Asymptotic Behavior of Solutions of Certain Elliptic Equations Near a Singular Manifold

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Communicated by P. R. GARABEDIAN

I. Introduction. We are concerned in this paper with the behavior as $t \rightarrow +\infty$ of solutions of elliptic equations of the form

$$(1) \quad \frac{\partial^2 u}{\partial t^2} + \frac{\mu}{t} \frac{\partial u}{\partial t} + t^{-2\alpha} \Delta u = k^2 u,$$

where Δ is the Laplace operator on a Riemannian manifold $\mathfrak{M}$ of dimension N . If R_+ is the half-line $t > 0$ and $\mathfrak{M}_1 = R_+ \times \mathfrak{M}$, with the metric

$$(ds_1)^2 = t^{2\alpha} (ds)^2 + (dt)^2,$$

where ds is the metric on $\mathfrak{M}$, the above equation takes the form

$$(2) \quad \Delta_1 u + \frac{c}{t} \frac{\partial u}{\partial t} = k^2 u,$$

where Δ_1 is the Laplacian on $\mathfrak{M}_1$ and $\mu = N\alpha + c$.

The manifold $\mathfrak{M}_\infty = \{+\infty\} \times \mathfrak{M}$ is a singular manifold for equation (2) if $k^2 \neq 0$. If the metric of $\mathfrak{M}$ is uniformly positive definite and if $\alpha \neq 0$, then (2) is not uniformly elliptic near $\mathfrak{M}_\infty$.

Thus our problem contrasts with those treated by Serrin and Myers (see [19] and [20]), where it is assumed that the equation is uniformly elliptic in the neighborhood of a singular point and that the coefficients of u and its first derivatives approach zero sufficiently rapidly near the singular point.

The case $\alpha = 1$, $c = 0$, $k^2 = -m^2 < 0$, $\mathfrak{M} = S_N$ (the N -sphere), was studied by Müller (see [15], [16]), who obtained results important for scattering theory. If $N = 2$, then

$$\lim_{r \rightarrow \infty} r^2 \int_r^{r+1} \int_{\mathfrak{M}} |u(t, x)|^2 dx dt$$

exists, and if it is finite, then as $t \rightarrow \infty$ we have essentially

$$u(t, x) \sim t^{-1} \{ e^{imt} f(x) + e^{-imt} f(-x) \}$$

where f is a function of class $\mathfrak{L}_2(\mathfrak{M})$.

In this paper we confine ourselves to the case $k^2 > 0$, which turns out to be simpler than the case investigated by Müller, since the expected factors $\exp(\pm kt)$ have different rates of growth. We should expect that our methods would yield results analogous to those of Müller for the case k^2 negative. We hope to return to this question in a future paper.

The critical values of α for the gross behavior of solutions turn out to be $\alpha = 0$ and $\alpha = 1$, with quite different behavior in the three intervals $\alpha < 0$, $0 < \alpha < 1$, and $\alpha > 1$. Preliminary computations indicate that changes in the finer structure of solutions take place as $1/(2\alpha)$ passes through positive integral values.

We shall confine ourselves here to the critical values $\alpha = 0$, and $\alpha = 1$, and shall leave the investigation of the other cases to future work.

Our problems fall within the general framework considered by Agmon and Nirenberg [1] and Lax [10], and also have relations to the work of Lions (see [12] and Lions and Magenes [13]). We are led to study Bessel functions of operators. In the case $\alpha = 0$ the order is scalar and the argument is an operator, while in the case $\alpha = 1$ both are operators. By using the general spectral theory developed in Hille and Phillips [9] and Dunford and Schwartz [5], we could carry out our analysis in a general Banach space, along lines similar to those of Agmon and Nirenberg. For the purpose of exhibiting the main features of the behavior it is sufficient for us to work with self-adjoint operators in Hilbert space, where the spectral theory is simpler and more completely known. For a general survey of the literature on operator differential equations see Nemytskii, Vainberg, and Gusarova [17].

II. Summary of results. After some remarks on the general problem, we focus our attention on the special cases $\alpha = 0$ and $\alpha = 1$, both of which can be reduced to the equation

$$(1) \quad v''(t) + t^{-1}v'(t) = (P^2 + t^{-2}Q^2)v(t),$$

where P and Q are commuting positive self-adjoint operators on a Hilbert space $\mathfrak{H}$, and v is a function to the half-line $t \geq a$ to $\mathfrak{H}$.

A fundamental system of solutions of (1) is constituted by $I_q(tP)$ and $K_q(tP)$, where I_q and K_q are the modified Bessel functions of order q . Roughly speaking, as $tp \rightarrow +\infty$ $I_q(tp)$ is large and $K_q(tp)$ is small, while as $q \rightarrow \infty$ the reverse is true. Hence for general Q and P we are concerned with the outcome of a "tug of war." In the problems which we treat, it turns out that as $t \rightarrow +\infty$ the former tendency wins out. This is illustrated by

Theorem 1. *If v satisfies the above equation for $t \geq t_0$ and $P \geq p_0 > 0$, $Q \geq 0$, then*

$$\lim_{t \rightarrow \infty} I_q(Pt)^{-1} I_q(Pa)v(t+c) = V(c, a)$$

exists for all $c \in R$, and

$$V(\lambda, a) = \exp (P\lambda)V(0, a)$$

for all λ , so that V is a solution of

$$\frac{\partial V}{\partial \lambda} = PV$$

such that

$$\|V(\lambda, a)\| \leq \exp (p_0\lambda) \|V(0, a)\| ,$$

for $\lambda \leq 0$. As a function of t , $V(\lambda, t)$ satisfies (1) for $t > 0$, and

$$\|v(t) - V(0, t)\| = O(\exp (-p_0t))$$

for $t \geq a$.

On every part of $\mathcal{H}$ where Q is bounded, the operator $I_Q(Pt)^{-1}$ behaves similarly to the simpler operator $\exp (-Pt)$. For the latter operator we have the following results:

Theorem 2. For all $c \in R$, the limit

$$A(c) = \lim_{t \rightarrow \infty} (2\pi t)^{1/2} \|\exp (-Pt)v(t+c)\|$$

exists. This limit is finite if and only if the limit

$$\lim_{t \rightarrow \infty} (2\pi t)^{1/2} \exp (-Pt)v(t+c) = W(c)$$

exists. If $W(\gamma)$ exists, then for any $c \in R$, $A(c)$ is finite if and only if $W(\gamma)$ is in the domain of $\exp (P(c-\gamma))$ and then

$$W(c) = \exp (P(c-\gamma))W(\gamma).$$

Corollary 2a. For all $c \in R$ and $\tau > 0$, we have

$$\lim_{t \rightarrow \infty} (2\pi t)^{1/2} P^{1/2} \exp (-P(\tau+t))I_Q(P\tau)v(t+\tau) = V(c-\tau, \tau),$$

where V is the function mentioned in Theorem 1. If $A(c)$ in Theorem 2 is finite, then

$$P^{-1/2}V(c, \tau) = I_Q(P\tau)W(c).$$

Corollary 2b. If Q is a bounded operator, then $A(c)$ is always finite, and therefore $W(c)$ always exists.

These results are vector analogues of certain theorems concerning the asymptotic behavior of integral transforms of scalar valued functions:

Theorem 4. If α is of bounded variation in $[c, \infty]$, $c \geq 0$, and

$$H(x) = \int_c^\infty I_\alpha(x) d\alpha(q),$$

then

$$\lim_{x \rightarrow \infty} (2\pi x)^{1/2} e^{-x} H(x) = \alpha(\infty) - \alpha(c).$$

Theorem 5. If α is of bounded variation in $[c, \infty]$, $c > 0$, and the integral

$$G(y) = \int_c^\infty \exp(-yq) q^a d\alpha(q)$$

converges for $R(y) > y_0$, and

$$F(x) = \int_c^\infty I_a(x)^{-1} d\alpha(q),$$

then

$$\lim_{x \rightarrow \infty} (2\pi x)^{-1/2} e^x F(x) = \alpha(\infty) - \alpha(c).$$

Theorem 4 is quite trivial, but Theorem 5 doesn't seem to be entirely so.

Let us apply these abstract results to some concrete problems. Suppose $\mathfrak{M}$ is either a closed C^∞ manifold or a compact C^∞ manifold with boundary, and suppose that L is an elliptic operator on $\mathfrak{M}$ with C^∞ coefficients which locally has the representation

$$(2) \quad Lf = \sum_{|\alpha| \leq n, |\beta| \leq n} X^\alpha a_{\alpha\beta}(x) X^\beta f,$$

where

$$X_i = i^{-1} \frac{\partial}{\partial x_i}, \quad X^\alpha = \prod_{i=1}^n X_i^{\alpha_i},$$

and that for some $c > 0$

$$(3) \quad (f, Lf)_{\mathfrak{M}} \geq c \sum_{|\alpha| \leq n} \|X^\alpha f\|^2$$

for all $f \in C_0^\infty(\mathfrak{M})$. (If $\mathfrak{M}$ is a compact manifold with boundary, then we take L to be the closure in $\mathcal{L}_2(\mathfrak{M})$ of the operator defined by (2) restricted to $C_0^\infty(\mathfrak{M})$.) Then we have

Theorem 6. If u is a solution in $\mathcal{L}_2(\mathfrak{M})$ of

$$(4) \quad u''(t) + (\mu/t)u'(t) = (k^2 + t^{-2}L)u$$

for $t > 0$, then the limit

$$A(0) = \lim_{t \rightarrow \infty} t^{\mu/2} e^{-kt} \|u(t)\|$$

exists. Furthermore, if $A(0)$ is finite, then

$$(5) \quad \lim_{t \rightarrow \infty} t^{\mu/2} e^{-kt} u(t) = f$$

exists in $\mathcal{L}_2(\mathfrak{M})$. If f is in the domain of L^k for some $k > 3N/(2n)$, then equation (5) also holds uniformly on $\mathfrak{M}$.

If L is of the second order, then a maximum principle holds for (4), and we can draw more precise conclusions.

Theorem 7. Suppose that u is a solution of (4) for $t > 0$ and that

$$\lim_{t \rightarrow 0} t^\sigma u(t) = 0,$$

uniformly on $\mathfrak{M}$, where $\sigma = m + (c^2 + m^2)^{1/2}$, c is the constant in (3), and $m = (\mu - 1)/2$. Suppose also that $g \in \mathcal{L}_2(\mathfrak{M})$ and that

$$\liminf_{t \rightarrow \infty} \max_{x \in \mathfrak{M}} \{(2\pi k)^{1/2} t^{\mu/2} e^{-kt} u(t, x) - g(x)\} \leq 0.$$

Then for $t > 0$, $x \in \mathfrak{M}$, the inequality

$$t^m u(t, x) \leq (I_Q(kt)g)(x) = v(t, x),$$

holds, where Q is the positive definite operator defined by $Q = (L^2 + m^2)^{1/2}$.

We note that for $t > 0$, v is the unique solution of (1) with $P = k$ such that $t^{\sigma-m} v(t) \rightarrow 0$ as $t \rightarrow 0$ and

$$(2\pi kt)^{1/2} e^{-kt} v(t) \rightarrow g \quad \text{as } t \rightarrow \infty.$$

This is an analogue for equation (4) of the Phragmén-Lindelöf theorem (see Ahlfors and Heins [2], Boas [3]).

If $\mathfrak{M}$ is the N -sphere S_N and $-L$ is the Laplacian on $\mathfrak{M}$, then (4) becomes

$$(6) \quad \Delta_1 u + cr^{-2} x \cdot \nabla u = k^2 u,$$

where Δ_1 is the ordinary Laplacian on E_{N+1} and $r = |x|$, and $\mu = N + c$. Let $M_2(r, u)^2$ be the integral of $|u|^2$ on the sphere of radius r . Theorem 6 reduces to

Theorem 8. If u is a solution of (6) for $r \geq a_1$, and if $c \neq 1 - N$, then

$$A = \lim_{r \rightarrow \infty} r^{c/2} e^{-kr} M_2(r, u)$$

exists. If A is finite, then

$$\lim_{r \rightarrow \infty} r^{(N+c)/2} e^{-kr} u(r\xi) = f(\xi)$$

exists in $\mathcal{L}_2(S_N)$.

(The operator Q is, in this case, positive definite except for the one value $c = 1 - N$.)

If $\mathfrak{M}$ is the compact manifold with boundary consisting of the hemisphere $S_N^+ = \{\xi \mid \xi \in S_N, \xi_{N+1} \geq 0\}$, then we obtain results for the solutions of (6) in the half-space $E_{N+1}^+ : x_{N+1} > 0$ which vanish on the boundary $x_{N+1} = 0$. In this case Q is positive definite with the lower bound $q_0 = (N^2 + m^2)^{1/2}$. We obtain

Theorem 10. Suppose that u is a solution of (6) in the half-space E_{N+1}^+ and continuous in $x_{N+1} \geq 0$ except possibly at the origin, and that $u(x) = 0$ for $x_{N+1} = 0$, $x \neq 0$. Let

$$M(r) = \max_{|x|=r} u(x),$$

and assume that

$$M(r) = o(r^{-\sigma}) \quad \text{as } r \rightarrow 0,$$

where $\sigma = m + q_0$, and that

$$\liminf_{r \rightarrow \infty} (2\pi k)^{1/2} r^{(N+c)/2} e^{-kr} M(r) \leq B < +\infty.$$

Then the inequality

$$r^m u(x) \leq Bv(x)$$

holds in E_{N+1}^+ , where v is the unique solution of (6) with $c = 1 - N$ for $r > 0$ such that

$$v(x) = o(r^{-(\sigma-m)}) \quad \text{as } r \rightarrow 0$$

and

$$\lim_{r \rightarrow \infty} (2\pi k r)^{1/2} e^{-kr} v(r\xi) = \operatorname{sgn} \xi_{N+1}$$

for $\xi \in S_N$. Applying this to u and to $-u$, we obtain the condition:

$$\max_{|x|=r} |u(x)| = -o(r^{-(N+c)/2} e^{kr}) \quad \text{as } r \rightarrow \infty$$

for the unique determination of the solution of (6) by its boundary values on $x_{N+1} = 0$.

Finally, if we take $\mathfrak{H} = E_N$ and $-L$ as the Laplacian on E_N , then equation (4) becomes

$$(7) \quad \frac{\partial^2 u}{\partial t^2} + \frac{\mu}{t} \frac{\partial u}{\partial t} + \Delta u = k^2 u,$$

and $t^m u$ satisfies (1) with $P = (k^2 - \Delta)^{1/2}$, $Q = |m|$, $m = (\mu - 1)/2$. The operator $\exp(-Pt)$ is then given by the formula

$$\exp(-Pt)f = H(t) * f,$$

where

$$H(t, x) = \frac{1}{2} \left(\frac{k}{2\pi} \right)^{\nu} \frac{t}{(t^2 + |x|^2)^{\nu/2}} K_{\nu}(k(t^2 + |x|^2)^{1/2}),$$

and $\nu = (N + 1)/2$. Theorem 2 implies

Theorem 11. If u is a solution of (7) in the half-space $t \geq a > 0$, and if the $\mathcal{L}_2$ norm $\|u(t)\|$ over E_N is bounded in every finite subinterval of $[a, \infty)$, then for all $c \in R$ the limit

$$A(c) = \lim_{t \rightarrow \infty} (2\pi)^{1/2} t^{\mu/2} \|H(t) * u(t + c)\|$$

exists and is finite. The limit

$$\lim_{t \rightarrow \infty} (2\pi)^{1/2} t^{\mu/2} H(t) * u(t + c) = U(c)$$

exists both in $\mathfrak{L}_2(E_N)$ and uniformly on compact sets. The function $U(t, x) = U(t)(x)$ is an entire solution of (7) with $\mu = 0$, and

$$\|U(t)\| = O(e^{kt}) \quad \text{for } t \leq 0.$$

Furthermore $t^{-\mu/2} \|U(t)\|$ is essentially of the same order of magnitude as $\|u(t)\|$ for $t \geq a$, with an error which is $O(t^{-m} e^{-kt})$ as $t \rightarrow +\infty$.

We use in our proofs the known uniform asymptotic representations of $I_q(x)$ and $K_q(x)$ which simultaneously exhibit the quite different behaviors as $q \rightarrow \infty$ and $x \rightarrow \infty$. We also use various well known properties of Bessel functions, as well as the inequality

$$I_q(t + a)I_q(c) \leq AI_q(t)I_q(c + a),$$

where

$$A \leq \exp(5/6q),$$

valid for $0 \leq a, 0 \leq c \leq t$, and $q > 0$.

It would be interesting to extend these results to other values of α , as well as to the case $k^2 < 0$, which would be important in scattering theory.

III. Preliminary remarks. We assume that $-\Delta$ is a non-negative self-adjoint operator on $\mathfrak{L}_2(\mathfrak{M})$. For example, $\mathfrak{M}$ may be a closed manifold, or it may be a compact smooth manifold with boundary. Then we may take $-\Delta$ to be the closure in $\mathfrak{L}_2(\mathfrak{M})$ of the restriction of the ordinary operator to the smooth functions with compact support.

If we think of $(I, 1)$ as an equation for a function of t with values in the Hilbert space $\mathfrak{H} = \mathfrak{L}_2(\mathfrak{M})$, then it takes the form

$$(1) \quad u''(t) + (\mu/t)u'(t) = (k^2 + t^{-2\alpha}L)u(t),$$

where L is a non-negative self-adjoint operator on $\mathfrak{H}$. If $\alpha = 0$, and we set $u(t) = t^{-m}v(t)$, $2m = \mu - 1$, $P = (k^2 + L)^{1/2}$, then we obtain

$$v''(t) + (1/t)v'(t) = (P^2 + t^{-2m^2})v(t),$$

so that both cases $\alpha = 0$ and $\alpha = 1$ are included in

$$u''(t) + (\mu/t)u'(t) = (P^2 + t^{-2}Q_1^2)u(t),$$

where P and Q_1 are commuting non-negative self-adjoint operators. The same substitution $u = t^{-m}v$, reduces this equation to

$$(2) \quad v''(t) + (1/t)v'(t) = (P^2 + t^{-2}Q^2)v(t),$$

where $Q = (Q_1^2 + m^2)^{1/2}$.

Let us consider briefly the general equation

$$(3) \quad v'' + t^{-1}v' = (P^2 + t^{-2\alpha}Q^2)v.$$

If P and Q are positive scalars, then the substitution

$$v(t) = w(\tau), \quad t = R\tau, \quad R = (Q/P)^{1/\alpha},$$

yields the equation

$$(4) \quad w'' + \tau^{-1}w' = S^2(1 + \tau^{-2\alpha})w,$$

where

$$S = PR = (P^{\alpha-1}Q)^{1/\alpha}.$$

Thus if $w = W_\alpha(S, \tau)$ is a solution of (4), then

$$v = W_\alpha(PR, R^{-1}t)$$

is a solution of (3).

If we make the substitution $\tau = r^\sigma$ in (4), and choose $\sigma = 1/(1 - \alpha)$, then we obtain

$$W_\alpha(p, \tau) = W_\beta(|\sigma| p, r),$$

where

$$\beta = 1 - \sigma = \alpha/(\alpha - 1).$$

This transformation interchanges the roles of P and Q in (3). It transforms the half-line $\alpha > 1$ into $\beta > 1$, and for such α 's relates $\tau \rightarrow +\infty$ to $r \rightarrow 0^+$ and $\tau \rightarrow 0^+$ to $r \rightarrow +\infty$. It maps the interval $0 < \alpha < 1$ onto the half-line $\beta < 0$, preserving the location of the singularities at 0 and ∞ . Hence, for the purposes of the general theory, it is sufficient to consider only the case $\alpha > 0$ and $t \rightarrow +\infty$.

By the general theory of equations of type (4) (see Erdélyi [6], Wasow [22]), we expect two solutions $w_\pm$ which, for large p or τ are represented approximately by

$$(5) \quad w_\pm \sim \tau^{-1/2} c(\tau)^{-1/2} \exp(\pm Sb(\tau)),$$

where

$$c(\tau) = (1 + \tau^{-2\alpha})^{1/2}$$

and

$$b(\tau) = \tau + \int_\tau^\infty (1 - c(s)) ds$$

if $2\alpha > 1$. (The formula for b is to be modified in an obvious way for $0 < \alpha \leq \frac{1}{2}$.) In the case $\alpha = 1$, the above equations are Bessel equations, and we can utilize the extensive information already available on their asymptotic behavior.

It is only necessary to work out the corresponding information regarding the asymptotic representation (5) of the solutions of (4). This would not involve any essentially new principles. Then one could apply the methods used below to obtain the desired results on the behavior of the solutions of (3) when P and Q are operators.

We shall for the present concentrate on the case $\alpha = 1$. (By the above remarks, the case $\alpha = 0$ can be transformed into this case.)

If $\alpha = 1$ and P and Q are scalars, then a fundamental system of solutions of (2) is constituted by

$$I_\nu(tP) \quad \text{and} \quad K_\nu(tP),$$

corresponding to w_+ and w_- in (5), where I_ν and K_ν are the modified Bessel functions of order ν . Hence the difference between $\alpha = 0$ and $\alpha = 1$ in the results obtained for equation (1) is explained by the difference in the asymptotic behavior of the Bessel functions as the argument or the order becomes large.

For general α , the corresponding phenomenon would be the way the behavior of the solutions depends on the boundedness or unboundedness of the operators R and S and their inverses.

IV. Application of spectral theory. We now focus attention on equation (III, 2), where P and Q are commuting self-adjoint positive definite operators on the Hilbert space $\mathcal{H}$, and v is a function on the ray $R_a : t \geq a$ to $\mathcal{H}$.

According to the spectral theorem (Segal and Kunze [18]), $\mathcal{H}$ may be represented as $\mathcal{L}_2(M)$, where M is a locally compact space with a regular measure m such that

$$\int_M f(\lambda) dm(\lambda) > 0$$

whenever $f \in \mathcal{C}_0(M)$, $f \geq 0$, $f \neq 0$, in such a way that P and Q are represented as multiplication by positive Baire functions p and q on M . Furthermore, to every Baire function F on R_+^2 , there corresponds a uniquely defined operator $F(P, Q)$, which is represented as multiplication by $F(p, q)$.

If $v(t)(\lambda) = v(t, \lambda)$ is the representative of v in $\mathcal{L}_2(M)$, then v is an ordinary scalar valued solution of (III, 2), with P and Q replaced by p and q . Hence

$$v(t) = A(t)v(a) + B(t)v'(a),$$

where A and B are the solutions of (2) determined by the initial conditions

$$A(a) = B'(a) = 1, \quad A'(a) = B(a) = 0.$$

A more useful form for our purposes is

$$(1) \quad v(t) = I_a(pt)C(a) - K_a(pt)D(a),$$

where

$$C(t) = (v(t) + K_a(pt)D(t))/I_a(pt)$$

and

$$D(t) = t(I_a(pt)v'(t) - pI'_a(pt)v(t)).$$

In fact, standard computations show that C and D are independent of t .

We now need to make use of certain uniform estimates for the Bessel functions. The information we need is summarized in the appendix, and we use the notations given there.

Let us examine the term $K_a(pt)D(a)$ for $t > a$. An easy computation yields

$$K_a(pt)I_a(pa) \leq k_a m_a e^{-p(t-a)}/(2p(ta)^{1/2}),$$

and

$$K_a(pt)paI'_a(pa) \leq k_a m_a e^{-p(t-a)}.$$

Thus if the operator P^{-1} is bounded, $P \geq p_0 > 0$, then the operators

$$K_a(Pt)I_a(Pa) \quad \text{and} \quad PaK_a(Pt)I'_a(Pa)$$

are bounded, with norms at most $C \exp(-p_0(t-a))$. Therefore we find that

$$\|v_2(t)\| \leq C \exp(-p_0(t-a))(\|v(a)\| + \|v'(a)\|).$$

We may infer that

$$\|I_a(Pt)C(a)\| \leq \|v(t)\| + O(\exp(-p_0(t-a))),$$

so that the element $C(a)$ is in the domain of the operator $I_a(Pt)$ for $t > a$. But the inequality

$$I_a(pt)^{-1}I_a(p\tau) \leq e^{-p(t-\tau)}(\tau/t)^a$$

for $0 \leq \tau \leq t$ shows that $C(a)$ is in the domain of $I_a(P\tau)$ for $\tau \leq t$, so that actually $C(a)$ is in the domain of $I_a(Pt)$ for all $t \geq 0$.

Since for non-negative p and q , $I_a(pt)$ is a non-decreasing function of t , then the norm $\|I_a(Pt)C(u)\|$ is non-decreasing.

Also since

$$e^{p(t-a)} \leq k_a^2 \frac{I_a(pt)}{I_a(pa)} \left(\frac{t}{a}\right)^{1/2}$$

for $0 < a < t$, we infer

$$(4) \quad (a/t)^{1/2} \|e^{p(t-a)}I_a(Pa)C(a)\| \leq k_a^2 \|v(t)\| + O(\exp(-p_0(t-a))),$$

so that $I_a(Pa)C(a)$ is in the domain of $\exp(Pc)$ for all $c > 0$.

Furthermore, in the estimate (see the appendix)

$$|I_a(pt)^{-1}I_a(p(t+c))I_a(pa)C(a)| \leq A_a |I_a(p(a+c))C(a),$$

the right-hand side is an element of $\mathfrak{L}_2(M)$, independent of t , so that the left-hand side approaches, in the $\mathfrak{L}_2$ sense, its pointwise limit, $\exp (pc)I_a(pa)C(a)$. Using equation (1) and the fact that

$$|I_a(pt)^{-1}I_a(at)K_a(p(t+c))D(a)| = O(e^{-p(2t+a-a)})$$

we obtain

Theorem 1. *If v satisfies equation (III, 2) for $t \geq a$, and $P \geq p_0 > 0$, $Q \geq 0$, then*

$$\lim_{t \rightarrow \infty} I_a(Pt)^{-1}I_a(Pa)v(t+c) = \exp (Pc)I_a(Pa)C(a)$$

for all $c \in R$.

The function

$$v(t, \tau, c) = I_a(P\tau)^{-1}I_a(Pt)v(\tau+c), \quad t \geq a, \quad \tau+c \geq a$$

has a simple interpretation. It is, as a function of t , the unique solution of (III, 2) in $0 < t < \tau$ such that

$$v(\tau, \tau, c) = v(\tau+c)$$

and

$$\lim_{t \rightarrow 0} t^Q v(t, \tau, c) = 0.$$

The function

$$\begin{aligned} v(t, \infty, \lambda) &= \exp (P\lambda)I_a(Pt)C(t) \\ &= \exp (P\lambda)I_a(Pt)C(a) \\ &= \exp (P\lambda)(v(t) + K_a(Pt)D(a)), \end{aligned}$$

is, as a function of t , also a solution of (III, 2) for all $t > 0$, and as a function of λ is a solution of

$$\frac{\partial v}{\partial \lambda} = Pv$$

for all $\lambda \in R$.

In many cases the operator $\exp (-Pt)$ is much simpler to work with than the operator $I_a(Pt)^{-1}$, which behaves similarly on any part of $\mathcal{H}$ where Q is bounded. For this operator we have the following result:

Theorem 2. *For every $c \in R$, the limit*

$$A(c) = \lim_{t \rightarrow \infty} (2\pi t)^{1/2} \|\exp (-Pt)v(t+c)\|$$

exists. This limit is finite if and only if $C(a)$ is in the domain of the operator $P^{-1/2} \exp (Pc)$, and then we have

$$\lim_{t \rightarrow \infty} (2\pi t)^{1/2} \exp (-Pt) v(t+c) = P^{-1/2} \exp (Pc) C(a).$$

Proof. Choose any A_1 such that

$$A_1 > A_0(c) = \liminf_{t \rightarrow \infty} (2\pi t)^{1/2} \|\exp (-Pt) v(t+c)\|,$$

and any $s > 0$. Then we have

$$(2\pi t)^{1/2} \exp (-pt) I_q(p(t+c)) |C(a)| \leq (2\pi t)^{1/2} \exp (-pt) |v(t+c)| \\ + O(t^{1/2} \exp (-p(t-a-c))),$$

so that for a sequence of t 's approaching $+\infty$, we obtain

$$(2\pi t)^{1/2} \|\exp (-Pt) I_q(P(t+c)) C(a)\| \leq A_1 + o(1).$$

For $q \leq s$ and $x \geq p_0 a$, there is a constant B such that

$$(2\pi x)^{1/2} I_q(x) \geq e^x (1 - B/x).$$

We infer that for $t+c \geq a$

$$(2\pi t) \|\exp (-Pt) I_q(P(t+c)) C(a)\|^2 \\ \geq \int_{q \leq s} p^{-1} \exp (2pc) (1 - B/(p(t+c))) |C(a)|^2 dm.$$

If we let $t \rightarrow \infty$ thru sequence for which the previous inequality occurs we obtain

$$\int_{q \leq s} p^{-1} \exp (2pc) |C(a)|^2 dm \leq A_1^2.$$

Now let $s \rightarrow \infty$, then let $A_1 \rightarrow A_0(c)$, and we conclude that $C(a)$ is in the domain of $P^{-1/2} \exp (Pc)$, and that

$$\|P^{-1/2} \exp (Pc) C(a)\| \leq A_0(c).$$

Consequently for any $t > a$, we have the estimates

$$(2\pi t)^{1/2} \exp (-Pt) v(t+c) = (2\pi t)^{1/2} \exp (-Pt) I_q(P(t+c)) C(a) \\ + O(t^{1/2} \exp (-p(2t+c-a))),$$

and

$$(2\pi t)^{1/2} \exp (-pt) I_q(p(t+c)) |C(a)| \leq k_q p^{-1/2} \exp (pc) |C(a)|,$$

the right-hand side of which is a fixed function in $\mathfrak{L}_2(M)$. Hence the integral of the square of the left-hand side approaches the integral of its pointwise limit as $t \rightarrow \infty$, and we obtain

$$\lim_{t \rightarrow \infty} \|(2\pi t)^{1/2} \exp(-Pt)v(t+c)\| = \|P^{-1/2} \exp(Pc)C(a)\| \leq A_0(c).$$

Therefore $A(c)$ exists and is finite if $A_0(c) < +\infty$. Furthermore, the difference

$$|(2\pi t)^{1/2} \exp(-pt)I_a(p(t+c)) - p^{-1/2} \exp(pc)| |C(a)|$$

is dominated by the fixed function

$$(k_a + 1)p^{-1/2} \exp(pc) |C(a)|$$

in $\mathfrak{L}_2(M)$, and approaches 0 pointwise as $t \rightarrow \infty$. Hence

$$\lim_{t \rightarrow \infty} (2\pi t)^{1/2} \exp(-Pt)I_a(P(t+c))C(a) = P^{-1/2} \exp(Pc)C(a),$$

and the conclusion of the theorem follows.

Corollary 2a. *If v is a solution of (III, 2) for $t \geq a$, then for all $c \in R$ and $\tau > 0$, we have*

$$\lim_{t \rightarrow \infty} (2\pi t)^{1/2} P^{1/2} \exp(-P(\tau+t))I_a(P\tau)v(t+c) = \exp(P(c-\tau))I_a(P\tau)C(a).$$

Proof. Since for some absolute constant B we have

$$(2\pi x)^{1/2} I_a(x)e^{-x} \leq B$$

for all $x \geq 0$, $q \geq 0$, the operator $P^{1/2} \exp(-P\tau)I_a(P\tau)$ is bounded, and has the bound $B/(2\pi\tau)^{1/2}$. Therefore

$$V(t) = P^{1/2} \exp(-P\tau)I_a(P\tau)v(t)$$

is also a solution of (III, 2) for $t \geq a$ and, in fact, $\|V(t)\| \leq B(2\pi\tau)^{-1/2} \|v(t)\|$. The function corresponding to $C(a)$ for V is

$$C_1(a) = p^{1/2} \exp(-P\tau)I_a(P\tau)C(a),$$

and this is in the domain of $P^{-1/2} \exp(Pc)$ since $I_a(P\tau)C(a) = I_a(P\tau)C(\tau)$ is in the domain of $\exp(P(c-\tau))$ for all $c \in R$. This was shown above in inequality (4) for $c \geq \tau$. But for $c < \tau$, $\exp(P(c-\tau))$ is a bounded operator, so that the conclusion is trivial. Therefore the function V satisfies the conditions of the theorem.

This result was suggested by Theorem 19, p. 100, in Müller [16].

Corollary 2b. *If Q is a bounded operator, the quantity $A(c)$ is finite, and therefore the conclusion always holds.*

We already know that $C(a)$ is in the domain of the operator $I_a(Pc)$. But the function $p^{-1/2} \exp(pc)I_a(pc)^{-1}$ is bounded by $(2\pi c)^{1/2}/k_a$, so the operator $P^{-1/2} \exp(Pc)I_a(Pc)^{-1}$ is bounded, so that $C(a)$ is in the domain of $P^{-1/2} \exp(Pc)$.

There is a related result for the solution $v(t) = f(t, \tau)h$ of the boundary value problem:

$$t^Q v(t) \rightarrow 0 \quad \text{as } t \rightarrow 0,$$

$$(5) \quad v(t) \text{ satisfies (III, 2) for } 0 < t < \tau,$$

$$v(\tau) = h,$$

where h is a given element of $\mathcal{H}$. The operator $f(t, \tau)$ is given explicitly by

$$f(t, \tau) = I_Q(P\tau)^{-1}I_Q(Pt).$$

By an argument similar to the above, we can prove

Theorem 3. *The element h is in the domain of $\exp(Pt)$ if and only if $f(t, \tau)h$ is in the domain of $\exp(P\tau)$ for all $\tau > t$ and*

$$\|\exp(P\tau)f(t, \tau)h\| = O(\tau^{1/2}),$$

and then

$$\lim_{\tau \rightarrow \infty} \tau^{-1/2} \exp(P\tau)f(t, \tau)h = (2\pi P)^{1/2} \exp(Pt)h.$$

Theorems 2 and 3 are analogues for Hilbert space to certain theorems of a classical type regarding integral transformations, which we formulate here only for the case where P is a positive scalar which we may, without loss of generality, take to be 1.

Theorem 4. *If $\alpha(q)$ is of bounded variation in $[c, +\infty]$, $c \geq 0$, and*

$$H(x) = \int_c^\infty I_q(x) d\alpha(q),$$

then

$$\lim_{x \rightarrow \infty} (2\pi x)^{1/2} e^{-x} H(x) = \alpha(\infty) - \alpha(c).$$

Proof. This is trivial since $(2\pi x)^{1/2} e^{-x} I_q(x)$ is bounded so that we may take the pointwise limit by Lebesgue's theorem.

Theorem 5. *If $\alpha(q)$ is of bounded variation in $[c, +\infty]$, $0 < c$, and the integral*

$$G(y) = \int_c^\infty e^{-yq} q^\sharp d\alpha(q)$$

converges for $\Re(y) > y_0$, and

$$F(x) = \int_c^\infty I_q(x)^{-1} d\alpha(q),$$

then

$$\lim_{x \rightarrow \infty} (2\pi x)^{-1/2} e^x F(x) = \alpha(+\infty) - \alpha(c).$$

Proof. By the inversion formula for Laplace transforms (see Widder [24]), we obtain

$$F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(k + iy) F(k + iy, x) dy$$

for $k > y_0$, $x > 2 \exp(k - 1)$ where

$$F(z, x) = \int_c^{\infty} I_a(x)^{-1} q^{-a} \exp(qz) dq.$$

The above formula is valid for $k > y_0$. We also note that $F(x)$ is analytic for

$$|x| > 2 \exp(y_0 - 1),$$

and that $F(z, x)$ is analytic for

$$|x| > 2 \exp(\Re(z) - 1).$$

If x is positive, then

$$|F(z, x)| \leq F(\Re(z), x),$$

and $F(k, x)$ is a decreasing function of x for $x > 0$, k real.

If $\Re(z) \leq k$, then we can estimate as follows:

$$(2\pi x)^{-1/2} |e^{x+az}| q^{-a} I_a(x)^{-1} \leq \left(k \gamma \left(\frac{x}{q} \right) \right)^{-1} \exp(q(k + \log(q^{-1} + x^{-1}))).$$

The right-hand side is a decreasing function of x , and is integrable, as a function of q , for $x \geq x_0$ if

$$k + \log(x_0^{-1}) < 0,$$

i.e.,

$$x_0 > e^k.$$

Hence we find that

$$(2\pi x)^{-1/2} e^x F(z, x) \rightarrow \int_c^{\infty} q^{-a} \exp(qz) dq$$

as $x \rightarrow \infty$, uniformly in every half-plane $\Re(z) \leq k$. Consequently we conclude that

$$\lim_{x \rightarrow \infty} (2\pi x)^{-1/2} e^x F(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} G(k + iy) \int_c^{\infty} q^{-a} \exp(q(k + iy)) dq dy$$

and this, again by the inversion formula for Laplace transforms is $\alpha(+\infty) - \alpha(c)$.

Each of Theorems 1 and 2 gives a condition on the growth of $v(t)$ as $t \rightarrow \infty$ which implies that $C(a) = 0$. But then we have

$$v(t) = -K_a(Pt)D(a),$$

which can also be expressed in the form

$$v(t) = K_a(Pa)^{-1} K_a(Pt)v(a) \quad \text{for } t > a.$$

We have already observed that $v(t) = O(\exp(-p_0(t-a)))$.

If we utilize the available information on the asymptotic behavior of $K_s(x)$, we can obtain more refined results, like Theorems 1 and 2, on the behavior of v . We leave this discussion for a future occasion.

V. Applications to partial differential equations. Consider equation (III, 1) with $\alpha = 1$, and L a positive definite operator on $\mathcal{H} = \mathcal{L}_2(\mathfrak{M})$. Then $u(t) = t^{-m}v(t)$, where v satisfies (2) with $P = k > 0$ and $Q = (L^2 + m^2)^{1/2}$, $m = (\mu-1)/2$. Theorem 2 yields

Theorem 6. *If u is a solution in $\mathcal{L}_2(\mathfrak{M})$ for $t \geq a$, then the limit*

$$A(0) = \lim_{t \rightarrow \infty} (2\pi)^{1/2} t^{\mu/2} e^{-kt} ||u(t)||$$

exists. Furthermore, if the limit is finite, then

$$\lim_{t \rightarrow \infty} (2\pi k)^{1/2} t^{\mu/2} e^{-kt} u(t) = C(a)$$

exists in the space $\mathcal{L}_2(\mathfrak{M})$.

This is the analogue for $k > 0$ to the theorem of Müller for k imaginary.

If L is a positive definite elliptic operator on $\mathfrak{M}$, i.e., if locally L has the representation

$$Lf = \sum_{|\alpha| \leq n, |\beta| \leq n} X^\alpha a_{\alpha\beta}(x) X^\beta f,$$

where

$$X_i = i^{-1} \frac{\partial}{\partial x_i}, \quad X^\alpha = \prod_{i=1}^N X_i^{\alpha_i},$$

and

$$(f, Lf)_{\mathfrak{M}} \geq C^2 \sum_{|\alpha| \leq n} ||X^\alpha f||^2$$

for all $f \in C^\infty(\mathfrak{M})$ with compact support, then we can apply known results (see Gårding [8]) on the asymptotic behavior of the spectral resolution of L . If we introduce the symbol of the principal part of L ,

$$L(x, \xi) = \sum_{|\alpha| = |\beta| = n} a_{\alpha\beta}(x) \xi^{\alpha+\beta},$$

then

$$L = \int_0^\infty q \, dE(q),$$

where

$$(E(q)f)(x) = \int_{\mathfrak{M}} E(q; x, y) f(y) \, dy,$$

and

$$E(q; x, y) = q^{N/n} \{ (2\pi)^N V(x) \delta(x, y) + O((\log q)^{-1}) \},$$

$$V(x) = \int_{L(x, \xi) \leq 1} d\xi,$$

and

$$\delta(x, y) = \begin{cases} 1, & x = y, \\ 0, & x \neq y. \end{cases}$$

Therefore we have the representation

$$(I_q(kt)f)(x) = \int_{\mathfrak{M}} K(kt; x, y) f(y) dy,$$

where

$$\begin{aligned} K(\tau; x, y) &= \int_c^\infty I_*(\tau) dE(q; x, y) \\ &= \int_0^\infty E(q; x, y) \frac{\partial}{\partial q} I_*(\tau) dq, \end{aligned}$$

with $\nu = \nu(q) = (q^2 + m^2)^{1/2}$. The integration by parts is justified and the integral converges uniformly since $E(q; x, y) = O(q^{N/n})$, while $I_*(\tau) = O(\exp(\nu \log(e\tau/2\nu)))$ and

$$\frac{\partial}{\partial \nu} I_*(\tau) = O((\log \nu) I_*(\tau)) \quad \text{as } \nu \rightarrow \infty.$$

Hence if the limit $A(0)$ in Theorem 6 is finite, then $f = C(a)$ is in $\mathfrak{L}_2(\mathfrak{M})$, and

$$\begin{aligned} t^m u(t)(x) &= \int_{\mathfrak{M}} K(kt; x, y) f(y) dy + O(e^{-kt}) \\ &= t^m u_0(t, x) + O(e^{-kt}). \end{aligned}$$

The function u_0 is the solution of equation (III, 1) which satisfies the boundary conditions at $t = 0$ and $t = +\infty$ analogous to (IV, 5):

$$(1a) \quad t^{m+q} u_0(t) \rightarrow 0 \quad \text{as } t \rightarrow 0$$

$$(1b) \quad (2\pi k)^{1/2} t^{\mu/2} e^{-kt} u_0(t) \rightarrow f \quad \text{as } t \rightarrow \infty.$$

Theorem 4 implies that (1) holds pointwise at every point x at which $(E(g)f)(x)$ is of bounded variation as a function of q , *i.e.*, where the Fourier series of f with respect to the eigen-functions of L is absolutely convergent. For example, if f is in the domain of L^k for some $k > 3N/(2n)$, then the Fourier series of f converges uniformly and then (1) above also holds uniformly. (We have not attempted to obtain the best possible results.)

If L is of the second order, so that a maximum principle holds for (III, 1), then we can obtain more precise information about u_0 . We assume that either $\mathfrak{M}$ is closed or that $\mathfrak{M}$ is compact with boundary, and that in the differential problems, we impose the condition that the solutions vanish on the boundary of $\mathfrak{M}$.

Theorem 7. *If u_0 is a solution of equation (III, 1) for $t > 0$ which satisfies condition (1a) above and*

$$(2) \quad \liminf_{t \rightarrow \infty} \max_{x \in \mathfrak{M}} \{(2\pi k)^{1/2} t^{\mu/2} e^{-kt} u_0(t, x) - g(x)\} \leq 0,$$

where $g \in \mathfrak{L}_2(\mathfrak{M})$, then

$$t^m u_0(t, x) \leq (I_{\mathbf{Q}}(kt)g)(x)$$

for $t > 0$, $x \in \mathfrak{M}$.

Proof. By hypothesis, there is a sequence $\{t_n\}$, $t_n \rightarrow +\infty$, such that for every $\epsilon > 0$ the inequality

$$(2\pi k)^{1/2} t^{\mu/2} e^{-kt} u_0(t, x) < g(x) + \epsilon$$

for $x \in \mathfrak{M}$ and $t = t_n$, n sufficiently large. By the maximum principle applied to (III, 1) on $[0, t_n] \times \mathfrak{M}$, we obtain

$$t^m u_0(t, x) \leq f(t, t_n) (2\pi k t_n)^{-1/2} e^{kt_n} (g + \epsilon),$$

for $0 < t < t_n$, where $f(t, \tau)$ is the operator $I_{\mathbf{Q}}(k\tau)^{-1} I_{\mathbf{Q}}(kt)$.

Now we could apply Theorem 5 to $f(t, \tau)(g + \epsilon)$, with

$$\alpha(\lambda) = \int_{\nu(\mathbf{Q}) \leq \lambda} I_{\nu(\mathbf{Q})}(kt) d(E(q)(g + \epsilon))(x),$$

but it is somewhat tedious, although not difficult, to show that α is of bounded variation in $[c, +\infty)$. It is perhaps simpler to apply to both sides of the above inequality the solution operator $\exp(-sL)$ for the heat equation

$$(3) \quad \frac{\partial v}{\partial s} = -Lv$$

on $\mathfrak{M}$. It is known that $\exp(-sL)$ is a positive operator (by the maximum principle) and has the representation

$$(\exp(-sL)h)(x) = \int_{\mathfrak{M}} G(s; x, y) h(y) dy,$$

where G is the Green's function for (3), and G is positive and

$$\int_{\mathfrak{M}} G(s; x, y) dy \leq 1$$

for $s > 0$, $x \in \mathfrak{M}$. Thus we obtain

$$t^m \exp(-sL)u_0(t, x) \leq \exp(-sL)(2\pi k\tau)^{-1/2} e^{k\tau} f(t, \tau)(g + \epsilon),$$

for $\tau = t_n > t$, $n > n_0$. Now we can apply Theorem 3, and observe that if $h_n \rightarrow h$ in the sense of $\mathfrak{L}_2(\mathfrak{M})$, then $\exp(-sL)h_n \rightarrow \exp(-sL)h$ uniformly on $\mathfrak{M}$. This yields

$$t^m \exp(-sL)u_0(t, x) \leq \exp(-sL)I_Q(kt)(g + \epsilon).$$

Now we let $\epsilon \rightarrow 0$, then let $s \rightarrow 0$, and arrive at the desired conclusion.

This result is an analogue for (III, 1) of the Ahlfors–Heins refinement of the Phragmén–Lindelöf theorem (see Ahlfors and Heins [2], Boas [3]).

Let us apply these results to the equation

$$(4) \quad \Delta_1 u + \frac{c}{r^2} \sum_{k=1}^{N+1} x_k \frac{\partial u}{\partial x_k} = k^2 u,$$

where Δ_1 is the ordinary Laplacian in E_{N+1} . This equation has the form (I, 2) with $t = r = |x|$, which is equivalent to (I, 1) with $t = r$, $\alpha = 1$, $\mu = N + c$, and $\Delta =$ the Laplacian on S_N , the unit N -sphere, or to (III, 1) with $\alpha = 1$ and $L = -\Delta$. Now the norm $\|u(r)\|$ is simply $r^{-N/2} M_2(r, u)$, where $M_2(r, u)^2$ is the integral of $|u|^2$ over the sphere of radius r . Theorem 6 implies

Theorem 8. *If u is a solution of (4) above for $r \geq a$, then*

$$A = \lim_{r \rightarrow \infty} (2\pi)^{1/2} r^{c/2} e^{-kr} M_2(r, u)$$

exists. If A is finite, then $r^{(N+c)/2} e^{-kr} u(r\xi)$ has a limit $f(\xi)$, as $r \rightarrow \infty$, in the sense of $\mathfrak{L}_2(S_N)$, and then

$$r^m u(r\xi) = (2\pi k)^{-1/2} (I_Q(kr)f)(\xi) + O(e^{-kr})$$

as $r \rightarrow \infty$, $\xi \in S_N$.

Corollary 8a. *If u is a solution of (4) for $r \geq a$, and $\rho > 0$, and U is defined by*

$$U(\rho, x) = (I_Q(k\rho)u(r))(\xi),$$

where $u(r)(\eta) = u(r\eta)$, $x = r\xi$, then

$$\lim_{r \rightarrow \infty} (2\pi k)^{1/2} r^{(N+c)/2} e^{-kr} U(\rho, r\xi) = v(\rho, \xi)$$

exists uniformly for $\xi \in S_N$. The function $u_0(r\xi) = r^m v(r, \xi)$ is a solution of (4) for $0 < r$ such that

$$u_0 = O(r^{m+|m|})$$

as $r \rightarrow 0$, and

$$u - u_0 = O(r^m e^{-kr})$$

as $r \rightarrow \infty$.

Proof. The fact that the limit defining v exists in the sense of $\mathfrak{L}_2(S_N)$ follows from Corollary 2a. To show the uniform convergence, take any $\sigma > \rho$. Then

$r^{(N+c)/2} e^{-kr} U(\sigma, r\xi)$ converges in the sense of $\mathfrak{L}_2(S_N)$ to $(2\pi k)^{-1/2} v(\sigma, \xi)$. We now observe, by the maximum principle that the operator $I_Q(k\sigma)^{-1} I_Q(k\rho)$ is a positive operator and is an integral operator on S_N with a certain kernel $K(\rho, \sigma; \xi, \eta)$ continuous for $(\xi, \eta) \in S_N \times S_N$. Hence $K(\rho, \sigma; \xi, \eta)$ is non-negative and integrable with respect to η over S_N , and consequently the integral transformation converts $\mathfrak{L}_2(S_N)$ convergence into uniform convergence. The function $v(\rho)$ is the function $I_Q(k\rho)C(a)$ in the representation (IV, 1) of $v(\rho) = \rho^m u(\rho)$.

We note that the eigenfunctions of L are the spherical harmonics, and the eigenvalue corresponding to the spherical harmonics of degree n is $\lambda_n = n(N + n - 1)$, the dimension of the eigenspace being

$$d(n, N) = \binom{2n + N - 1}{n + N - 1} \binom{n + N - 1}{n}.$$

The operator $I_Q(\tau)$ can be expressed in the form

$$(5) \quad (I_Q(\tau)f)(\xi) = \int_{S_N} K(\tau, \xi \cdot \eta) f(\eta) d\eta,$$

where

$$K(\tau, \lambda) = \sum I_Q(\tau) C_n^{(N-1)/2}(\lambda) a(n, N),$$

C_n^r being the Gegenbauer polynomial of degree n and order ν , $q = (\lambda_n^2 + m^2)^{1/2}$, and

$$\begin{aligned} a(n, N) &= d(n, N) / (\omega C_n^{(N-1)/2}(1)) \\ &= (n + (N - 1)/2) \Gamma((N - 1)/2) / (2\pi^{(N+1)/2}). \end{aligned}$$

(See Erdélyi et al. [7].) For further discussion of the most important case $c = 0$, see Sommerfeld [21], Magnus [14].

We note further that Q is positive definite if and only if $m \neq 0$, i.e., $c \neq 1 - N$. Hence the previous theory applies to equation (4) except for this one value of c . The simplest sufficient condition for (1a) above is that

$$(1a') \quad r^{m+|m|} u_0(x) \rightarrow 0 \quad \text{as } r \rightarrow 0.$$

Hence we obtain from Theorem 7 the following:

Theorem 9. *If u_0 is a solution of equation (4) which satisfies conditions (1a') and (2) above with $\mu = N + c = 2m + 1 \neq 0$, then*

$$r^m u_0(r\xi) \leq (I_Q(kr)g)(\xi)$$

for $r > 0$, $\xi \in S_N$.

The solutions of (4) in the half-space $E_{N+1}^+ : x_{N+1} > 0$, which vanish on the boundary $x_{N+1} = 0$ correspond to the case where $\mathfrak{M}$ is a compact manifold with boundary, the hemisphere S_N^+ . By the reflection principle we can obtain the desired information about this case from the above results applied to solutions

of (4) which are odd functions of x_{N+1} . In this case L is positive definite with the lower bound $\lambda_1 = N$, and Q is always positive definite with the lower bound $q_0 = (N^2 + m^2)^{1/2}$. Simplifying condition (1a) as before, and applying Theorem 9 to the simplest case, namely $g = \text{constant}$ on $\mathfrak{M}$, we obtain

Theorem 10. *Suppose u is a solution of (4) in the half-space $x_{N+1} > 0$ and is continuous in $x_{N+1} \geq 0$, $x \neq 0$, and $u(x) = 0$ for $x_{N+1} = 0$, $x \neq 0$, and let*

$$M(r) = M(r, u) = \max_{|x|=r} u(x).$$

If $\sigma = m + q_0$ and

$$M(r) = o(r^{-\sigma})$$

as $r \rightarrow 0$ and

$$\liminf_{r \rightarrow \infty} (2\pi k)^{1/2} r^{(N+c)/2} e^{-kr} M(r) \leq B < +\infty,$$

then the inequality

$$r^m u(x) \leq B(I_Q(kr)g_1)(\xi)$$

holds for $x = r\xi \in E_{N+1}^+$, where

$$g_1(\xi) = \text{sgn } \xi_{N+1}, \quad \xi \in S_N,$$

and $I_Q(kr)$ is defined by (5) above.

Corollary 10a. *Assume that u is as in the theorem. If*

$$M(r, |u|) = o(r^{-\sigma})$$

as $r \rightarrow 0$, and

$$M(r, |u|) = o(e^{kr} r^{-(N+c)/2})$$

as $r \rightarrow +\infty$, then $u \equiv 0$.

Therefore the growth conditions of the corollary are sufficient to insure uniqueness of the solution of (4) in the half-space with prescribed values on the hyperplane $x_{N+1} = 0$. Note that if $c \neq 0$, then this boundary hyperplane is also a singular manifold for the equation.

Now let us consider equation (I, 1) for the case $\alpha = 0$, $\mathfrak{M} = E_N$. This is closely related to the generalized axially symmetric potential theory studied by Weinstein and his associates (see [23]). This corresponds to equation (III, 2) for the case $P = (k^2 - \Delta)^{1/2}$, $Q = |m|$. We consider this equation in the space $\mathcal{H} = \mathcal{L}_2(E_N)$. An easy calculation with Fourier transforms yields the following formula for $\exp(-Pt)$:

$$\exp(-Pt)f = H(t) * f,$$

where

$$H(t, x) = \frac{1}{2} \left(\frac{k}{2\pi} \right)^\nu \frac{t}{(t^2 + |x|^2)^{\nu/2}} K_\nu(k(t^2 + |x|^2)^{1/2}),$$

and $\nu = (N + 1)/2$. (Note that H is an elementary function for even N .) Theorem 2 now yields

Theorem 11. *If u is a solution of*

$$(6) \quad \frac{\partial^2 u}{\partial t^2} + \frac{\mu}{t} \frac{\partial u}{\partial t} + \Delta u = k^2 u$$

in the half-space $t \geq a > 0$, and if the $\mathcal{L}_2$ norm $\|m(t)\|$ over E_N is bounded in every finite subinterval of $[a, \infty)$, then for all $c \in \mathbb{R}$ the limit

$$A(c) = \lim_{t \rightarrow \infty} (2\pi)^{1/2} t^{\mu/2} \|H(t) * u(t + c)\|$$

exists and is finite.

The limit

$$(7) \quad \lim_{t \rightarrow \infty} (2\pi)^{1/2} t^{\mu/2} H(t) * u(t + c) = U(c)$$

exists both in the sense of $\mathcal{L}_2(E_N)$ and uniformly on compact sets. The function $U(t, x)$ is an entire solution of

$$(8) \quad \frac{\partial^2 U}{\partial t^2} + \Delta U = k^2 U$$

in E_{N+1} . For $t \leq 0$, $\|U(t)\| = O(e^{kt})$, while for $t \geq a$, U satisfies

$$A_2 t^{\mu/2} \|u(t)\| - B_2 t^{1/2} e^{-kt} \leq \|U(t)\| \leq A_1 t^{\mu/2} \|u(t)\| + B_1 t^{1/2} e^{-kt},$$

where A_1 , A_2 , B_1 , and B_2 are positive constants.

We conclude with some remarks regarding the operator

$$f(t, \tau) = I_Q(P\tau)^{-1} I_Q(Pt)$$

in this case. This operator can be expressed as a convolution operator

$$f(t, \tau)g = H_Q(t, \tau) * g,$$

where

$$H_Q(t, \tau, x) = \frac{1}{(2\pi)^N} \int_{E_N} e^{i(x \cdot y)} \frac{I_Q(tp(y))}{I_Q(\tau p(y))} dy,$$

where $p(y) = (k^2 + |y|^2)^{1/2}$. An alternative formula is

$$H_Q(t, \tau, x) = (4\pi)^{(1/2)-\sigma} |x|^{1-\sigma} \int_0^\infty r^\sigma J_{\sigma-1}(r|x|) h(r) dr,$$

where $\sigma = N/2$, and

$$h(r) = I_0(\tau(k^2 + r^2)^{1/2})^{-1} I_0(t(k^2 + r^2)^{1/2}).$$

As a function of $x \in C^N$, H_Q is analytic for $|gx| < \tau - t$. For real x , H_Q is positive and satisfies for a constant $C(t, \tau)$ depending on t and τ the inequality

$$0 < H_Q(t, \tau, x) \leq C(t, \tau) \exp(-k\rho),$$

where

$$\rho = ((\tau - t)^2 + |x|^2)^{1/2}.$$

These properties of H_Q are proved in the appendix.

The function $v(t) = f(t, \tau)g$ is the unique solution of (III, 2) in the strip $0 < t < \tau$ such that

$$v(t) \in \mathcal{L}_2(E_N) \quad \text{for } 0 < t < \tau,$$

$$v(\tau) = g,$$

and

$$t^Q \|v(t)\| = o(1) \quad \text{as } t \rightarrow 0, \quad \text{if } Q > 0.$$

(If $Q = 0$, i.e., $\mu = 1$, then this condition is to be replaced by $\|v(t)\| = o(|\log t|)$ as $t \rightarrow 0$.) Then Theorem 1 implies

Theorem 12. *If u satisfies the conditions of Theorem 11, then for $t \geq a$ and all $c \in R$,*

$$\lim_{\tau \rightarrow \infty} \tau^m H_Q(t, \tau) * u(\tau + c) = V(c, t)$$

exists both in the sense of $\mathcal{L}_2(E_N)$ and uniformly on compact subsets. The function $V(s, t, x)$ satisfies the differential equations

$$(9) \quad \frac{\partial^2 V}{\partial t^2} + \frac{1}{t} \frac{\partial V}{\partial t} = \frac{\partial^2 V}{\partial s^2} = k^2 V - \Delta V$$

for $t > 0$ and all s . It also has the properties:

$$\|t^m u(t) - V(0, t)\| = O(e^{-kt}) \quad \text{for } t \geq a,$$

$$\|V(s, t)\| \leq e^{ks} (\|V(0, t)\| + O(e^{-kt})) \quad \text{for } s \leq 0, \quad t \geq a,$$

$$t^{1/2} \|V(s, t)\| \leq A(s + t)^{1/2} \|V(0, s + t)\| + O(e^{-ks}) \quad \text{for } s \geq 0, \quad s + t \geq a,$$

and $\|V(s, t)\|$ is an increasing function of s and t .

Thus the function V contains a good deal of information about the behavior of u as $t \rightarrow \infty$.

VI. Appendix. We give here, for the convenience of the reader, a brief summary of the properties of Bessel functions used in this paper. Almost all of these are known or are easy consequences of known results. We have found

most of these results in Erdélyi et al. [7], where full references to further literature are given.

For $x > 0$, $q > 0$, the function I_q can be represented by the formula

$$(2\pi x)^{1/2} I_q(x) = k_q Y_q(x/q),$$

where $k_q = k_q(x)$ satisfies

$$-1/12 \leq q \log k_q(x) \leq 1/3.$$

These explicit bounds can be obtained by the method given in Erdélyi [6]. For $0 \leq q \leq 1$, the factor $k_q(x)$ is bounded from above by a constant, which we have not evaluated explicitly. (For $q = 0$, the right hand side is to be interpreted as its limit

$$\lim_{q \rightarrow 0} Y_q(x/q) = e^x.)$$

In the above formula, we have

$$Y_q(x) = \gamma(x) \exp(qb(x/q)),$$

where

$$b'(x) = (1 + x^{-2})^{1/2},$$

$$b(x) = (x^2 + 1)^{1/2} + \log(x/(1 + (x^2 + 1)^{1/2})),$$

and

$$\gamma(x) = b'(x)^{-1/2} = x^{1/2}(1 + x^2)^{-1/4}.$$

We note the following elementary properties of $b(x)$:

$$b(x) - x - \log(x/(1 + x)) \text{ decreases from } \log(e/2) \text{ to } 0,$$

so that

$$0 < b(x) - x - \log(x/(1 + x)) < \log(e/2)$$

for $0 < x < +\infty$;

$$x - x^{-1}/2 < b(x) < x$$

for $0 < x < +\infty$; for $0 < x < y$, we have

$$b(y) - b(x) > y - x$$

and

$$b(y) - b(x) > \log(y/x).$$

Hence we also have for $0 < x < y$

$$e^{y-x} \leq (k_q(x)/k_q(y))(y/x)^{1/2} I_q(y)/I_q(x).$$

Since $I_q(x)$ and $I'_q(x)$ are positive for $x > 0$, and

$$I_{q-1}(x) - I_{q+1}(x) = (q/x)I'_q(x),$$

we have

$$I_{q+1}(x) \leq I_{q-1}(x)$$

for $q \geq 0$, $x \geq 0$. But then the formula

$$\int_0^\infty I_{2q}(x^{1/2}t) \exp(-t^2) dt = (\pi^{1/2}/2) \exp(x/8) I_q(x/8)$$

shows that

$$I_{q+1}(x) \leq I_q(x) \quad \text{for } x \geq 0,$$

so that

$$\begin{aligned} I'_q(x) &= I_{q+1}(x) + (q/x)I_q(x) \\ &\leq (1 + (q/x))I_q(x). \end{aligned}$$

Thus the previous estimates can be used for $I'_q(x)$.

The formula

$$\pi^{1/2} \Gamma(q + \frac{1}{2}) I_q(x) e^{-x} = (x/2)^q \int_{-1}^1 e^{-x(1+t)} (1-t^2)^{q-1/2} dt$$

shows that $I_q(x) e^{-x}$ is a decreasing function of x .

For $q \leq s$ and $x \geq a > 0$, there is a constant $C = C(a, s)$ such that

$$|(2\pi x)^{1/2} I_q(x) e^{-x} - 1| \leq C/x.$$

Similarly, the function K_q is represented by

$$(2x/\pi)^{1/2} K_q(x) = m_q \gamma(x/q) \exp(-qb(x/q)),$$

where $m_q = m_q(x)$ satisfies

$$q \log(1/4) - 1/3 \leq q \log m_q \leq 1/12.$$

For $0 \leq q \leq 1$, the factor m_q is bounded from above.

In the proof of Theorem 1 we need

Lemma. *If $0 \leq a$ and $0 \leq c \leq t$ and $q > 0$, then*

$$I_q(t+a)I_q(c) \leq AI_q(t)I_q(c+a),$$

where

$$\begin{aligned} A &= k_q(t+a)k_q(c)k_q(t)^{-1}k_q(c+a)^{-1} \\ &\leq \exp(5/6q). \end{aligned}$$

Proof. If $r = 1/q$, then we have

$$\frac{\gamma(r(t+a))\gamma(rc)}{\gamma(rt)\gamma(r(c+a))} = \left(\frac{c(t+a)}{t(c+a)} \right)^{1/2} \left(\frac{(1+r^2t^2)(1+r^2(c+a)^2)}{(1+r^2(t+a)^2)(1+r^2c^2)} \right)^{1/4},$$

but since

$$\frac{t^2 + (c + a)^2}{(t + a)^2 + c^2} \leq 1 \leq \frac{t^2(c + a)^2}{(t + a)^2 c^2},$$

it follows that

$$\frac{(1 + r^2 t^2)(1 + r^2(c + a)^2)}{(1 + r^2(t + a)^2)(1 + r^2 c^2)} \leq \frac{t^2(c + a)^2}{(t + a)^2 c^2},$$

and therefore

$$\gamma(r(t + a))\gamma(rc) \leq \gamma(rt)\gamma(r(c + a)).$$

Let

$$\varphi(r) = r^{-1}(b(r(t + a)) + b(rc) - b(rt) - b(r(c + a))).$$

Then when r is small, we have

$$\varphi(r) = r^{-1} \log \left(\frac{c(t + a)}{t(c + a)} \right) + O(r) \rightarrow -\infty,$$

and when r is large we have

$$\varphi(r) = -\frac{1}{2r^2} \left(\frac{1}{t + a} + \frac{1}{c} - \frac{1}{t} - \frac{1}{c + a} \right) < 0.$$

Also we have

$$\begin{aligned} \varphi'(r) &= r^{-2}[\psi(r(t + a)) + \psi(rc) - \psi(rt) - \psi(r(c + a))] \\ &= r^{-2} \int_0^{ra} \{ \psi'(rt + \lambda) - \psi'(rc + \lambda) \} d\lambda, \end{aligned}$$

where

$$\psi(x) = xb'(x) - b(x).$$

But since

$$\psi'(x) = xb''(x) = -1/x^2(1 + x^2)^{1/2}$$

is an increasing function, we see that $\varphi'(r) > 0$, and therefore $\varphi(r)$ is increasing and is negative for $r > 0$. The lemma now immediately follows.

Finally, we discuss briefly some properties of

$$(1) \quad H_{\mathbf{q}}(t, \tau, x) = (2\pi)^{-N} \int_{\mathbb{R}^N} \exp(i(x \cdot \xi)) F(\xi) d\xi,$$

where

$$F(\xi) = I_{\mathbf{q}}(\tau p(\xi))^{-1} I_{\mathbf{q}}(t p(\xi)),$$

and $p(\xi) = (k^2 + \sum \xi_j^2)^{1/2}$. Since

$$|F(\xi)| = O(\exp(-(\tau - t)r)),$$

it follows that H_Q is analytic for $|g(x)| < \tau - t$ and, in fact, that

$$|H_Q(t, \tau, x)| \leq C(t, \tau)(\tau - t - |g(x)|)^{-N}.$$

If $\zeta = \xi + i\eta$, where ξ and η are in E_N , then F is analytic for

$$|\eta|^2 < |\xi|^2 + k^2 + (\gamma_a^2/\tau^2),$$

where γ_a is the smallest positive zero of $J_a(x)$. Furthermore since

$$\Re(p(\zeta)) \geq (k^2 + |\xi|^2 - |\eta|^2)^{1/2} - C,$$

for a certain constant C , when $|\eta| \leq k$, and

$$|F(\zeta)| \leq (\tau/t)^{1/2} \exp(-(\tau - t)\Re p(\zeta))(1 + O(|p(\zeta)|^{-1})),$$

we find that F is analytic in the tube $|\eta| \leq k$, and is uniformly in $\mathfrak{L}_1(E^N) \cap \mathfrak{L}_2(E_N)$ with respect to ξ for $|\eta| \leq k$. Consequently, the integral (1) can be taken over any of the parallel spaces $E_N + i\eta$, $|\eta| \leq k$ (see Leray [11], Bochner and Martin [4]), and we obtain

$$(2\pi)^N |H_Q(t, \tau, x)| \leq (\tau/t)^{1/2} \exp(-x \cdot \eta + (\tau - t)C) \int_{E_N} \exp(-|\tau - t| \alpha(|\xi|)) d\xi,$$

where $\alpha(\xi) = (k^2 + |\xi|^2 - |\eta|^2)^{1/2} = (\rho^2 + |\xi|^2)^{1/2}$.

Now the integral

$$\int_{E_N} \exp(-\lambda \alpha(\xi)) d\xi = \omega_N \int_0^\infty \exp(-\lambda(\rho^2 + r^2)^{1/2}) r^{N-1} dr$$

can be expressed explicitly in terms of $K_{N/2}$, but it is sufficient for our purposes to use the standard estimate

$$\int_0^\infty e^{-s^t}(s^{k-1} + O(s^k)) = \Gamma(k)s^{-k}(1 + O(s^{-1})).$$

In the resulting estimate of H_Q , in which η may still be chosen freely in the sphere $|\eta| \leq k$, we set $\eta = \lambda x/|x|$, $\lambda = k|\xi|/((\tau - t)^2 + |\xi|^2)^{1/2}$. We then find that

$$|H_Q(t, \tau, x)| \leq C(\tau/t)^{1/2}(\tau - t)^{-N} \exp(C(\tau - t) - k((\tau - t)^2 + |x|^2)^{1/2}),$$

where C is a constant.

If g is a continuous non-negative function with compact support, then we can apply the maximum principle to the solution

$$v(t, x) = (H_Q(t, \tau) * g)(x)$$

of equation (8) with $\mu = 1$, and find that

$$0 < v(t, x) \leq \max g(y)$$

for $0 < t < \tau$ and all x . This implies that $H_Q(t, \tau, x)$ is positive.

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This research was partially supported by National Science Foundation Grant GP 29704.

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and

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Date communicated: DECEMBER 11, 1972