A Hilbert Space Approach to the Localization Problem of the Roots of Analytic Functions

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I. Introduction. The classical problem of the localization of the roots of polynomials of a complex variable has been examined by many authors. The first important works on the topic have been those of Cauchy and Gauss [1-2]. Research continued during the second half of the 19th century by Laguerre [3] and during our century by many mathematicians [4].

The study of the problem of the localization of the roots of any class of analytic functions obviously contains the above problem as a particular case.

In the present work we present a Hilbert Space approach to the localization problem of the roots of analytic functions that belong to the Hardy-Lebesgue class. The localization problem is reduced to a perturbation problem of a non-normal isometry V , perturbed with a compact operator C . The approach followed is based on the representation of the Hardy-Lebesgue space by means of the shift operator [5]. Part of the problem consists of finding conditions under which $r(T)$, the spectral radius of $T = V + C$, is greater than unity or equal to unity. In particular for a class of interesting polynomials it is proven that all the roots lie on the boundary of the unit disc if and only if $r(T) = 1$. Upper bounds for the spectral radius lead also to useful conclusions.

Attempts for finding conditions in order that $V + C$ and V are similar and consequently $r(T) = r(V) = 1$ have been made recently [6, 7] from a different point of view. The approach is useful because it checks these conditions and other theorems concerning perturbation of non-normal isometries.

With respect to the applications we restrict ourselves to the class of polynomials and derive different results which have been found classically by different authors.

The above results are obtained from one source, the spectral radius of an operator in Hilbert space.

In paragraph 2 the approach is described and some theorems are stated and proven. In paragraph 3 some applications are given.

II. The approach. We use the representation of the Hardy-Lebesgue space by means of the unilateral shift operator [5]. Denote by $\mathcal{H}$ an abstract separable Hilbert space over the complex field, by $\{e_n\}_1^\infty$ an orthonormal basis in $\mathcal{H}$ and by V the unilateral shift operator $V : Ve_n = e_{n+1}$, $n = 1, 2, \dots$.

Consider the eigenelements $f_z = \sum_{n=1}^\infty z^{n-1}e_n$ of V^* , the adjoint of V . The mapping

$$(1) \quad f(z) = (f_z, f), \quad f \in \mathcal{H},$$

is an isomorphism from $\mathcal{H}$ onto the Hardy-Lebesgue space $\mathcal{H}_2(D)$, consisting of the functions $f(z) = \sum_{n=1}^\infty \bar{\alpha}(n)z^{n-1}$ which are analytic in the unit disc D and satisfy the condition $\sum_{n=1}^\infty |\alpha(n)|^2 < \infty$.

Let $h = \sum_{n=1}^\infty c_n e_n$ be a fixed element of $\mathcal{H}$. Define the operator C as follows

$$(2) \quad Cf = (f, h)e_1, \quad f \in \mathcal{H},$$

and consider the operator $T^* = V^* + C^*$.

By the correspondence (1) we have

$$(3) \quad \begin{aligned} zf(z) &= (f_z, Vf), \\ f(z) - f(0) &= z(f_z, V^*f). \end{aligned}$$

From (2) we also have

$$(4) \quad (f_z, C^*f) = (Cf_z, f) = (f_z, h)(e_1, f) = f(0) \sum_{n=1}^\infty \bar{c}_n z^{n-1}.$$

Thus, by (1), (3) and (4) the eigenvalue problem

$$(5) \quad T^*f = \bar{\lambda}f$$

is equivalent [5] to the problem of finding the values of λ for which equation

$$(6) \quad f(z) - f(0) + f(0) \sum_{n=1}^\infty \bar{c}_n z^n = \lambda z f(z)$$

has solutions $f(z)$ in $\mathcal{H}_2(D)$.

We observe that if $f \neq 0$ is an eigenelement of T^* , then $(e_1, f) = f(0) \neq 0$. In fact if $(e_1, f) = 0$, then from (5) it follows that $(e_2, f) = (e_3, f) = \dots = 0$, i.e., $f = 0$.

From (6) we obtain

$$(7) \quad f(z) = f(0) \left(1 - \sum_{n=1}^\infty \bar{c}_n z^n \right) = -f(0)P(z)$$

for $\lambda = 0$ and

$$(8) \quad f(z) = f(0) \frac{1 - \sum_{n=1}^\infty \bar{c}_n z^n}{1 - \lambda z} = f(0) \frac{P(z)}{\lambda(z - 1/\lambda)}$$

for $\lambda \neq 0$, where

$$(9) \quad P(z) = \sum_{n=1}^{\infty} \bar{c}_n z^n - 1.$$

Obviously every function in $\mathcal{H}_2(D)$ can be normalized as the above function $P(z)$. From (8) we can easily derive the following theorems:

Theorem 1. *The function $P(z)$ has roots inside the unit disc if and only if T^* has eigenvalues $\bar{\lambda}$ such that $|\lambda| > 1$.*

Proof. If z_1 is a root of $P(z)$ inside the unit disc, then $\bar{\lambda} = 1/\bar{z}_1$ is an eigenvalue of T^* because from (8) $f(z)$ belongs to $\mathcal{H}_2(D)$. Conversely if $\bar{\lambda}$ is an eigenvalue of T^* with $|\lambda| > 1$, then $f(z) \notin \mathcal{H}_2(D)$ and this means (due to (8)) that $P(1/\lambda) = 0$.

Since C^* is a compact operator we shall use below the following theorems [8, 9]:

Theorem I. *For C^* compact the open unit disc $|z| < 1$ belongs to the point spectrum of T^* .*

In the present special case of C^* , Theorem I follows easily from (7) and (8).

Theorem II. *For C^* compact the spectrum of T^* outside the unit disc, if not empty, consists only of eigenvalues.*

This follows easily from the well known Weyl's theorem for non self-adjoint operators (9).

Using the algebraic definition of the spectral radius we obtain from Theorem I

$$(10) \quad r(T) = r(T^*) \geq 1.$$

Remark. The above relation can also be found easily as follows: We call $C^* \equiv A$. From (2) we have $Af = (f, e_1)h$. Hence

$$(V^* + A)^n e_n = h,$$

$$\|(V^* + A)^n e_n\| = \|h\| \leq \|(V^* + A)^n\|.$$

Hence

$$\|h\|^{1/n} \leq \|(V^* + A)^n\|^{1/n}$$

and for $n \rightarrow \infty$

$$(10) \quad 1 \leq r(T^*) = r(T).$$

This is a characteristic relation of the Hilbert space formalism.

Corollary 1. *A lower bound for the roots ρ of $P(z)$ can be obtained from the relation*

$$|\rho| \geq \frac{1}{r(T^*)} \geq \frac{1}{\|T^*\|}$$

or since

$$(11) \quad \begin{aligned} ||T^*|| &= (1 + ||h||^2)^{1/2}, \\ |\rho| &\geq \frac{1}{r(T^*)} \geq \frac{1}{(1 + ||h||^2)^{1/2}}. \end{aligned}$$

Proof. If ρ is a root of $P(z)$ with $|\rho| < 1$, then due to Theorem 1 $1/\bar{\rho}$ is an eigenvalue of T^* , i.e., $1/\bar{\rho}$ belongs to the spectrum of T^* . Hence

$$\left| \frac{1}{\rho} \right| \leq r(T^*).$$

If ρ is a root of $P(z)$ with $|\rho| \geq 1$, then $1/|\rho| \leq 1$ and by (10) $1/|\rho| \leq r(T^*) = r(T)$.

Remark. In the case of polynomials we can use the fact that the inverse numbers of lower bounds of the absolute values of the non-zero roots of $P(z)$ are upper bounds of the absolute values of the roots of $P_1(W) = P(1/W)$, $z = 1/W$.

The following theorems follow immediately from Theorem II and the fact that $\bar{\lambda}$: $|\lambda| > 1$ is an eigenvalue of T^* if and only if $1/\lambda$ is a root of $P(z)$ (Theorem 1).

Theorem 2. *The function $P(z)$ has at least one root inside the unit disc if and only if $r(T) > 1$.*

Theorem 3. *The function $P(z)$ has no roots inside the unit disc if and only if $r(T) = 1$.*

Corollary 2. *If T and V are similar, then $P(z)$ has no roots inside the unit disc.*

In problems of statistical mechanics it is interesting to know conditions under which all the roots of a polynomial lie on the unit disc [10, 11]. We formulate within the framework of Hilbert space theory a necessary and sufficient condition in order that a polynomial has all the roots on the unit circle.

Consider the two polynomials

$$(12) \quad P(z) = a_0 + a_1 z + \cdots + a_n z^n, \quad a_0 \neq 0, a_n \neq 0,$$

$$(13) \quad R(z) = a_n + a_{n-1} z + \cdots + a_0 z^n,$$

so that the roots of $R(z)$ are the inverse roots of $P(z)$.

In the case of the above polynomials the operator C is defined as follows:

$$C_1 f = (f, h_1) e_1, \quad f \in \mathcal{H},$$

$$C_2 f = (f, h_2) e_1, \quad f \in \mathcal{H},$$

where

$$h_1 = \frac{\bar{a}_1}{\bar{a}_0} e_1 + \frac{\bar{a}_2}{\bar{a}_0} e_2 + \cdots + \frac{\bar{a}_n}{\bar{a}_0} e_n ,$$

$$h_2 = \frac{\bar{a}_{n-1}}{\bar{a}_n} e_1 + \frac{\bar{a}_{n-2}}{\bar{a}_n} e_2 + \cdots + \frac{\bar{a}_0}{\bar{a}_n} e_n .$$

According to Theorem 3, $r(V - C_1) = 1$ is a necessary and sufficient condition in order that $P(z)$ has no roots inside or in order that $R(z)$ has no roots outside the unit disc. Thus, we obtain:

Theorem 4. *The polynomial $P(z)$ has all the roots on the unit disc if and only if*

$$r(V - C_1) = r(V - C_2) = 1.$$

For symmetric polynomials of the form

$$(14) \quad \begin{aligned} P(z) &= 1 + P_1 z + \cdots + P_{n-1} z^{n-1} + z^n , \\ P_{n-k} &= P_k , \quad k = 1, \cdots, n \end{aligned}$$

we have $h_1 = h_2$, and consequently

$$T_1 = T_2 = V - C = T_0 .$$

Thus we obtain:

Theorem 5. *The symmetric polynomials (12) have all the roots on the unit circle if and only if $r(T_0) = 1$.*

III. Applications. For the applications we restrict ourselves to polynomials.

(a) **Bound obtained from the norm. The Carmichael-Mason inequality.** Consider the polynomial

$$(15) \quad P(z) = -1 + \bar{c}_1 z + \bar{c}_2 z^2 + \cdots + \bar{c}_m z^m, \quad c_m \neq 0.$$

If we put $z = 1/W$ we have

$$(16) \quad \bar{c}_m + \bar{c}_{m-1} W + \bar{c}_{m-2} W^2 + \cdots + \bar{c}_1 W^{m-1} - W^m = 0.$$

The above has the same roots as the polynomial

$$(17) \quad -1 - \frac{\bar{c}_{m-1}}{\bar{c}_m} W - \frac{\bar{c}_{m-2}}{\bar{c}_m} W^2 \cdots - \frac{\bar{c}_1}{\bar{c}_m} W^{m-1} + \frac{W^m}{\bar{c}_m} = 0.$$

From Corollary 1 we get for the roots of (17)

$$|W| \geq \left(1 + \frac{1}{|c_m|^2} + \frac{|c_1|^2}{|c_m|^2} + \cdots + \frac{|c_{m-1}|^2}{|c_m|^2} \right)^{-1/2}$$

or

$$(18) \quad |z| \leq \frac{(1 + |c_1|^2 + \cdots + |c_{m-1}|^2 + |c_m|^2)^{1/2}}{|c_m|}.$$

On the other hand

$$(19) \quad |z| \geq (1 + |c_1|^2 + |c_2|^2 + \cdots + |c_m|^2)^{-1/2}.$$

Condition (18) with a different normalization for the polynomial is the same as the one found classically by Carmichael and Mason [12]. Condition (18) which has been easily derived here from the norm of $T^* = V^* + A$ is sometimes stronger from some conditions found by classical methods by several authors. For example Montel [13] has found that given the polynomial

$$(20) \quad \phi(x) = a_0 x^m + a_1 x^{m-1} + a_2 x^{m-2} + \cdots + a_\lambda x^{m-\lambda}$$

the following relation holds for its roots

$$|\rho| \leq \max \left\{ 1, \sum_{i=1}^{\lambda} |a_i| \right\}.$$

In our case the above criterion becomes for (15)

$$(21) \quad |\rho| \leq 1 + |c_1| + \cdots + |c_{m-2}| + |c_{m-1}|.$$

It is obvious that criterion (18) is stronger than (21) if $|c_m|$ is sufficiently large.

(b) **Bounds obtained from the spectral radius.**

1. Derivation of some general formulae.

Proposition 1. *If $h = \sum_{i=1}^m c_i e_i$, then for every normalized element $f = \sum_{i=1}^{\infty} \lambda_i e_i$ in $\mathcal{H}$ we have*

$$(22) \quad (V^* + A)^n f = \sum_{i=1}^m \alpha_n^{(i)} e_i + \sum_{k=1}^{\infty} \lambda_{n+m+k} e_{m+k},$$

where A is the adjoint of the operator (2),

$$(23a) \quad \alpha_n^{(1)} = \sum_{i=1}^m c_i \alpha_{n-i}^{(1)} + \lambda_{n+1},$$

$$(23b) \quad \alpha_n^{(2)} = \sum_{i=1}^{m-1} c_{i+1} \alpha_{n-i}^{(1)} + \lambda_{n+2},$$

$$(23c) \quad \alpha_n^{(m)} = c_m \alpha_{n-1}^{(1)} + \lambda_{n+m}$$

and

$$(24) \quad \begin{aligned} \alpha_0^{(1)} &= \lambda_1, & \alpha_1^{(1)} &= \lambda_1 c_1 + \lambda_2, & \alpha_2^{(1)} &= (\lambda_1 c_1 + \lambda_2) c_1 + \lambda_1 c_2 + \lambda_3, \\ \alpha_3^{(1)} &= c_1 \alpha_2^{(1)} + c_2 \alpha_1^{(1)} + c_3 \alpha_0^{(1)} + \lambda_4, & & \cdots \end{aligned}$$

Proof. From (22) we get

$$\begin{aligned}
\alpha_n^{(1)} &= [(V^* + A)^n f, e_1] = [(V^* + A)^{n-1} f, (V + A^*)e_1] = [(V^* + A)^{n-1} f, e_2] \\
&\quad + [(V^* + A)^{n-1} f, \bar{c}_1 e_1] = c_1 \alpha_{n-1}^{(1)} + [(V^* + A)^{n-2} f, (V + A^*)e_2] \\
&= c_1 \alpha_{n-1}^{(1)} + c_2 \alpha_{n-2}^{(1)} + [(V^* + A)^{n-2} f, e_3] = \dots \\
&= c_1 \alpha_{n-1}^{(1)} + c_2 \alpha_{n-2}^{(1)} + \dots + c_m \alpha_{n-m}^{(1)} + [(V^* + A)^{n-m} f, e_{m+1}] \\
&= \sum_{i=1}^m c_i \alpha_{n-i}^{(1)} + [(V^* + A)^{n-m-1} f, e_{m+2}] + [(V^* + A)^{n-m-1} f, A^* e_{m+1}] \\
&= \sum_{2 \leq i}^m c_i \alpha_{n-i}^{(1)} + (f, e_{n+1}) = \sum_{2 \leq i}^m c_i \alpha_{n-i}^{(1)} + \lambda_{n+1} .
\end{aligned}$$

The proof for (23b), (23c) is similar.

Proposition 2. *The coefficients $\alpha_n^{(i)}$ satisfy the following inequalities:*

$$\begin{aligned}
(25a) \quad & |\alpha_1^{(1)}| \leq 1 + |c_1|, \\
(25b) \quad & |\alpha_2^{(1)}| \leq 1 + |c_1| + |c_2| + |c_1|^2, \\
(25c) \quad & |\alpha_3^{(1)}| \leq 1 + |c_1| + |c_2| + |c_3| + 2|c_1| \cdot |c_2| + |c_1|^2 + |c_1|^3, \\
& \dots \dots \dots \\
(25d) \quad & |\alpha_n^{(1)}| \leq \sum_{i=1}^m |c_i| \cdot |\alpha_{n-i}^{(1)}| + 1, \\
& \dots \dots \dots \\
(25e) \quad & |\alpha_n^{(2)}| \leq \sum_{i=1}^{m-1} |c_{i+1}| \cdot |\alpha_{n-i}^{(1)}| + 1, \\
& \dots \dots \dots \\
(25f) \quad & |\alpha_n^{(m)}| \leq |c_m| |\alpha_{n-1}^{(1)}| + 1.
\end{aligned}$$

The proof is obvious in virtue of (23), (24).

By the above proposition we can now find by complete induction different upper bounds for $|\alpha_n^{(i)}|$, $i = 1, 2, \dots, m$, and therefore for $\| (V^* + A)^n f \|$ and draw conclusions for the spectral radius $r(T^*)$ which in virtue of Corollary 1 lead to upper bounds for all the roots of the polynomial (16) or to lower bounds for all the roots of (15).

In order to illustrate the method we derive here two relations found classically by Montel as well as the well known relations of Cauchy, Walsh and Parodi.

2. The Montel inequalities.

(a) Call $R = |c_1| + |c_2| + \cdots + |c_m|$. From (25) we can easily obtain:

$$\begin{aligned} (27a) \quad & |\alpha_1^{(1)}| \leq 1 + R, \\ (27b) \quad & |\alpha_2^{(1)}| \leq 1 + R + R^2, \\ (27c) \quad & |\alpha_3^{(1)}| \leq 1 + R + R^2 + R^3 \end{aligned}$$

and by complete induction we obtain from (25d)

$$(27d) \quad |\alpha_n^{(1)}| \leq 1 + R + R^2 + \cdots + R^n.$$

Hence from (25) and (27)

$$(28) \quad |\alpha_n^{(1)}| \leq \frac{R^{n+1} - 1}{R - 1}, \quad |\alpha_n^{(2)}| \leq |c_2| \frac{R^n - 1}{R - 1} + |c_3| \frac{R^{n-1} - 1}{R - 1} + \cdots \\ + |c_m| \frac{R^{n-m+2} - 1}{R - 1} + 1, \cdots, |\alpha_n^{(m)}| \leq |c_m| \frac{R^n - 1}{R - 1} + 1.$$

From (22) we obtain

$$(29) \quad \|(V^* + A)^n f\| = \left\{ \sum_{i=1}^m |\alpha_n^{(i)}|^2 + \sum_{i=n+m+1}^{\infty} |\lambda_i|^2 \right\}^{1/2}.$$

If $R > 1$, then from (29) and (28) we obtain, after some algebra: $r(T^*) = \lim_{n \rightarrow \infty} \|(V^* + A)^n\|^{1/n} \leq R$. If $R < 1$, then from (28) and (29): $\lim_{n \rightarrow \infty} \|(V^* + A)^n\|^{1/n} \leq 1$ and in virtue of (10) $r(T^*) = 1$.

Finally, if $R = 1$ from (29) and (27) $r(T^*) \leq 1$ and in virtue of (10) $r(T^*) = 1$. Hence from Corollary 1 $|1/\rho| \leq r(T^*) \leq \max \{1, R\}$, or for the roots of polynomial (16)

$$(30) \quad |W| \leq \max \left\{ 1, \sum_{i=1}^m |c_i| \right\}.$$

This is the well known criterion found by Montel [13].

(b) Consider the numbers $|c_1|, (|c_2|)^{1/2}, (|c_3|)^{1/3}, \cdots, (|c_m|)^{1/m}$ and let $(|c_i|)^{1/i} = \max_{i=1,2,\dots,m} |c_i|^{1/i}$. From (25) we obviously get:

$$(31a) \quad |\alpha_1^{(1)}| \leq 1 + (|c_i|)^{1/i},$$

$$(31b) \quad |\alpha_2^{(1)}| \leq 2(1 + (|c_i|)^{1/i} + (|c_i|^2)^{1/i}),$$

$$(31c) \quad |\alpha_3^{(1)}| \leq 4(1 + (|c_i|)^{1/i} + (|c_i|^2)^{1/i} + (|c_i|^3)^{1/i}),$$

and by complete induction

$$(31d) \quad |\alpha_n^{(1)}| \leq 2^{n-1}(1 + (|c_i|)^{1/i} + \cdots + (|c_i|^n)^{1/i}) = 2^{n-1} \frac{(|c_i|^{n+1})^{1/i} - 1}{(|c_i|)^{1/i} - 1}.$$

Finally

$$(32) \quad |\alpha_n^{(2)}| \leq |c_2| |\alpha_{n-1}^{(1)}| + |c_3| \cdot |\alpha_{n-2}^{(1)}| + \cdots + |c_m| \cdot |\alpha_{n-m+1}^{(1)}| + 1 \\ \dots\dots\dots$$

From (29), (31), (32) we find that:

(i) If $(|c_i|)^{1/i} \leq 1$, then $r(T^*) \leq 2$.

(ii) If $(|c_i|)^{1/i} > 1$, then $r(T^*) \leq 2 (|c_i|)^{1/i}$.

Hence if $(|c_i|)^{1/i} > 1$ all the roots of (16) satisfy, by Corollary 1, the relation

$$|W| \leq 2 (|c_i|)^{1/i}.$$

If $(|c_i|)^{1/i} \leq 1$ we divide both members of (16) by $(|c_i|^m)^{1/i}$ and obtain

$$(33) \quad \frac{\bar{c}_m}{(|c_i|^m)^{1/i}} + \frac{\bar{c}_{m-1}}{(|c_i|^{m-1})^{1/i}} \cdot \left(\frac{W}{(|c_i|)^{1/i}} \right) + \frac{\bar{c}_{m-2}}{(|c_i|^{m-2})^{1/i}} \cdot \left(\frac{W}{(|c_i|)^{1/i}} \right)^2 + \dots - \left(\frac{W}{(|c_i|)^{1/i}} \right)^m = 0.$$

From (33) we get

$$\max \left(\frac{|c_i|}{(|c_i|^i)^{1/i}} \right)^{1/i} = \frac{\max (|c_i|)^{1/i}}{(|c_i|)^{1/i}} = \frac{(|c_i|)^{1/i}}{(|c_i|)^{1/i}} = 1.$$

Hence $(|c_i|/(|c_i|^i)^{1/i})^{1/i} \leq 1$ and therefore $r(T^*) \leq 2$ or $|W|/(|c_i|)^{1/i} \leq 2$. Thus, $|W| \leq 2 (|c_i|)^{1/i}$. Hence we have

$$(34) \quad |W| \leq 2 \max_{i=1,2,\dots,m} |c_i|^{1/i},$$

which is the well known bound found by Montel [14].

3. The Cauchy inequality. Let $|c| = \max_{i=1,2,\dots,m} |c_i|$. Obviously from (25) we obtain:

$$(35a) \quad |\alpha_1^{(1)}| \leq 1 + |c|,$$

$$(35b) \quad |\alpha_2^{(1)}| \leq (1 + |c|)^2, \\ \dots\dots\dots$$

and by induction due to (25d)

$$(35c) \quad |\alpha_n^{(1)}| \leq (1 + |c|)^n$$

and from (25e)

$$(35d) \quad |\alpha_n^{(2)}| \leq (1 + |c|)^n \\ \dots\dots\dots$$

In virtue of (29), (35c), (35d)

$$\lim_{n \rightarrow \infty} |(V^* + A)^n|^{1/n} \leq 1 + |c|.$$

Hence $r(T^*) \leq \max \{1 + |c_i|\}$ and therefore for the roots of (16) we have

$$(36) \quad |W| \leq \max_{i=1,2,\dots,m} (1 + |c_i|).$$

This is the well known condition found by Cauchy [1].

4. The Walsh inequality. Let

$$(37) \quad R = |c_1| + (|c_2|)^{1/2} + (|c_3|)^{1/3} + \dots + (|c_m|)^{1/m}$$

From (25) we obviously get:

$$(38a) \quad |\alpha_1^{(1)}| \leq 1 + R,$$

$$\begin{aligned}
 (38b) \quad & |\alpha_2^{(1)}| \leq 1 + R + R^2, \\
 & \dots\dots\dots \\
 (38c) \quad & |\alpha_n^{(1)}| \leq 1 + R + R^2 + \dots + R^n.
 \end{aligned}$$

The last relation is easily verified by complete induction using (25d).

We now find from (29), (25), (38)

$$\begin{aligned}
 (39) \quad & r(T^*) = \lim_{n \rightarrow \infty} ||(V^* + A)^n||^{1/n} \leq R, \quad \text{if } R > 1; \\
 (40) \quad & r(T^*) = \lim_{n \rightarrow \infty} ||(V^* + A)^n||^{1/n} \leq 1, \quad \text{if } R \leq 1,
 \end{aligned}$$

and in virtue of (10), $r(T^*) = 1$.

From (39), (40) and Corollary 1 we conclude that if $R > 1$, then $|W| \leq R$ where W are the roots of (16). If $R \leq 1$, then $|W| \leq 1$. If $R \leq 1$, dividing both members of (16) by R^m , we have

$$(41) \quad \frac{\bar{c}_m}{R^m} + \frac{\bar{c}_{m-1}}{R^{m-1}} \left(\frac{W}{R}\right) + \frac{\bar{c}_{m-2}}{R^{m-2}} \left(\frac{W}{R}\right)^2 + \dots + \frac{\bar{c}_1}{R} \left(\frac{W}{R}\right)^{m-1} - \left(\frac{W}{R}\right)^m = 0.$$

Since

$$\left(\frac{|\bar{c}_m|}{R^m}\right)^{1/m} + \left(\frac{|\bar{c}_{m-1}|}{R^{m-1}}\right)^{1/(m-1)} + \dots + \frac{|c_1|}{R} \leq 1 \quad \text{is true,}$$

applying the result we found to (41), we conclude that if $R \leq 1$, $|W/R| \leq 1$ and therefore $|W| \leq R$. Hence in all cases

$$(42) \quad |W| \leq R = |c_1| + (|c_2|)^{1/2} + \dots + (|c_m|)^{1/m},$$

where W are the roots of equation (16). This is the well known inequality of Carmichael and Walsh [15–16].

5. The Parodi inequality. Let

$$(43) \quad R = \frac{|c_1| + \left(|c_1|^2 + 4 \sum_{i=2}^m |c_i|\right)^{1/2}}{2}.$$

From (25) we obtain:

$$\begin{aligned}
 (44a) \quad & |\alpha_1^{(1)}| \leq 1 + R, \\
 (44b) \quad & |\alpha_2^{(1)}| \leq 1 + R + R^2, \\
 & \dots\dots\dots \\
 (44c) \quad & |\alpha_n^{(1)}| \leq 1 + R + R^2 + \dots + R^n.
 \end{aligned}$$

(44a) and (44b) are obvious. (44c) can be proven from (25d) by complete induction. Since in that case the proof of (44c) is not straightforward, it is outlined here.

Using (25d) and accepting (44c) for $n - 1, n - 2, \dots, n - m$ we have

$$(45) \quad |\alpha_n^{(1)}| \leq |c_1| (1 + R + \cdots + R^{n-1}) + |c_2| (1 + R + \cdots + R^{n-2}) \\ + |c_3| (1 + R + \cdots + R^{n-3}) + \cdots + |c_m| (1 + R + \cdots + R^{n-m}) + 1.$$

In order to prove (44c) it is sufficient to prove that

$$(46) \quad 1 + |c_1| \frac{R^n - 1}{R - 1} + |c_2| \frac{R^{n-1} - 1}{R - 1} + |c_3| \frac{R^{n-2} - 1}{R - 1} + \cdots \\ + |c_m| \frac{R^{n-m+1} - 1}{R - 1} \leq \frac{R^{n+1} - 1}{R - 1}.$$

Suppose first that $R > 1$. We can substitute in (46) $R^{n-2}, \dots, R^{n-m+1}$ by the greater quantity R^{n-1} . Then it is sufficient to prove that

$$(47) \quad |c_1| (R^n - 1) + \sum_{i=2}^m |c_i| (R^{n-1} - 1) + R \leq R^{n+1}.$$

From (43) we have

$$(48) \quad R^2 - |c_1| R - \sum_{i=2}^m |c_i| = 0.$$

Hence, from (48) and (47) we obtain

$$(R - 1) (|c_1| - R) \leq 0$$

which is true because $R > 1$ by hypothesis and $R > |c_1|$ due to (43).

If $R < 1$, then we have from (46)

$$|c_1| \frac{1 - R^n}{1 - R} + |c_2| \frac{1 - R^{n-1}}{1 - R} + |c_3| \frac{1 - R^{n-2}}{1 - R} + \cdots \\ + |c_m| \frac{1 - R^{n-m+1}}{1 - R} + 1 \leq \frac{1 - R^{n+1}}{1 - R}.$$

Therefore it is sufficient to prove that

$$|c_1| \frac{1 - R^n}{1 - R} + |c_2| \frac{1 - R^{n-1}}{1 - R} + |c_3| \frac{1 - R^{n-2}}{1 - R} + \cdots \\ + |c_m| \frac{1 - R^{n-m+1}}{1 - R} + 1 \leq \frac{1 - R^{n+1}}{1 - R}$$

or by (48)

$$(1 - R) (|c_1| - R) \leq 0$$

which is obviously true because $R < 1$ by hypothesis and $R > |c_1|$ from (43).

Finally for $R = 1$ (46) in the expanded form is easily verified using (48).

From (25), (29), (44) we find: If $R > 1$, then $r(T^*) = \lim_{n \rightarrow \infty} \|(V^* + A)^n\|^{1/n} \leq R$ and if $R \leq 1$, then $r(T^*) = \lim_{n \rightarrow \infty} \|(V^* + A)^n\|^{1/n} \leq 1$ and in virtue of (10) $r(T^*) = 1$. Hence from Corollary 1

$$(49) \quad |W| \leq \max \left\{ 1, \frac{|c_1| + \left(|c_1|^2 + 4 \sum_{i=2}^m |c_i| \right)^{1/2}}{2} \right\},$$

where W are the roots of equation (16).

This is the well known inequality found by Parodi [17, 18]. This is obviously stronger than the inequality (30) of Montel.

The above applications indicate that the study of perturbation of non-normal isometries is not only of mere theoretical interest.

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