A Characterization of T_3 -Spaces

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A well known result in general topology is the characterization of $T_{3\frac{1}{2}}$ (i.e., completely regular Hausdorff) spaces as precisely those which can be embedded in compact Hausdorff spaces. In this paper, we establish a similar characterization for T_3 (i.e., regular Hausdorff) spaces. We then utilize this characterization to prove that if a continuous function has a T_3 range and a Wallman extension, then it must be a *WO* function.

By an extension of a topological space X we will mean a pair $\langle Y, \varphi \rangle$ where Y is a topological space and φ is a dense embedding of X in Y . Clearly the above characterization of $T_{3\frac{1}{2}}$ spaces can be rephrased: A space is $T_{3\frac{1}{2}}$ if and only if it has a compact Hausdorff extension. We will say that an extension $\langle Y, \varphi \rangle$ of a space X is relatively Hausdorff provided that for any point $x \in X$ and any point $y \in Y$ distinct from $\varphi(x)$, there exist disjoint open subsets U and V of Y containing $\varphi(x)$ and y respectively.

Proposition 1. *If a space X has a compact relatively Hausdorff extension, then X is T_3 .*

Proof. It is obvious that any space which has a relatively Hausdorff extension must be a Hausdorff space. Suppose now that $\langle Y, \varphi \rangle$ is a compact relatively Hausdorff extension of X , that A is a closed subset of X , and that $x \in X \sim A$. Since φ is an embedding, $\text{cl}_Y(\varphi[A])$ does not contain $\varphi(x)$. Therefore, since $\langle Y, \varphi \rangle$ is relatively Hausdorff, given any $y \in \text{cl}_Y(\varphi[A])$ there exist disjoint open subsets U_y, V_y of Y such that $y \in U_y$ and $\varphi(x) \in V_y$. Because $\text{cl}_Y(\varphi[A])$ is a closed subset of a compact space it is compact. $\{U_y : y \in \text{cl}_Y(\varphi[A])\}$ is an open cover of $\text{cl}_Y(\varphi[A])$, and so contains a finite subcover $\{U_{y_i} : i = 1, \dots, n\}$. Then $\bigcup_{i=1}^n U_{y_i}$ and $\bigcap_{i=1}^n V_{y_i}$ are disjoint open subsets of Y containing $\varphi[A]$ and $\varphi(x)$ respectively. Therefore $\varphi^{-1}[\bigcup_{i=1}^n U_{y_i}]$ and $\varphi^{-1}[\bigcap_{i=1}^n V_{y_i}]$ are disjoint open subsets of X containing A and x respectively. Hence X must be T_3 .

Recall (from [1], [3], or [4]) that for any T_1 space X , if $W(X)$ denotes the collection of all ultrafilters in the lattice of all closed subsets of X with topology generated by $\{C(A) = \{u \in W(X) : A \in u\} : A \text{ a closed subset of } X\}$ as a base

for the closed subsets, and if for each $x \in X$, $\varphi_x(x) = \{A: A \text{ a closed subset of } X \text{ and } x \in A\}$, then $\langle W(X), \varphi_x \rangle$ is a compact T_1 extension of X .

Proposition 2. *If X is a T_3 space, then $\langle W(X), \varphi_x \rangle$ is a compact relatively Hausdorff extension of X .*

Proof. Suppose u and $\varphi_x(x)$ are distinct points of $W(X)$. Then there exists an element $A \in u$ such that $x \notin A$; so, since X is T_3 , there exist disjoint open U, V in X such that $x \in U$ and $A \subseteq V$. Ultrafilters are union prime (that is, if F, G are closed subsets of X and $F \cup G \in v \in W(X)$, then $F \in v$ or $G \in v$) and X is an element of each element of $W(X)$; so, since $X \sim U$ and $X \sim V$ are closed subsets of X whose union is X , $W(X) = C(X \sim U) \cup C(X \sim V)$. A is disjoint from $X \sim V$, and hence $u \notin C(X \sim V)$. Because $x \in U$, $\varphi_x(x) \notin C(X \sim U)$. Therefore $W(X) \sim C(X \sim V)$ and $W(X) \sim C(X \sim U)$ are open subsets of $W(X)$ containing u and $\varphi_x(x)$ respectively. Furthermore,

$$\begin{aligned} (W(X) \sim C(X \sim U)) \cap (W(X) \sim C(X \sim V)) \\ = W(X) \sim (C(X \sim U) \cup C(X \sim V)) = W(X) \sim W(X) = \emptyset. \end{aligned}$$

Hence $\langle W(X), \varphi_x \rangle$ is relatively Hausdorff.

Clearly, combining Propositions 1 and 2 gives the following characterization.

Theorem 3. *A space is T_3 if and only if it has a compact relatively Hausdorff extension.*

If f is a continuous function from a T_1 space X to a T_1 space Y and $g: W(X) \rightarrow W(Y)$ is continuous, we will say g is a Wallman extension of f if $g \circ \varphi_x = \varphi_Y \circ f$. One consequence of the previous work is that if f is a continuous function into a T_3 space and has a Wallman extension, then the extension is unique, and f is, in fact, a WO function (as defined by Harris in [2]).

Lemma 4. *If $\langle Y, \varphi \rangle$ is a relatively Hausdorff extension of X and A is a compact subset of Y , then $\varphi^{-1}[A] = \varphi^{-1}[\text{cl}_Y(A)]$.*

Proof. Since $A \subseteq \text{cl}_Y(A)$, $\varphi^{-1}[A] \subseteq \varphi^{-1}[\text{cl}_Y(A)]$. If, however, $\varphi(x) \notin A$, then (as in the proof of Proposition 1) there exist disjoint open sets U and V in Y containing $\varphi(x)$ and A respectively. Hence $\varphi(x) \notin \text{cl}_Y(A)$.

In [2] Harris proved that any WO function has a unique Wallman extension. It is not difficult to show that a WO function is a continuous function $f: X \rightarrow Y$ between two T_1 spaces with the property that f has a Wallman extension $f^*: W(X) \rightarrow W(Y)$ such that for any $u \in W(X)$ and any closed set A of Y not in $f^*(u)$ there is an open subset V of X containing an element of u such that if B is any closed subset of X contained in V , then $\text{cl}_Y(f[B]) \cap A = \emptyset$.

Theorem 5. *If f is a continuous function from a T_1 space X into a T_3 space Y , and if f has a Wallman extension f^* , then f is a WO function.*

Proof. For any $u \in W(X)$ and A a closed subset of Y which is not in $f^*(u)$, since f^* is continuous $f^{*-1}[W(Y) \sim C(A)]$ is an open subset of $W(X)$ containing

u. Therefore there exists some closed subset B of X which is not in u and is such that $W(X) \sim C(B) \subseteq f^{*-1}[W(Y) \sim C(A)]$. If D is a closed subset of X disjoint from B , $C(D)$ is a closed subset of $W(X)$ disjoint from $C(B)$, and hence $f^*[C(D)]$ is a compact subset of $W(Y)$ contained in $W(Y) \sim C(A)$. Since f^* is a Wallman extension of f , $f[D] \subseteq \varphi_Y^{-1}[f^*[C(D)]]$, and since $\langle W(Y), \varphi_Y \rangle$ is relatively Hausdorff (Proposition 2) $\varphi_Y^{-1}[f^*[C(D)]]$ is closed by Lemma 4.

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