Lipschitz Spaces and Exponentially Integrable Functions

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1. We let $\mathbf{R}^n$ be the Euclidean n -space, and we consider the Lipschitz spaces $\Lambda(\sigma; p, s; \mathbf{R}^n)$ as defined in [2, 5]. Our main result will imply an imbedding of the Lipschitz spaces $\Lambda(n/p; p, s; \mathbf{R}^n)$ into the class $\exp(L^s)$ consisting of all functions $f: \mathbf{R}^n \rightarrow \mathbf{R}$ with compact support satisfying $\int \exp(\alpha |f(x)|^{s'}) dx < \infty$ for some $\alpha > 0$, $1/s + 1/s' = 1$, where the integration is extended over the support of f .

In view of the fundamental inclusion relationships between the space $\Lambda(\sigma; p, s; \mathbf{R}^n)$ and the space of Bessel potentials $L^p_\sigma(\mathbf{R}^n)$ (see [2, 5]), this result extends, for $p \geq 2$, the inequality obtained by Strichartz [4].

The proof is accomplished by estimating $\|f\|_q$ in terms of the L^p -modulus of continuity of f , and the proof of this estimate is an n -dimensional adaptation and refinement of an argument of Storozenko [3] whose main result, for $n = 1$, is the exponential integrability of f if its L^p -modulus of continuity $\omega_p(f; h) = O(h^{1/p})$.

2. For $f \in L^p(\mathbf{R}^n)$ we define

$$\omega_p(f; h) = \sup_{|t| \leq h} \left\{ \int_{\mathbf{R}^n} |f(x+t) - f(x)|^p dx \right\}^{1/p}.$$

It is well-known that $\omega_p(f; h)$ is non-decreasing, $\omega_p(f; h) = o(1)$ as $h \rightarrow 0$, and, if c is a positive integer, $\omega_p(f; ch) \leq c \omega_p(f; h)$.

Let $Q(x; h)$ be the cube $[x_1, x_1 + h] \times \cdots \times [x_n, x_n + h]$, where $x = (x_1, \dots, x_n)$. If F is the integral of f , i.e., $F(x) = \int_0^{x_1} \cdots \int_0^{x_n} f$, we define, in analogy with $n = 1$, the first and second differences of F as $\Delta F(x; h) = \int_{Q(x; h)} f$, $\Delta^2 F(x; h) = \Delta F(x; 2h) - 2\Delta F(x; h)$.

Lemma. Let $f \in L^p(\mathbf{R}^n)$ for some $p \geq 1$. Then for $q \geq p$, $\|\Delta^2 F(\cdot; h)\|_q \leq ch^{n(1+1/q-1/p)} \omega_p(f; h)$.

Proof. We consider the case $1 \leq p < \infty$. The $p = \infty$ situation is similar and easier. By an obvious change of variables we can write

$$(1) \quad \Delta^2 F(x; h) = \sum_{i=1}^{2^n-1} \int_{Q(x; h)} [f(u + h^i) - f(u)] du,$$

where $h^i \in \mathbf{R}^n$ has components 0 or h . Each integral can be written as a convolution with the characteristic function of a cube with side-length h , and hence by Young's inequality $\|\Delta^2 F(\cdot; h)\|_p \leq ch^n \omega_p(f; n^{1/2}h) \leq ch^n \omega_p(f; h)$.

By Hölder's inequality we obtain from (1),

$$|\Delta^2 F(x; h)| \leq ch^{n(1-1/p)} \omega_p(f; h),$$

and consequently for $q \geq p$,

$$\begin{aligned} \|\Delta^2 F(\cdot; h)\|_q^q &= \int_{\mathbf{R}^n} |\Delta^2 F(x; h)|^p |\Delta^2 F(x; h)|^{q-p} dx \\ &\leq ch^{n(1-1/p)(q-p)} \omega_p^{q-p}(f; h) \cdot h^{np} \omega_p^p(f; h) \\ &= ch^{n(1+q-p/p)} \omega_p^q(f; h). \end{aligned}$$

Theorem 1. Let $f \in L^p(\mathbf{R}^n)$ for some $1 \leq p < \infty$. Then for $p < q < \infty$,

$$\|f\|_q \leq c \int_0^\infty \frac{\omega_p(f; t)}{t^{1+n(1/p-1/q)}} dt,$$

where c depends on n only.

Proof. From the relation $\Delta^2 F(x; h) = \Delta F(x; 2h) - 2^n \Delta F(x; h)$ we obtain

$$\begin{aligned} \sum_{m=0}^{r-1} \frac{1}{2^{n(m+1)}} \Delta^2 F(x; 2^m t) &= \sum_{m=0}^{r-1} \frac{1}{2^{n(m+1)}} \Delta F(x; 2^{m+1} t) - \sum_{m=0}^{r-1} \frac{1}{2^{nm}} \Delta F(x; 2^m t) \\ &= \frac{1}{2^{nr}} \Delta F(x; 2^r t) - \Delta F(x; t). \end{aligned}$$

Let $\lambda > 0$, and let $t = \lambda/2^r$. Then

$$\Delta F\left(x; \frac{\lambda}{2^r}\right) = - \sum_{m=0}^{r-1} \frac{1}{2^{n(m+1)}} \Delta^2 F(x; 2^{m-r} \lambda) + \frac{1}{2^{nr}} \Delta F(x; \lambda).$$

Let $k = r - m$. Then

$$\Delta F\left(x; \frac{\lambda}{2^r}\right) = - \frac{1}{2^{n(r+1)}} \sum_{k=1}^r 2^{nk} \Delta^2 F\left(x; \frac{\lambda}{2^k}\right) + \frac{1}{2^{nr}} \Delta F(x; \lambda).$$

The lemma implies that

$$\frac{2^{nr}}{\lambda^n} \left\| \Delta F\left(\cdot; \frac{\lambda}{2^r}\right) \right\|_q \leq c \sum_{k=1}^r \frac{2^{nk}}{\lambda^n} \left(\frac{\lambda}{2^k}\right)^{n(1+1/q-1/p)} \omega_p\left(f; \frac{\lambda}{2^k}\right) + \frac{1}{\lambda^n} \|\Delta F(\cdot; \lambda)\|_q.$$

We estimate the last term by observing that $\Delta F(x; \lambda) = (\chi * f)(x)$, where χ is the characteristic function of $Q(\tilde{\lambda}; \lambda)$, $\tilde{\lambda} = (-\lambda, -\lambda, \dots, -\lambda)$. Since $q > p$, we can choose $1 < s < p'$ such that $1/q = 1/p + 1/s - 1$, and hence by Young's inequality

$$\frac{1}{\lambda^n} \|\Delta F(\cdot; \lambda)\|_q \leq \|f\|_p \cdot \frac{1}{\lambda^{n/s'}}.$$

The first term we denote by A and observe that

$$\begin{aligned} A &= c \sum_1^r \left(\frac{2^k}{\lambda}\right)^{n+1} \left(\frac{\lambda}{2^k}\right)^{n(1+1/q-1/p)} \omega_p\left(f; \frac{\lambda}{2^k}\right) \cdot \frac{\lambda}{2^k} \\ &\leq c \sum_1^r \frac{\omega_p(f; \lambda/2^k)}{(\lambda/2^k)^{1+n(1/p-1/q)}} \frac{\lambda}{2^k}. \end{aligned}$$

Since $\omega_p(f; \cdot)$ is non-decreasing, the sum is majorized by

$$c \int_0^\lambda \frac{\omega_p(f; t)}{t^{1+n(1/p-1/q)}} dt.$$

Finally, since $(2^{nr}/\lambda^n)\Delta F(x; \lambda/2^r) \rightarrow f(x)$, a.e. x , as $r \rightarrow \infty$ we apply Fatou's lemma and obtain

$$\|f\|_q \leq c \int_0^\lambda \frac{\omega_p(f; t)}{t^{1+n(1/p-1/q)}} dt + \frac{\|f\|_p}{\lambda^{n/s'}}.$$

If we let $\lambda \rightarrow \infty$ we obtain the desired estimate.

Remark. When $\lambda = 1$ we obtain that

$$\|f\|_q \leq c \int_0^1 \frac{\omega_p(f; t)}{t^{1+n(1/p-1/q)}} dt + \|f\|_p$$

is valid for $q \geq p$, and this is how the L^q -estimate will be applied below.

Corollary 1. Let $f \in L^p(\mathbb{R}^n)$ for some $1 \leq p < \infty$, $\sigma \leq n/p$, and $1 < s \leq \infty$.

Then, for $p \leq q < n/(n/p - \sigma)$,

$$\|f\|_q \leq c \left\{ \int_0^1 \frac{\omega_p^*(f; t)}{t^{1+\sigma s}} dt \right\}^{1/s'} \cdot \left\{ \frac{1}{(n/q + \sigma - n/p)s'} \right\}^{1/s'} + \|f\|_p.$$

Proof. We apply Hölder's inequality to

$$\begin{aligned} \int_0^1 \frac{\omega_p(f; t)}{t^{1+n(1/p-1/q)}} dt &= \int_0^1 \frac{\omega_p(f; t)}{t^{1-n/q+\alpha}} \cdot \frac{1}{t^{n/p-\alpha}} dt \\ &\leq \left\{ \int_0^1 \frac{\omega_p^*(f; t)}{t^{(1-n/q+\alpha)s}} dt \right\}^{1/s'} \left\{ \int_0^1 \frac{dt}{t^{(n/p-\alpha)s'}} \right\}^{1/s'}. \end{aligned}$$

We choose $\alpha = -1/s' + n/q + \sigma$. Then $(n/p - \alpha)s' = (n/p - n/q - \sigma + 1/s')s' = 1 + s'(n/p - n/q - \sigma) < 1$.

Remark. If $s = 1$ in Corollary 1, then $n/p - n/q \leq \sigma$ implies

$$\|f\|_q \leq c \int_0^1 \frac{\omega_p(f; t)}{t^{1+\sigma}} dt + \|f\|_p.$$

3. We are now ready to apply the results in §2 to the exponential integrability of functions in the Lipschitz space $\Lambda(n/p; p, s; \mathbf{R}^n)$ [2, 5].

Theorem 2. *Let $1 \leq p < \infty$, $1 < s \leq \infty$, and let $f \in \Lambda(n/p; p, s; \mathbf{R}^n)$ have compact support. Then $f \in \exp(L^{s'})$.*

Proof. We first consider the case $0 < n/p < 1$. Then (see [5]) $f \in \Lambda(n/p; p, s; \mathbf{R}^n)$ if and only if

$$\int_0^\infty \frac{\omega_p^s(f; t)}{t^{t + (n/p)s}} dt < \infty.$$

Corollary 1 implies now that $\|f\|_a = O(q^{1/s'})$, $q \geq p$, and it is well-known that this estimate implies that $f \in \exp(L^{s'})$ (see [4]).

If $n/p \geq 1$, then we choose $p_0 > n$, and set $\alpha = n/p$, $\alpha_0 = n/p_0$. Since $\alpha - n/p = \alpha_0 - n/p_0$, $\alpha \geq \alpha_0$, we have the inclusion [2, 5]

$$\Lambda\left(\frac{n}{p}; p, s; \mathbf{R}^n\right) \subset \Lambda\left(\frac{n}{p_0}; p_0, s; \mathbf{R}^n\right).$$

Thus the first case applies and the proof is complete.

Remark. If $s = 1$ in the above theorem, then $f \in L^\infty(\mathbf{R}^n)$. This follows from the observation made at the end of §2.

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