Poincaré Metrics on Real Projective Space

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Communicated by S. S. CHERN

In this note we consider Riemannian metrics on 3-dimensional real projective space, P_3 , which are induced by Riemannian metrics on the 2-dimensional sphere, S_2 , (cf. below for details) and seek upper bounds for the quotient L^3/V , where (henceforth) L is the length of the shortest closed geodesic in the non-trivial free homotopy class of continuous, sectionally smooth $\xi: [0, 1] \rightarrow P_3$ and V is the volume of P_3 relative to the metric in question. It is conjectured (cf. [1–6] for background and related results) that for *all* Riemannian metrics on P_3 , $L^3/V \leq \pi$ the value achieved when the Riemannian metric on P_3 has constant sectional curvature, and that equality above characterizes the Riemannian metric of constant sectional curvature on P_3 . We are content to limit the class of Riemannian metrics to much less than the class of all Riemannian metrics as there are sufficiently many difficulties yet to be resolved in this restricted case.

Our class of Riemannian metrics is constructed as follows: the unit tangent bundles of different Riemannian metrics on the 2-sphere, S_2 , are diffeomorphic, and the unit tangent bundle of the Riemannian metric of constant curvature 1 on S_2 can be explicitly written as $SO(3)$, the special orthogonal group acting on 3-dimensional Euclidean space. But $SO(3)$ is diffeomorphic to P_3 ([16, p. 115]). For any Riemannian metric g on S_2 one constructs in the standard manner the three global, pointwise linearly independent, differential 1-forms, $\omega_1, \omega_2, \omega_{12}$ on P_3 now viewed as the oriented frame bundle of S_2 , where ω_1, ω_2 are the canonical forms of the bundle, and ω_{12} is the Levi-Civita connection form on P_3 of the Riemannian metric g . The forms $\omega_1, \omega_2, \omega_{12}$ satisfy the Cartan structure equations

$$d\omega_1 = -\omega_2 \wedge \omega_{12},$$

$$d\omega_2 = \omega_1 \wedge \omega_{12},$$

$$d\omega_{12} = -K\omega_1 \wedge \omega_2,$$

where K is the Gaussian curvature of g on S_2 (lifted to P_3 via the standard projection of the unit tangent bundle). The induced Riemannian metric on P_3

is then defined by $(dS_o)^2 = (\omega_1)^2 + (\omega_2)^2 + (\omega_{12})^2$, i.e., the global linear frame E_1, E_2, E_{12} on P_3 dual to the basis of 1-forms $\{\omega_1, \omega_2, \omega_{12}\}$ at each point is declared to be orthonormal. Such Riemannian metrics on the unit tangent bundle of a Riemannian manifold were introduced by H. Poincaré [12] and subsequently studied by S. Sasaki [14]. For the details of the construction of the above differential forms, and related material necessary for the sequel we refer the reader to [11, Chap. III, IV; 15, Chap. 7; 14]. In this paper we prove

Theorem 1. *Assume $L = 2\pi$. Then $L^3/V \leq \pi$ with equality if and only if K is identically 1 on S_2 , in which case dS_o on P_3 has constant sectional curvature $1/4$.*

Theorem 2. *Assume the Gauss curvature, K , satisfies: $1 < K < 2$ on all of S_2 . Then L is realized by the unit velocity vector field of a simple closed geodesic on S_2 whose length is L and satisfies: $2^{1/2}\pi < L < 2\pi$.*

We first remark that E_{12} is tangent to the fiber, i.e., a circle, which traversed once is non-null homotopic [16, p. 115] and has length 2π ; thus for all Riemannian metrics g on S_2 , $L \leq 2\pi$. Also, if we let v denote the volume of S_2 relative to g , then $V = 2\pi v$. Our proof of Theorem 1 will then consist of showing that $K \leq 1$ on all of S_2 if $L = 2\pi$. The Gauss-Bonnet theorem will then imply that $v \geq 4\pi$ (with equality if and only if $K = 1$ on all of S_2) which will imply the claim.

We next remark that the unit vector field tangent to a regular simple closed $\omega: [0, 1] \rightarrow S_2$ is easily seen to be homotopic to the fiber traversed once, and is thus non-null homotopic. If ℓ (henceforth) denotes the length of the shortest simple closed geodesic $\gamma: [0, 1] \rightarrow S_2$ on S_2 (which always exists [1, p. 243]), then $\gamma'/\|\gamma'\|: [0, 1] \rightarrow P_3$, the unit velocity vector field tangent to γ , is a geodesic of length ℓ ; thus $L \leq \ell$. In general, it will not suffice to estimate ℓ alone from above, e.g., in the case of the Zoll surfaces $v = 4\pi$, $\ell = 2\pi$ and $L < 2\pi$ as one can easily check from the calculations of [1, pp. 145] and Theorem 1.

It is worth noting that even if g has constant curvature κ on S_2 the induced metric on P_3 is of constant sectional curvature if and only if $\kappa = 1$, in which case P_3 has constant sectional curvature $1/4$. For all κ , however, we have $L^3/V \leq \pi$. Indeed, if $\kappa \leq 1$, then $v \geq 4\pi$ and $L^3/V \leq \pi$. Also, if $\kappa > 1$, then $L \leq \ell = 2\pi/\kappa^{1/2}$ and $v = 4\pi/\kappa$ which implies $L^3/V \leq \pi/\kappa^{1/2}$. Obviously, if $\kappa \neq 1$, $L^3/V < \pi$.

We turn to the proofs of Theorem 1 and Theorem 2.

Henceforth we let M denote the 2-sphere S_2 with Riemannian metric g , and $S(M)$ its unit tangent bundle with standard projection $\pi: S(M) \rightarrow M$ satisfying $\pi(x) = p$ for all $x \in M_p$ (M_p is the tangent space to M at p) of length 1. Lengths of tangent vectors will be denoted by $\|\cdot\|$ and inner products by $\langle \cdot, \cdot \rangle$ for both Riemannian metrics on M and $S(M)$. Covariant differentiation on M and $S(M)$ will be denoted by D and ∇ respectively; for paths $\omega: I \rightarrow M$, $\xi: I \rightarrow S(M)$, where I is some interval on the real line, the respective velocity vectors will be denoted by ω', ξ' ; and for any $x \in S(M)$, ix will denote the unit vector in $M_{\pi(x)}$ such that $\{x, ix\}$ form an oriented orthonormal frame in $M_{\pi(x)}$.

We recall the facts necessary for proving Theorem 1. Let e_1, e_2 be a local oriented orthonormal frame on M , with local 1-forms σ_1, σ_2 dual to e_1, e_2 at every point for which the frame is defined. Then there exists a unique local 1-form σ_{12} on M such that

$$\begin{aligned} d\sigma_1 &= -\sigma_2 \wedge \sigma_{12}, \\ d\sigma_2 &= \sigma_1 \wedge \sigma_{12}. \end{aligned}$$

For any $x = \cos \varphi \cdot e_1 + \sin \varphi \cdot e_2 \in S(M)$, the differential forms $\omega_1, \omega_2, \omega_{12}$ at x are related to $\sigma_1, \sigma_2, \sigma_{12}$ at $\pi(x)$ by

$$\begin{aligned} \omega_1 &= \pi^* \sigma_1 \cdot \cos \varphi + \pi^* \sigma_2 \cdot \sin \varphi, \\ \omega_2 &= \pi^* \sigma_1 \sin \varphi + \pi^* \sigma_2 \cdot \cos \varphi, \\ \omega_{12} &= \pi^* \sigma_{12} + d\varphi. \end{aligned}$$

As a result, we have $\omega_1 \wedge \omega_2 \wedge \omega_{12} = \pi^*(\sigma_1 \wedge \sigma_2) \wedge d\varphi$; but $\sigma_1 \wedge \sigma_2$ is the local element of area. Therefore, by integration over the fibers, we have $V = 2\pi v$.

For any $x \in S(M)$, let $E_{1|x}, E_{2|x}, E_{12|x}$ be the respective restrictions of the vector fields E_1, E_2, E_{12} on $S(M)$ to the tangent space to $S(M)$ at x . Then one has $\pi_*(E_{1|x}) = x$, $\pi_*(E_{2|x}) = ix$, and $\pi_*(E_{12|x}) = 0$. Also, if I is an interval in the real numbers, $\xi: I \rightarrow S(M)$ a smooth path, and $\omega = \pi \circ \xi$, then the co-variant derivative of ξ along ω , $D_i \xi$, is given by

$$(1) \quad D_i \xi = \omega_{12}(\xi') i\xi.$$

Therefore if

$$\xi'(t) = \alpha_1(t)E_{1|\xi(t)} + \alpha_2(t)E_{2|\xi(t)} + \alpha_{12}(t)E_{12|\xi(t)}$$

we have

$$\omega'(t) = \alpha_1(t)\xi(t) + \alpha_2(t)i\xi(t),$$

which implies

$$(2) \quad \|\xi'(t)\|^2 = \|\omega'(t)\|^2 + \|D_i \xi(t)\|^2.$$

Now let $p \in M$, and U an open neighborhood of p in which geodesic polar coordinates $(r, \theta): U \rightarrow \mathbb{R}^2$ centered at p are valid, where for $q \in U$, $r(q)$ denotes the distance from p to q and $\theta(q)$ denotes the angle (mod 2π) from a fixed unit vector in M_p to the initial unit velocity vector in M_p of the unique geodesic connecting p to q . For $q \in U$, $q \neq p$ we let $\partial_r, \partial_\theta$ denote the natural basis of M_q relative to the coordinates (r, θ) , and set $G = \|\partial_\theta\|^2$. As is well-known, $\|\partial_r\| = 1$, $\langle \partial_r, \partial_\theta \rangle = 0$, and $D_r \partial_r = 0$, and

$$(3) \quad \frac{\partial^2}{\partial r^2} (G^{1/2}) + KG^{1/2} = 0,$$

$$\lim_{r \rightarrow 0} G^{1/2} = 0, \quad \lim_{r \rightarrow 0} \frac{\partial}{\partial r} (G^{1/2}) = 1$$

on all of U (K is the Gaussian curvature on M). For each $\alpha > 0$, consider the geodesic circle $\omega_\alpha : [0, 1] \rightarrow U$ given by $r(\omega_\alpha(t)) = \alpha$, $\theta(\omega_\alpha(t)) = 2\pi t$, and the unit vector field along ω_α , $\xi_\alpha : [0, 1] \rightarrow S(M)$ given by $\xi_\alpha(t) = \partial_r(\omega_\alpha(t))$. Then ξ_α is not homotopic to zero in $S(M)$, and one easily checks that

$$||\xi'_\alpha(t)|| = 2\pi \left\{ G(\alpha, 2\pi t) + \left(\frac{\partial}{\partial r} G^{1/2}(\alpha, 2\pi t) \right)^2 \right\}^{1/2}.$$

Assume the Gauss curvature at p satisfies $K(p) > 1$. Then there exists $r_0 > 0$ and $\lambda > 1$ such that $K(q) \geq \lambda > 1$ for all $q \in U$ with $r(q) < r_0$. A standard Sturmian argument, applied to (3) and to the equation $\eta'' + \lambda\eta = 0$ with initial conditions $\eta(0) = 0$, $\eta'(0) = 1$, implies that for all α satisfying $0 < \alpha \leq \min(r_0, \pi/2\lambda^{1/2})$ we have

$$||\xi'_\alpha(t)|| \leq 2\pi \left\{ \frac{\sin^2(\lambda^{1/2}\alpha)}{\lambda} + \cos^2(\lambda^{1/2}\alpha) \right\}^{1/2} < 2\pi.$$

Thus for all such α the length of ξ_α is less than 2π , which implies $L < 2\pi$. We conclude that $K \leq 1$ on all of M if $L = 2\pi$, and therefore $L^3/V \leq \pi$.

To complete the proof of Theorem 1 we calculate the curvature tensor of $S(M)$. For the moment relabel E_{12} , ω_{12} by E_3 , ω_3 respectively, and let (θ_{ij}) denote the unique skew-symmetric matrix of 1-forms on $S(M)$ satisfying

$$d\omega_i = \sum_{j=1}^3 \omega_j \wedge \theta_{ij},$$

where $j = 1, 2, 3$. One easily checks that

$$\begin{aligned} \theta_{12} &= (1 - K/2)\omega_3, \\ \theta_{23} &= (K/2)\omega_1, \\ \theta_{31} &= (K/2)\omega_2. \end{aligned} \tag{4}$$

The Riemann curvature tensor is given by

$$d\theta_{ij} - \sum_{k=1}^3 \theta_{ik} \wedge \theta_{kj} = -\frac{1}{2} \sum_{k,l=1}^3 R_{ijkl} \omega_k \wedge \omega_l$$

from which one obtains

$$\begin{aligned} R_{1212} &= K(1 - 3K/4), \\ R_{2123} &= -K_2/2, \\ R_{2113} &= -K_1/2, \\ R_{3123} &= 0, \\ R_{1313} &= R_{2323} \\ &= K^2/4, \end{aligned}$$

where $dK = K_1\omega_1 + K_2\omega_2$ (we are actually speaking of the lift of K to $S(M)$). One now checks that $S(M)$ has constant sectional curvature if and only if K is identically 1 or 0. Since M is a 2-sphere we have: $S(M)$ has constant sectional curvature if and only if K is identically 1 on M , in which case the sectional curvature of $S(M)$ is identically 1/4. Theorem 1 is therefore proven.

Now let I be an interval in the real numbers, $\xi: I \rightarrow S(M)$ a smooth path in $S(M)$, and let $\alpha_1, \alpha_2, \alpha_{12}: I \rightarrow \mathbb{R}$ be the coordinate functions of ξ' , *i.e.*, $\xi' = \alpha_1 E_1 + \alpha_2 E_2 + \alpha_{12} E_{12}$. Then from (4) one explicitly calculates that $\nabla_i \xi' = 0$ if and only if

$$\begin{pmatrix} \alpha'_1 \\ \alpha'_2 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_{12}(1 - K) \\ -\alpha_{12}(1 - K) & 0 \end{pmatrix} \begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}$$

$$\alpha'_{12} = 0.$$

Therefore ξ is a geodesic in $S(M)$ if and only if

$$\begin{pmatrix} \alpha_1(t) \\ \alpha_2(t) \end{pmatrix} = \begin{pmatrix} \cos \psi(t) & \sin \psi(t) \\ -\sin \psi(t) & \cos \psi(t) \end{pmatrix} \begin{pmatrix} \alpha_1(t_0) \\ \alpha_2(t_0) \end{pmatrix}$$

$$\alpha_{12}(t) = \text{const.}$$

where $t_0 \in I$ and

$$(5) \quad \psi(t) = \alpha_{12} \int_0^t (1 - K(\pi \circ \xi(\tau))) d\tau.$$

In particular if $\omega: I \rightarrow M$ is a geodesic in M , then by (1) the path $\xi = \omega'/||\omega'||: I \rightarrow S(M)$ is a geodesic in $S(M)$, and (2) implies that ξ has the same length as ω .

Let $\xi: I \rightarrow S(M)$ be a geodesic and $\omega = \pi \circ \xi$. Then $||\omega'(t)||^2 = \alpha_1^2(t) + \alpha_2^2(t) = \alpha_1^2(t_0) + \alpha_2^2(t_0)$ for all t . Also we can write

$$\begin{aligned} \omega'(t) &= (\alpha_1(t) + i\alpha_2(t))\xi(t) \\ &= \exp(-i\psi(t))(\alpha_1(t_0) + i\alpha_2(t_0))\xi(t) \\ &= \exp(-i(\psi(t) - \psi_0)) ||\omega'|| \xi(t), \end{aligned}$$

where $i = (-1)^{1/2}$ in the scalar expressions is interchangeable with i as the complex structure mapping ξ to $i\xi$. Therefore

$$\xi(t) = \exp(i(\psi(t) - \psi_0))\omega'(t)/||\omega'||$$

and

$$\begin{aligned} \alpha_{12}i\xi(t) &= \omega_{12}(\xi'(t))i\xi(t) \\ &= D_i\xi(t) \\ &= i\psi'(t)\xi(t) + \exp(i(\psi(t) - \psi_0))D_t(\omega'/||\omega'||)(t) \\ &= [\psi'(t) + ||\omega'|| \kappa_\sigma(\omega(t))]i\xi(t), \end{aligned}$$

where $\kappa_\sigma(\omega(t))$ is the geodesic curvature of ω at $\omega(t)$. From (5) one then concludes (cf. [12, p. 266]) that

$$0 = -\alpha_{12}K(\omega(t)) + \|\omega'\|_{\kappa_\sigma(\omega(t))}.$$

Therefore we have

Lemma 3. *If $\xi: I \rightarrow S(M)$ is a geodesic, $\omega = \pi \circ \xi$, then the geodesic curvature of ω is proportional to the Gauss curvature of M at ω . If $\langle \xi', E_{12} \rangle = 0$, then ω is a geodesic.*

We now prove Theorem 2. Assume $1 < K < 2$ on all of M ; then by Theorem 1, $L < 2\pi$ and if $\xi: [0, 1] \rightarrow S(M)$ is the shortest closed path in the non-trivial free homotopy class of continuous sectionally smooth paths in $S(M)$ then ξ is of course a geodesic. Since the speed of a geodesic is constant, we have $\|\xi'(t)\| = L$ for all t , which implies $|\alpha_{12}| \leq L < 2\pi$. Since ξ is closed we have for $\omega = \pi \circ \xi$

$$\begin{aligned} \psi(1) &= \alpha_{12} \int_0^1 (1 - K(\omega(t))) dt \\ &= 2\pi k \end{aligned}$$

for some integer k . From $|\alpha_{12}| < 2\pi$ and $0 < |1 - K| < 1$ for all t we obtain $k = 0$. But $1 - K(\omega(t)) < 0$ for all t . Therefore $\alpha_{12} = 0$. The projection of ξ onto M , $\omega = \pi \circ \xi$, is therefore a closed geodesic of length L . If ω is not simple, it is the union of two geodesic loops, each of which must have length $\geq 2^{1/2}\pi$ by [8, Theorem 1(b)]. Therefore $L \geq 2^{1/2}2\pi > 2\pi$ which is a contradiction. Thus ω is a simple closed geodesic of length L satisfying $2^{1/2}\pi < L < 2\pi$. Theorem 2 is proven.

Actually we have given a proof of the existence of at least one simple closed geodesic for any Riemannian metric on S_2 with pinching greater than $1/2$. For the most recent definitive results on closed geodesics on spheres we refer the reader to [9; 10].

We remark that by the Gauss-Bonnet theorem if $K \leq \kappa$, for some constant κ , on all of S_2 , then $L^3/V \leq \kappa\pi$. This can be slightly improved by using an argument of N. Grossman [7, p. 261] to obtain $L^3/V \leq \kappa^{1/2}\pi$ under the assumption: $K \leq \kappa$ on all of S_2 .

Appendix. By our remarks at the beginning of our proof of Theorem 1, one has that $\omega_1 \wedge \omega_2 \wedge \omega_{12}$ is the so-called *kinematic density* on $S(M)$. A quick proof of its invariance under the geodesic flow is obtained by noting that E_1 generates the geodesic flow and

$$\begin{aligned} \mathcal{L}_{E_1}(\omega_1 \wedge \omega_2 \wedge \omega_{12}) &= i(E_1) d(\omega_1 \wedge \omega_2 \wedge \omega_{12}) + di(E_1)(\omega_1 \wedge \omega_2 \wedge \omega_{12}) \\ &= d(\omega_2 \wedge \omega_{12}) \\ &= 0 \end{aligned}$$

by the Cartan structure equations. One can also prove Theorem 1 by directly checking that if Φ_τ denotes the geodesic flow and $\xi: [0, 1] \rightarrow S(M)$ is a fiber traversed once then the energy of $\Phi_\tau \circ \xi$ has power series expansion: $4\pi^2 + 4\pi^2\tau^2(1 - K) + \dots$.

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Date communicated: DECEMBER 29, 1972