Automorphisms and Equivalence in von Neumann Algebras II

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1. Introduction. A problem which is often quite difficult to decide is whether the crossed product between a von Neumann algebra and a group of $*$ -automorphisms is semi-finite or not. In the present paper we shall study an aspect of this problem by giving characterizations for the canonical image of a projection in the von Neumann algebra $\mathfrak{A}$ to be finite in the crossed product $\mathfrak{B}$ of $\mathfrak{A}$ and the group G . Our approach is closely related to that in [8] in that we shall use the equivalence relation $\sim_a$ on the projections in $\mathfrak{A}$ defined in [8] by the automorphisms in G . The main result (Theorem 4.1) states that a projection in $\mathfrak{A}$ is finite relative to the partial ordering defined by $\sim_a$ if and only if its canonical image in $\mathfrak{B}$ is a finite projection in $\mathfrak{B}$. As a consequence we obtain a characterization of such projections in terms of estimates (Theorem 4.2). When $\mathfrak{A}$ is abelian these theorems can be viewed as results in ergodic theory. Note that if $\mathfrak{A}$ is countably decomposable, then the equivalence relation $\sim_a$ coincides with the one introduced by Hopf in [2], see [8]. In the final section we comment on the applications to ergodic theory and obtain a sharper version (Theorem 5.3) of Theorem 4.2, which appears to be new, and thus gives a new criterion for the existence of G -invariant σ -finite measures. The result is related to one of Markov [3, 4.8.7] on the existence of finite invariant measures (see also [7]).

2. The crossed product. Let $\mathfrak{A}$ be a von Neumann algebra of operators on a Hilbert space $\mathfrak{H}$, and let G be a (discrete) group of $*$ -automorphisms of $\mathfrak{A}$. The crossed product of $\mathfrak{A}$ with G is the von Neumann algebra generated by $\mathfrak{A}$ and G in a certain covariant representation. For convenience and in order to fix some notation we indicate here the construction of $\mathfrak{A} * G$ from [9]. This construction differs slightly from the one usually employed when G acts as a unitary transformation group on $B(\mathfrak{H})$ (see [1, Chap. I, §9] and [8, §3]), but gives an isomorphic result.

For each s in G let $\mathfrak{H}_s$ be a copy of $\mathfrak{H}$ and define $\mathfrak{K} = \sum_{s \in G}^{\oplus} \mathfrak{H}_s$. We define a normal isomorphism Φ of $\mathfrak{K}$ into $B(\mathfrak{K})$ by

$$\Phi(T) \sum_{s \in G}^{\oplus} x_s = \sum_{s \in G}^{\oplus} g_s(T) x_s.$$

Letting G act as a permutation group in $\mathfrak{K}$ we obtain a unitary representation $t \rightarrow U_t$ of G , where

$$U_t \sum_{s \in G}^{\oplus} x_s = \sum_{s \in G}^{\oplus} x_{st}.$$

One can then verify the covariance formula

$$U_t \Phi(T) U_t^* = \Phi(g_t(T))$$

for all t in G and T in $\mathfrak{K}$.

Let $\mathfrak{B}$ denote the von Neumann algebra on $\mathfrak{K}$ generated by $\Phi(\mathfrak{K})$ and U_G . Using the covariance formula it is easy to see that the set $\mathfrak{B}_0$ of operators of the form

$$B = \sum_{s \in G} U_s \Phi(T_s),$$

where $T_s = 0$ except for finitely many s , form a $*$ -subalgebra in $\mathfrak{B}$, containing $\Phi(\mathfrak{K})$ and U_G ; and therefore $\mathfrak{B}_0$ is σ -weakly dense in $\mathfrak{B}$. If E_s denotes the projection of $\mathfrak{K}$ on $\mathfrak{H}_s$, then for each B in $\mathfrak{B}_0$

$$E_s B E_t = E_s U_{t^{-1}s} \Phi(T_{t^{-1}s}) E_t.$$

By the σ -weak continuity of the map $B \rightarrow E_s B E_t$ there is then for each B in $\mathfrak{B}$ an element $T_{t^{-1}s}$ in $\mathfrak{K}$ such that

$$E_s B E_t = E_s U_{t^{-1}s} \Phi(T_{t^{-1}s}) E_t.$$

This gives a matrix representation $(B_{s,t})$ of B , where $B_{s,t} = g_t(T_{t^{-1}s})$.

With $s = t = e$ we see that the map $B \rightarrow E_e B E_e$ defines a faithful normal projection Ψ of norm one from $\mathfrak{B}$ into $\mathfrak{K}$. Straightforward computations in $\mathfrak{B}_0$ followed by a continuity argument yield the formulas

$$(*) \quad \Psi(B^* B) = \sum T_s^* T_s, \quad \Psi(B B^*) = \sum g_s(T_s T_s^*),$$

$$(**) \quad \Psi(U_t B U_t^*) = g_t(\Psi(B))$$

for each B in $\mathfrak{B}$ with $B_{s,t} = g_t(T_{t^{-1}s})$.

3. Equivalence in $\mathfrak{K}$ and $\mathfrak{B}$. If E and F are projections in $\mathfrak{B}$ we write $E \sim F$ if $E = V^* V$, $F = V V^*$ for some V in $\mathfrak{B}$. This is the usual Murray-von Neumann equivalence of projections. In [4] another equivalence relation, applicable to all positive elements was considered. If A and B are elements in $\mathfrak{B}_+$ we write $A \approx B$ if there is a set $\{C_i\}$ in $\mathfrak{B}$ such that

$$A = \sum C_i^* C_i, \quad B = \sum C_i C_i^*.$$

It was proved in [4, Theorem 4.1] that if E and F are projections, then $E \sim F$ if and only if $E \approx F$. Also a projection E is $\sim$ -finite if and only if it is $\approx$ -finite.

In [8] the second author introduced the notion of $\sim_g$ -equivalence of projections in a von Neumann algebra $\mathfrak{R}$ with a group G of automorphisms. If E and F are projections in $\mathfrak{R}$, then $E \sim_g F$ if for each s in G there is an operator T_s in $\mathfrak{R}$ such that

$$E = \sum T_s^* T_s, \quad F = \sum g_s(T_s T_s^*).$$

With $\mathfrak{B}$ the crossed product of $\mathfrak{R}$ with G it was proved that $E \sim_g F$ if and only if $\Phi(E) \sim \Phi(F)$ [8, Lemma 1]. Moreover, $\mathfrak{R}$ is $\sim_g$ -semi-finite if and only if there is a faithful normal G -invariant semi-finite trace on $\mathfrak{R}_+$ [8, Theorem 2].

We now define a relation $\approx_g$ in $\mathfrak{R}_+$. If S and T are elements in $\mathfrak{R}_+$ we write $S \approx_g T$ if there is a set $\{R_{i,s} \mid i \in I, s \in G\}$ in $\mathfrak{R}$, where I is an arbitrary index set, such that

$$S = \sum R_{i,s}^* R_{i,s}, \quad T = \sum g_s(R_{i,s} R_{i,s}^*).$$

Lemma 3.1. *If $A \approx B$ with A and B in $\mathfrak{B}_+$, then $\Psi(A) \approx_g \Psi(B)$.*

Proof. We have $A = \sum C_i^* C_i$, $B = \sum C_i C_i^*$. Thus by the normality of Ψ and formula (*) $\Psi(A) \approx_g \Psi(B)$.

Proposition 3.2. *If S and T are elements of $\mathfrak{R}_+$, then $S \approx_g T$ if and only if $\Phi(S) \approx \Phi(T)$.*

Proof. Applying Φ to the relation $S \approx_g T$ we get

$$\begin{aligned} \Phi(S) &= \sum \Phi(R_{i,s})^* \Phi(R_{i,s}) = \sum (U_s \Phi(R_{i,s}))^* (U_s \Phi(R_{i,s})), \\ \Phi(T) &= \sum \Phi(g_s(R_{i,s} R_{i,s}^*)) = \sum (U_s \Phi(R_{i,s})) (U_s \Phi(R_{i,s}))^*, \end{aligned}$$

so that $\Phi(S) \approx \Phi(T)$. The converse follows from Lemma 3.1.

Corollary 3.3. *The relation $\approx_g$ is a completely additive equivalence relation in $\mathfrak{R}_+$.*

Corollary 3.4. *If E and F are projections in $\mathfrak{R}$, then $E \approx_g F$ if and only if $E \sim_g F$.*

We denote by $\mathfrak{C}^G$ the set of elements in the center of $\mathfrak{R}$ which are fixed points for G . This von Neumann algebra plays the same rôle for the $\sim_g$ -equivalence theory as the center does for the ordinary equivalence.

Lemma 3.5. (Cf. [8, Lemma 2]). *If $T \in \mathfrak{C}^G$, then $\Phi(T)$ belongs to the center of $\mathfrak{B}$. Conversely, if S is central in $\mathfrak{B}$, then $\Psi(S) \in \mathfrak{C}^G$.*

Proof. Since $\Phi(T)$ commutes with both $\Phi(\mathfrak{R})$ and U_g it belongs to the center of $\mathfrak{B}$. Conversely, if S is central in $\mathfrak{B}$, then since Ψ is a projection we have

$$R\Psi(S) = \Psi(\Phi(R)S) = \Psi(S\Phi(R)) = \Psi(S)R$$

for each R in $\mathfrak{R}$. Moreover, by (**)

$$g_*(\Psi(S)) = \Psi(U_* S U_*^*) = \Psi(S),$$

so that $\Psi(S) \in \mathfrak{C}^G$.

It is clear from the definition that if R and S are elements of $\mathfrak{R}_+$ with $R \approx_G S$ and $T \in \mathfrak{C}_+^G$, then $RT \approx_G ST$. Writing $R \prec_G S$ if R is G -equivalent with an element majorized by S , we have the following version of the comparison theorem.

Lemma 3.6. *For any two elements S and T in $\mathfrak{R}_+$ there is a projection P in $\mathfrak{C}^G$ such that*

$$PS \prec_G PT \quad \text{and} \quad (I - P)T \prec_G (I - P)S.$$

Proof. Let P be a maximal projection in $\mathfrak{C}^G$ such that $PS \prec_G PT$. If $(I - P)T \prec_G (I - P)S$ was false, then from Proposition 3.2 and the ordinary comparison theorem (see [4, Proposition 2.9]) there would be a central element C in $\mathfrak{R}_+$ such that

$$C\Phi((I - P)S) \prec C\Phi((I - P)T).$$

Since $\Psi(C) \in \mathfrak{C}^G$ we would then have an element R in $\mathfrak{C}^G$ with $0 \leq R \leq I - P$ and $RS \prec_G RT$ in contradiction with the maximality of P . Therefore $(I - P)T \prec_G (I - P)S$.

4. $\sim_G$ -finite projections. In a set with an ordering and an equivalence relation an element is finite if it is not equivalent to any other smaller element. It is therefore not obvious that a $\sim_G$ -finite projection in $\mathfrak{R}$ is necessarily $\approx_G$ -finite, even though the two equivalence relations coincide for projections. Instead of establishing this fact with the aid of a G -invariant trace using [8, Theorem 2], we give here a direct proof.

Theorem 4.1. *If E is a projection in $\mathfrak{R}$, then the following conditions are equivalent:*

- i) E is $\sim_G$ -finite,
- ii) E is $\approx_G$ -finite,
- iii) $\Phi(E)$ is finite (in $\mathfrak{R}$).

Proof. i) $\Rightarrow$ ii). If E is not $\approx_G$ -finite there is an S in $\mathfrak{R}_+$ with $S \leq E$ and $S \approx_G E$. By assumption S is not a projection and we can therefore find a non-zero spectral projection F of S corresponding to an interval $]0, \alpha]$ with $\alpha < 1$. For sufficiently large n this implies that $F \leq n(E - S)$. Thus $F \prec_G n(E - S)$ and clearly $nE \approx_G nS$. Adding these expressions we get $nE + F \prec_G nE$. Repeated use of this gives

$$nE + nF \prec_G nE + (n - 1)F \prec_G \cdots \prec_G nE.$$

Thus $E + F \prec_G E$ and clearly $E + F \succ_G E$; hence $E + F \approx_G E$ by [4, Proposition 2.7] combined with Proposition 3.2.

Take a maximal projection P in $\mathfrak{C}^G$ with $PF = 0$ and put $E_0 = (I - P)E$, $F_0 = (I - P)F \neq 0$. Then $E_0 + F_0 \approx_G E_0$. We claim that $\Phi(E_0)$ is properly infinite in $\mathfrak{B}$. For if $Q_0 = Q\Phi(E_0)$ is finite for some central projection Q in $\mathfrak{B}$, then the matrix

$$\begin{bmatrix} Q_0 & 0 \\ 0 & Q_0 \end{bmatrix}$$

is finite in $\mathfrak{B} \otimes M_2$. With $P_0 = Q\Phi(F_0)$ we have $Q_0 + P_0 \approx Q_0$ by Proposition 3.2; hence

$$\begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} Q_0 + P_0 & 0 \\ 0 & 0 \end{bmatrix} \approx \begin{bmatrix} Q_0 & 0 \\ 0 & P_0 \end{bmatrix}.$$

It follows from [4, Theorem 4.1] that

$$\begin{bmatrix} Q_0 & 0 \\ 0 & 0 \end{bmatrix} \sim \begin{bmatrix} Q_0 & 0 \\ 0 & P_0 \end{bmatrix}$$

and since they are majorized by the finite projection

$$\begin{bmatrix} Q_0 & 0 \\ 0 & Q_0 \end{bmatrix}$$

we conclude that $P_0 = 0$. Then $\Psi(Q)F_0 = \Psi(P_0) = 0$ and $\Psi(Q) \in \mathfrak{C}^G$ by Lemma 3.5. By the maximality of P this gives $\Psi(Q) = 0$; hence $Q = 0$ and $\Phi(E_0)$ is properly infinite.

We can therefore find projections Q_1 and Q_2 in $\mathfrak{B}$ such that $Q_1 + Q_2 = \Phi(E_0)$ and $Q_1 \sim Q_2 \sim \Phi(E_0)$. Then $\Phi(E_0) \approx 2\Phi(E_0)$, and $E_0 \approx_G 2E_0$ by Proposition 3.2.

Take a projection $G \leq E_0$. Then there is, by Lemma 3.6, a projection H in $\mathfrak{C}^G$ such that

$$HG \prec_G H(E_0 - G), \quad (I - H)(E_0 - G) \prec_G (I - H)G.$$

Therefore

$$2HE_0 \approx_G HE_0 = H(E_0 - G) + HG \prec_G 2H(E_0 - G),$$

so that $HE_0 \approx_G H(E_0 - G)$. Likewise $(I - H)E_0 \approx_G (I - H)G$. Since $\approx_G$ and $\sim_G$ coincide for projections, and since E_0 is $\sim_G$ -finite this implies that $HG = 0$ and $(I - H)E_0 = (I - H)G$ so that $G = (I - H)E_0$. It follows that $E_0 \mathfrak{B} E_0 = \mathfrak{C}^G E_0$.

Now $E_0 \approx_G 2E_0$, so that

$$E_0 = \sum R_{is}^* R_{is}, \quad 2E_0 = \sum g_s(R_{is} R_{is}^*).$$

From the above $R_{is}^* R_{is} = C_{is} E_0$ and $g_s(R_{is} R_{is}^*) = D_{is} E_0$; with C_{is} and D_{is} in $\mathfrak{C}^G$. Let $R_{is} = V_{is} |R_{is}|$ be the polar decomposition of R_{is} . Then

$$D_{is} E_0 = g_s(V_{is} C_{is} E_0 V_{is}^*) \leq C_{is}$$

so that $D_{is}E_0 \leq C_{is}E_0$. By a symmetric argument $C_{is}E_0 \leq D_{is}E_0$. It follows that $C_{is}E_0 = D_{is}E_0$; hence $E_0 = 2E_0$ and $E_0 = 0$, a contradiction since it would imply that $F = 0$.

ii) $\Rightarrow$ iii). If $0 \leq B \leq \Phi(E)$ and $B \approx \Phi(E)$, then $\Psi(B) \approx_g E$ by Lemma 3.1. Since $\Psi(B) \leq E$ and E is $\approx_g$ -finite this implies that $\Psi(B) = E$ and thus $B = \Phi(E)$ by the faithfulness of Ψ .

iii) $\Rightarrow$ i). If F is a projection in $\mathfrak{R}$ with $F \leq E$ and $F \sim_g E$, then $\Phi(F) \leq \Phi(E)$ and $\Phi(F) \sim \Phi(E)$. Thus $\Phi(F) = \Phi(E)$ and $F = E$.

Using Theorem 4.1 one may give another proof of the result from [8, Theorem 2] that $\sim_g$ -semi-finiteness of $\mathfrak{R}$ is equivalent with the existence of a faithful normal G -invariant semi-finite trace on $\mathfrak{R}_+$. We leave the details to the reader. Instead we give another characterization of countably decomposable $\sim_g$ -finite projections in $\mathfrak{R}$.

Theorem 4.2. *A countably decomposable projection E in $\mathfrak{R}$ is $\sim_g$ -finite if and only if for some (hence any) normal state ω of $\mathfrak{R}$ with support E and each $\epsilon > 0$ there is a $\delta > 0$ such that $\omega(T) < \delta$ and $S \approx_g T$ implies $\omega(S) < \epsilon$ for all elements S and T in $\mathfrak{R}_+$ majorized by E .*

Proof. To prove the necessity of the condition we note that if E is countably decomposable in $\mathfrak{R}$, then $\Phi(E)$ is countably decomposable in $\mathfrak{B}$ (see [8, Lemma 5] or use the faithfulness of Ψ). Thus if E is $\sim_g$ -finite, then by Theorem 4.1 $\Phi(E)$ is finite and countably decomposable in $\mathfrak{B}$ and there is a faithful normal finite trace τ on $\mathfrak{B}_E (= \Phi(E)\mathfrak{B}\Phi(E))$. Let ω be a normal state of $\mathfrak{R}$ with support E . Then $\tilde{\omega} = \omega \circ \Psi$ is a faithful normal state of $\mathfrak{B}_E$ and there is therefore a self-adjoint positive operator H affiliated with $\mathfrak{B}_E$ such that $\tilde{\omega}(B) = \tau(HB)$ for all B in $(\mathfrak{B}_E)_+$. Let P_n be the spectral projection of H corresponding to the interval $[n, \infty]$ and put $H_n = (I - P_n)H$. If now S and T are elements in $\mathfrak{R}_+$ with $S \approx_g T$, then $\Phi(S) \approx \Phi(T)$ and therefore

$$\begin{aligned} \omega(S) &= \tilde{\omega}(\Phi(S)) = \tau(H\Phi(S)) \\ &= \tau(H_n\Phi(S)) + \tau(HP_n\Phi(S)) \\ &\leq n\tau(\Phi(S)) + \|S\| \tau(HP_n) \\ &= n\tau(\Phi(T)) + \|S\| \tilde{\omega}(P_n). \end{aligned}$$

Since $P_n \searrow 0$ as $n \rightarrow \infty$, we see that if S and T are majorized by E and $\omega(T) \rightarrow 0$, then $T \rightarrow 0$ σ -weakly since ω is faithful on $E\mathfrak{R}E$ by [1, I.4.6. Proposition 4]; hence $\tau(\Phi(T)) \rightarrow 0$ and thus $\omega(S) \rightarrow 0$.

The condition is also sufficient. For if E is not $\sim_g$ -finite, then by Theorem 4.1 $\Phi(E)$ is not finite and there is therefore a sequence of non-zero orthogonal projections G_n with sum $F \leq \Phi(E)$ such that $G_n \sim F$ for each n (see [1, III.8.6. Corollaire 2]). Put $T_n = \Psi(G_n)$ and $S = \Psi(F)$. Then $T_n \approx_g S$ by Lemma 3.1 and $T_n \rightarrow 0$ σ -weakly. If therefore ω is a normal state of $\mathfrak{R}$ with support E , then $\omega(S) > 0$ while $\omega(T_n) \rightarrow 0$. This completes the proof.

It is conceivable that one may weaken the condition in Theorem 4.2 with the requirement that S and T be projections. However, we have only been able to prove this in the case where $\mathfrak{A}$ is abelian (see Theorem 5.3 below).

5. Abelian von Neumann algebras. We apply the results in §4 to the case when $\mathfrak{A}$ is abelian to study an equivalence relation introduced by Hopf [2] in ergodic theory. Let $\mathfrak{A}$ be an abelian von Neumann algebra and G a group of $*$ -automorphisms of $\mathfrak{A}$. We say that two projections E and F in $\mathfrak{A}$ are *equivalent in the sense of Hopf*, written $E \sim F$, if there is an orthogonal family $\{E_s\}_{s \in G}$ of projections in $\mathfrak{A}$ such that $E = \sum_{s \in G} E_s$ and $F = \sum_{s \in G} g_s(E_s)$ (so in particular $\{g_s(E_s)\}$ is also an orthogonal family of projections). We say that E is *bounded* (or *finite* in the sense of Hopf) if $F \leq E$ and $F \sim E$ implies $F = E$. It was shown in [8; Theorem 5] that if $\mathfrak{A}$ is countably decomposable, then $E \sim F$ if and only if $E \sim_g F$ (note that it is trivial that $E \sim F$ implies $E \sim_g F$ and that $E \sim_g F$ if and only if $E \approx_g F$). In our previous notation we therefore have the following corollary to Theorem 4.1.

Corollary 5.1. *Suppose $\mathfrak{A}$ is abelian and countably decomposable. Then a projection E in $\mathfrak{A}$ is bounded if and only if $\Phi(E)$ is a finite projection in $\mathfrak{A}$.*

The above result can be shown without the assumption that $\mathfrak{A}$ is countably decomposable. A proof can be obtained by modifying the arguments of Singer [6]. (We are indebted to W. Arveson for pointing out the relevance of this reference.) We next prove a sharper version of Theorem 4.2 in the case when $\mathfrak{A}$ is abelian. Again we shall for simplicity make the unnecessary assumption that $\mathfrak{A}$ is countably decomposable. For this we need a lemma which is also a consequence of [6]. For completeness we indicate the proof.

Lemma 5.2. *Let $\mathfrak{A}$ be abelian and E a projection in $\mathfrak{A}$. Then there is a group G_E of $*$ -automorphisms of the von Neumann algebra $\mathfrak{A}_E$ such that if P and Q are projections in $\mathfrak{A}_E$ and $\alpha \in G_E$ with $\alpha(P) = Q$, then $P \sim Q$. Conversely, if P and Q are mutually orthogonal projections in $\mathfrak{A}_E$ such that $P \sim Q$, then there is α in G_E such that $\alpha(P) = Q$.*

Proof. Let G_E consist of all maps

$$\alpha(T) = \sum_{s \in G} g_s(E_s T),$$

where $\{E_s\}_{s \in G}$ is an orthogonal family of projections in $\mathfrak{A}$ with sum E such that $\{g_s(E_s)\}$ has the same properties. It is a straightforward but somewhat tedious computation to show that α is a $*$ -automorphism of $\mathfrak{A}_E$ and that G_E is a group. Note that if $\alpha \in G_E$ and $\alpha(P) = Q$ for two projections P and Q in $\mathfrak{A}_E$, then $P = \sum E_s P$ and $Q = \sum g_s(E_s P)$, so $P \sim Q$. Conversely, if P and Q are mutually orthogonal projections in $\mathfrak{A}_E$ and $P \sim Q$, then there is an orthogonal family $\{F_s\}_{s \in G}$ of projections in $\mathfrak{A}$ such that $P = \sum F_s$ and $Q = \sum g_s(F_s)$. Put $E_e = E - P - Q$ and $E_s = F_s + g_{s^{-1}}(F_{s^{-1}})$ for $s \neq e$. Let $\alpha(T) = \sum g_s(E_s T)$. Then another straightforward computation shows that $\alpha \in G_E$ and $\alpha(P) = Q$.

Theorem 5.3. *Let $\mathfrak{R}$ be abelian and countably decomposable. Then a projection E in $\mathfrak{R}$ is bounded if and only if for some (hence any) normal state ω of $\mathfrak{R}$ with support E and each $\epsilon > 0$ there is a $\delta > 0$ such that $\omega(P) < \delta$ and $Q \sim P$ implies $\omega(Q) < \epsilon$ for all projections P and Q in $\mathfrak{R}$ majorized by E .*

Proof. Since $P \sim Q$ if and only if $P \approx_G Q$ the necessity follows from Theorem 4.2. Conversely, let ω be a normal state of $\mathfrak{R}$ with support E such that given $\epsilon > 0$ there is $\delta > 0$ such that if P is a projection in $\mathfrak{R}_E$ with $\omega(P) < \delta$, then $\omega(Q) < \epsilon$ for all projections Q in $\mathfrak{R}_E$ for which $Q \sim P$. By Lemma 5.2 and [7, Corollary] there is a faithful normal G_E -invariant state ρ of $\mathfrak{R}_E$. We show that if P and Q are projections in $\mathfrak{R}_E$ with $P \sim Q$, then $\rho(P) = \rho(Q)$. By Lemma 5.2 this holds if P is orthogonal to Q . Let P and Q be arbitrary in $\mathfrak{R}_E$ and $P \sim Q$, say $P = \sum P_s$ and $Q = \sum g_s(P_s)$ with P_s projections in $\mathfrak{R}_E$. Since ρ is normal it suffices to show that $\rho(P_s) = \rho(g_s(P_s))$, or in general, if P is a projection in $\mathfrak{R}_E$ such that $g_s(P) \in \mathfrak{R}_E$ for a given s in G , then $\rho(P) = \rho(g_s(P))$. Let E_s and F_s be projections in $\mathfrak{R}$ with sum I such that $g_s(T) = T$ for all T in $\mathfrak{R}_{E_s}$, and the automorphism g_s is freely acting on $\mathfrak{R}_{F_s}$, see e.g., [5]. By freely acting we mean that given a projection $F \neq 0$ there is a subprojection $H \neq 0$ of F such that H and $g_s(H)$ are orthogonal. Clearly $\rho(PE_s) = \rho(g_s(PE_s))$, so it suffices to consider PF_s , i.e., we may assume g_s is freely acting. By Zorn's Lemma let $\{P_\alpha\}$ be a maximal orthogonal family of subprojections of P such that $g_s(P_\alpha)$ is orthogonal to P_α . Since g_s is freely acting $\sum P_\alpha = P$. Now $P_\alpha \sim g_s(P_\alpha)$, and they are orthogonal and contained in $\mathfrak{R}_E$. Thus $\rho(P_\alpha) = \rho(g_s(P_\alpha))$, and by the normality of ρ ,

$$\rho(P) = \rho(\sum P_\alpha) = \sum \rho(P_\alpha) = \sum \rho(g_s(P_\alpha)) = \rho(g_s(P)).$$

Hence we have shown that if P and Q are projections in $\mathfrak{R}_E$ such that $P \sim Q$, then $\rho(P) = \rho(Q)$. Now suppose F is a projection in $\mathfrak{R}$ such that $F \leq E$ and $F \sim E$. Then $F \in \mathfrak{R}_E$ so $\rho(F) = \rho(E)$; hence $\rho(E - F) = 0$. Since ρ is faithful on $\mathfrak{R}_E$, $E = F$, and E is bounded as we wanted to show.

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