} \leq \text{const.} \|f(\cdot, t + h) - f(\cdot, t)\|_{0,p},$$

which shows that $\|I_1\|_{2,p} \rightarrow 0$ as $h \rightarrow 0$.

To complete a proof of the lemma, it remains to show that $\|I_2\|_{2,p} \rightarrow 0$ as $h \rightarrow 0$. For a fixed number h , denote by $A_{x,t}$ and $A_{x,t+h}$ the operator A with its coefficients evaluated at (x, t) and $(x, t + h)$ respectively. It follows from Theorem 3.1' that for every $f : R \rightarrow L^p(R^m)$,

$$(A_{x,t+h} - I) \int G(x, y; t + h)f(y, t) dy = -f(x, t),$$

$$(A_{x,t} - I) \int G(x, y; t)f(y, t) dy = -f(x, t),$$

in $L^p(R^m)$ for all t in R . Consequently,

$$\begin{aligned} (A_{x,t+h} - A_{x,t}) \int G(x, y; t + h)f(y, t) dy \\ + (A_{x,t} - I) \int [G(x, y; t + h) - G(x, y; t)]f(y, t) dy = 0 \end{aligned}$$

in $L^p(R^m)$ for all t in R . By applying the operator $(A_{x,t} - I)^{-1}$ to the both sides of the above identity, we find

$$(4.7) \quad I_2 = \int G(x, y; t) \left\{ (A_{u,t+h} - A_{u,t}) \int G(y, z; t + h)f(z, t) dz \right\} dy.$$

In this way we have shifted the variations of $G(x, y; t)$ due to the increment in t to the corresponding variations of the coefficients of the operator A so as

to facilitate our estimation for I_2 . Since the norm $\|(A - I)^{-1}\|$ has a bound independent of t and the coefficients of A are bounded over $R^m \times R$, it follows from (4.7) and (3.13) that

$$\begin{aligned} \|I_2\| &\leq \text{const.} \left\| (A_{\cdot, t+h} - A_{\cdot, t}) \int G(\cdot, z; t+h) f(z, t) dz \right\|_{0,p} \\ &\leq \text{const.} \epsilon \left\| \int G(\cdot, z; t+h) f(z, t) dz \right\|_{2,p} \\ &\leq \epsilon \text{const.} \|f\|_{0,p}, \end{aligned}$$

where $\epsilon \rightarrow 0$ as $h \rightarrow 0$. This proves the continuity of $\|(A - I)^{-1}f\|_{2,p}$ as a function of t .

For simplicity, we shall write $u(t)$ for the value of the function $u: R \rightarrow W^{k,p}(R^m)$ for some integer k .

Lemma 4.2. *Under the hypotheses in Theorem 4.1, the integral equation,*

$$(4.8) \quad u(\cdot, t) = u_0 - \int_0^t ((A - I)^{-1}Bu)(\sigma) d\sigma + \int_0^t ((A - I)^{-1}f)(\sigma) d\sigma,$$

for the function $u: R \rightarrow W^{2,p}(R^m)$ has a solution given by the formulas,

$$(4.9) \quad u(t) = \sum_{n=0}^{\infty} (-1)^n (K_n v)(t)$$

where

$$(4.10) \quad v(t) = u_0 + \int_0^t ((A - I)^{-1}f)(\sigma) d\sigma \equiv (K_0 v)(t),$$

$$(4.11) \quad (K_n v)(t) \equiv \int_0^t ((A - I)^{-1}BK_{n-1}v)(\sigma) d\sigma, \quad n \geq 1,$$

and where the time integral is understood to be the limit in $W^{2,p}(R^m)$ of the Riemannian sums.

Proof. Let us first prove that for all integers $n \geq 1$, $K_n v$ is a well-defined continuous function mapping R into $W^{2,p}(R^m)$.

By Lemma 4.1, $\|(A - I)^{-1}f\|_{2,p}$ is a continuous function of t . Consequently, $v: R \rightarrow W^{2,p}(R^m)$ is well-defined and continuous. For every number $T > 0$, let $C[T, W^{k,p}(R^m)]$ be the space of continuous functions $u: [-T, T] \rightarrow W^{k,p}(R^m)$. By Theorem 3.1',

$$(A - I)^{-1}: C[T, L^p(R^m)] \rightarrow C[T, W^{2,p}(R^m)]$$

is a bounded linear map. Hence there are constants c_1 and c_2 such that for all $|t| \leq T$

$$(4.12) \quad \|(A - I)^{-1}\| \leq c_1, \quad \|f\|_{0,p} \leq c_2.$$

Thus, we have the estimate,

$$\begin{aligned}
\|v(t)\|_{2,p} &\leq \|u_0\|_{2,p} + \int_0^{|t|} \|(A - I)^{-1}f(\sigma)\|_{2,p} d\sigma \\
&\leq \|u_0\|_{2,p} + c_1 c_2 |t| \quad \text{for all } |t| \leq T.
\end{aligned}$$

Since the coefficients of the operator B are bounded measurable in x for all t in R and are continuous in t uniformly in x except on a set of measure zero in R^m , $Bv : R \rightarrow L^p(R^m)$ is continuous. By Lemma 4.1, $(A - I)^{-1}Bv : R \rightarrow W^{2,p}(R^m)$ is thus continuous. Consequently, $K_1 v : R \rightarrow W^{2,p}(R^m)$ is well-defined and continuous. Consider the operator $(A - I)^{-1}B$ which maps $C[T, W^{2,p}(R^m)]$ into itself. By Theorem 3.1' and the restrictions on B , there exists a constant c such that

$$(4.14) \quad \|(A - I)^{-1}B\|_{2,p} \leq c \quad \text{for all } |t| \leq T.$$

Hence, it follows from (4.11)–(4.13) that

$$\begin{aligned}
\|(K_1 v)(t)\|_{2,p} &\leq c \int_0^{|t|} \|v(\sigma)\|_{2,p} d\sigma \\
&\leq c(\|u_0\|_{2,p} |t| + c_1 c_2 |t|^2/2!).
\end{aligned}$$

An induction on n shows that for all $n \geq 1$, $K_n v : R \rightarrow W^{2,p}(R^m)$ is well-defined and continuous. Moreover, for all $|t| \leq T$

$$(4.15) \quad \|K_n v(t)\|_{2,p} \leq c^n (|t|^n/n!) \|u_0\|_{2,p} + (c_1 c_2/c) c^{n+1} |t|^{n+1}/(n+1)!.$$

Using the estimate in (4.15), we obtain, for the function $u(t)$ defined in (4.9)–(4.11), the estimate

$$(4.16) \quad \|u(t)\|_{2,p} \leq e^{c|t|} \|u_0\|_{2,p} + c_1 c_2 (e^{c|t|} - 1)/c \quad \text{for all } |t| \leq T,$$

$$(4.17) \quad \|u(t_1) - u(t_2)\|_{2,p} \leq c_1 c_2 (e^{c|t_1 - t_2|} - 1)/c \quad \text{for all } |t_1|, |t_2| \leq T.$$

Thus, $u : R \rightarrow W^{2,p}(R^m)$ is well-defined and continuous.

To show that $u(t)$ satisfies the integral equation (4.8), we apply the operator $-K_1$ to the both sides of equation (4.9) to obtain

$$(4.18) \quad -(K_1 u)(t) = \sum_{n=1}^{\infty} (-1)^n (K_n v)(t).$$

The change of the order of summation and integration is permissible because of the uniform convergence of the series. Using the definition of $u(t)$, we can write (4.18) as

$$-(K_1 u)(t) = u(t) - v(t),$$

which is, as is easily seen, the integral equation (4.8). The proof is now complete.

Proof of Theorem 4.1. Any function $u : R \rightarrow W^{2,p}(R^m)$, which satisfies the integral equation (4.8) for all t in R , is a solution of our initial-value problem. This is true, because by differentiating the both sides of equation (4.8) with

respect to t in the sense described in the statement of the Cauchy problem, we find that for all t in R

$$(4.19) \quad u_t = -(A - I)^{-1}Bu + (A - I)^{-1}f \quad \text{in } W^{2,p}(R^m).$$

After applying $(A - I)$ to the both sides, we obtain equation (4.8). Moreover, it follows from (4.9)–(4.11) and (4.15) that

$$(4.20) \quad \|u(t) - u_0\|_{2,p} \leq \int_0^{|t|} \|((A - I)^{-1}f)(\sigma)\|_{2,p} d\sigma + \sum_{n=1}^{\infty} \|K_n v\|_{2,p} \\ \leq (e^{|t|} - 1) \|u_0\|_{2,p} + c_1 c_2 (e^{|t|} - 1)/c$$

for all t in R . In particular, this shows that the initial condition (4.5) is satisfied by u .

Finally, we check the uniqueness of the solution. If u_1 and u_2 are two solutions of the same initial-value problem, then $u_1 - u_2$ is a solution of the initial-value problem described by equations (4.4) and (4.5) with $u_0 = 0$ in $W^{2,p}(R^m)$ and $f = 0$ in $L^p(R^m)$ for all t . Since every solution of our Cauchy problem is also a solution of the integral equation (4.8), it follows from the estimate in (4.20), that $\|(u_1 - u_2)(t)\|_{2,p} = 0$ for all t in R . The theorem is now proved.

5. Regularity of solutions. In this section we wish to show how the regularity of the solution $u(x, t)$ in §4 depends on that of the initial data, the coefficients of the operators A and B and the non-homogeneous term f . In particular, our results will show that the seemingly over-prescribed initial condition (4.5) is actually a natural one.

Theorem 5.1. *Let $n \geq 0$ be an integer. Suppose that, in addition to the assumptions (A_1) – (A_3) in §4, the coefficients of the operators A and B together with their first n derivatives with respect to the space variable x are continuous and bounded in $R^m \times R$, that $f : R \rightarrow W^{n,p}(R^m)$ is continuous and bounded and that u_0 belongs to $W^{n+2,p}(R^m)$. Then, as a function of x , the solution $u(x, t)$ of the initial-value problem described by equations (4.4) and (4.5) belongs to $W^{n+2,p}(R^m)$ for all t in R .*

Proof. First, we prove the theorem for the case $n = 1$. Our method is to consider the difference quotients of the solution and show that they form a bounded set in a Sobolev's space. To this end let $x + h \equiv (x_1 + h, x_2, \dots, x_m)$ and for any two functions $v(x)$ and $v(x, t)$ let

$$v^h(x) = h^{-1}[v(x + h) - v(x)], \quad v^h(x, t) = h^{-1}[v(x + h, t) - v(x, t)],$$

respectively. Let $L_{x+h,t}$ denote the differential operator defined in (4.4) with its coefficients evaluated at $(x + h, t)$. Then

$$L_{x+h,t}u(x + h, t) = f(x + h, t) \quad \text{in } L^p(R^m) \quad \text{for all } t \text{ in } R, \\ \lim_{t \rightarrow 0} u(x + h, t) = u_0(x + h) \quad \text{in } W^{2,p}(R^m),$$

and

$$\begin{aligned} L_{x,t} u(x, t) &= f(x, t) \quad \text{in } L^p(R^m) \quad \text{for all } t \text{ in } R, \\ \lim_{t \rightarrow 0} u(x, t) &= u_0(x) \quad \text{in } W^{2,p}(R^m). \end{aligned}$$

If $L_{x,t}^h$ denotes the differential operator,

$$\begin{aligned} &\left(\sum a_{i,i}^h(x, t) \frac{\partial^2}{\partial x_i \partial x_i} + \sum a_i^h(x, t) \frac{\partial}{\partial x_i} + a_0^h(x, t) \right) \frac{\partial}{\partial t} \\ &\quad + \sum b_{i,i}^h(x, t) \frac{\partial^2}{\partial x_i \partial x_i} + \sum b_i^h(x, t) \frac{\partial}{\partial x_i} + b_0^h(x, t) \end{aligned}$$

then it is easy to check that for all t in R

$$(5.1) \quad L_{x+h,t} u^h(x, t) = f^h(x, t) - L_{x,t}^h u(x, t) \quad \text{in } L^p(R^m),$$

and

$$(5.2) \quad \lim_{t \rightarrow 0} u^h(x, t) = u_0^h(x) \quad \text{in } W^{2,p}(R^m).$$

This shows that for every fixed number h , the difference quotient $u^h(x, t)$ is also a solution of an initial-value problem completely similar to that described by (4.4) and (4.5). If we denote the right-hand side by $g^h(x, t)$, then $g^h : R \rightarrow L^p(R^m)$ is continuous and hence it is bounded over every compact subset of R . Accordingly, all the reasoning used in §4 can be applied here and we conclude that $u^h(x, t)$ is given by

$$(5.3) \quad u^h(x, t) = \sum_{n=0}^{\infty} (-1)^n (K_n v)(x, t)$$

where

$$(5.4) \quad (K_0 v)(x, t) \equiv v(x, t) \equiv u_0^h(x, t) + \int_0^t [(A_{x+h,\sigma} - I)^{-1} g^h(x, \sigma)] d\sigma,$$

$$(5.5) \quad (K_n v)(x, t) \equiv \int_0^t [(A_{x+h,\sigma} - I)^{-1} B_{x+h,\sigma} (K_{n-1} v)(x, \sigma)] d\sigma, \quad n \geq 1.$$

We wish to show that $\|u^h(\cdot, t)\|_{2,p}$ has a bound independent of the number h . To this end, we start with the representation formulas in (5.3)–(5.5) and proceed in the same way as that for deriving estimate (4.15) to obtain

$$(5.6) \quad \|(K_n v)(\cdot, t)\|_{2,p} \leq c^n \frac{|t|^n}{n!} \|u_0^h\|_{2,p} + \frac{c_1}{c} c^{n+1} \frac{|t|^{n+1}}{(n+1)!} G^h(t),$$

where c_1 and c are, respectively, the constants defined in (4.12) and (4.14) and where

$$(5.7) \quad G^h(t) \equiv \max \{ \|g^h(\cdot, \sigma)\|_{0,p}, \quad |\sigma| \leq |t| \}.$$

Note that both c and c_1 are independent of h and $G^h(t)$. By combining (5.3) and (5.6) we find that

$$(5.8) \quad \|u^h(\cdot, t)\|_{2,p} \leq \|u_0^h\|_{2,p} e^{c_1 t} + (c_1/c)(e^{c_1 t} - 1)G^h(t).$$

We assert that for every fixed t in R and for all small $|h|$ the right-hand side in (5.8) is less than a constant. In fact, that $u_0 \in W^{2,p}(R^m)$ implies that $\|u_0^h\|_{2,p} \rightarrow \|\partial u_0 / \partial x_1\|_{2,p}$ as $h \rightarrow 0$. Hence our assertion is true for the first term of the right-hand side in (5.8). By definition of $g^h(x, t)$, we have

$$(5.9) \quad \begin{aligned} \|g^h(\cdot, t)\|_{0,p} &\leq \|f^h(\cdot, t)\|_0 + \|L^h u(\cdot, t)\|_{0,p} \\ &\leq \epsilon(h, t) + \|\partial f(\cdot, t) / \partial x_1\|_{0,p} + \|L^h u(\cdot, t)\|_{0,p}, \end{aligned}$$

where $\epsilon_1(h, t) \rightarrow 0$ for all t in R as $|h| \rightarrow 0$. Since L^h is the differential operator L with its coefficients replaced by the corresponding difference quotients with respect to x_1 and since the coefficients of L have first derivatives with respect to the space variables, which are continuous and bounded on $R^m \times R$, we conclude that

$$(5.10) \quad \|L^h u(\cdot, t)\|_{0,p} \leq c_3 \|u(\cdot, t)\|_{2,p},$$

where c_3 is a constant independent of h and t . From (5.7), (5.9) and (5.10) we see that our assertion is also true for the second term on the right-hand side of (5.8).

We have shown that for all t in R there exists a constant h_0 such that for all $|h| \leq h_0$ the set of difference quotients $\{u^h(x, t), |h| \leq h_0\}$, as functions of x , is bounded in $W^{2,p}(R^m)$. Since $W^{2,p}(R^m)$, $1 < p < \infty$, is a reflexive Banach space and hence weakly compact, there exists a sequence of numbers $\{\mu\}$ in the interval $[-h_0, h_0]$ such that as $\mu \rightarrow 0$ the sequence of difference quotients $\{u^\mu(x, t)\}$, as functions of x , converges weakly in $W^{2,p}(R^m)$ to a function $u_1(x, t)$ for all t in R . For all t in R , $u_1(x, t)$, as a function of x , belongs to $W^{2,p}(R^m)$ and is the distribution derivative and hence strong derivative $\partial u(x, t) / \partial x_1$. The same reasoning may be applied to prove that all the first derivatives of $u(x, t)$ with respect to the space variables x belong to $W^{2,p}(R^m)$ for all t in R . The theorem is proved for the case $n = 1$.

Before proceeding further we note from equation (4.19) and Theorem 3.1' that if $f : R \rightarrow W^{n,p}(R^m)$ and $u : R \rightarrow W^{n+2p}(R^m)$, then $u_t : R \rightarrow W^{n+2,p}(R^m)$. In short, $u(x, t)$ is always differentiable in t and the regularity of $u(x, t)$ in x is preserved under time differentiation.

We now prove the general case by induction on n . Assume, therefore, the statement to be true for the case $n - 1$. The induction hypothesis ensures that as functions of x , both $u(x, t)$ and $u_t(x, t)$ belong to $W^{n+1}(R^m)$ for all t in R . Thus, we may differentiate both sides of $Lu = f$, $(n - 1)$ times with respect to the space variable x . This leads to

$$(5.11) \quad L D_x^\alpha u(x, t) = g(x, t), \quad |\alpha| = n - 1,$$

where $g(x, t)$ is the sum of $D_x^\alpha f(x, t)$ and products of derivatives of $u(x, t)$ with the coefficients of the operator L up to the orders $(n - 2)$ and $(n - 1)$, respectively. Clearly, as a function of x , $g(x, t)$ belongs to $W^{1,p}(R^m)$ for all t in R and its $\|\cdot\|_{1,p}$ -norm is a continuous function of t for all t in R . Moreover, induction hypothesis states that

$$(5.12) \quad \lim_{t \rightarrow 0} \|D_x^\alpha u(\cdot, t) - D^\alpha u_0(\cdot)\|_{2,p} = 0, \quad |\alpha| = n - 1.$$

Now equations (5.11) and (5.12) describe an initial-value problem which is completely similar to that in §4 for the determination of $D_x^\alpha u(x, t)$. What has been shown for the case $n = 1$ ensures that as a function of x , $D_x^\alpha u(x, t)$ belongs to $W^{3,p}(R^m)$ for all t in R and $u(x, t)$ belongs to $W^{n+2}(R^m)$ for all t in R . The theorem is now proved.

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