Two Sublattices of Weighted Shift Invariant Subspaces

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Introduction.

1. A weighted shift is an operator in Hilbert space sending

$$(1.1) \quad e_n \text{ to } a_{n+1}e_{n+1},$$

where $\{e_n\}_0^\infty$ is an orthonormal basis and $\{a_n\}_0^\infty$ is a never-zero sequence.

The unweighted shift S (all a 's = 1) has been much studied since Beurling proved in [1] that

- 1) S restricted to an invariant subspace is unitarily equivalent to S (acting on the whole space) and
- 2) every invariant subspace of S is the range of an isometry commuting with S .

There has been some interest in whether some generalization of these properties still hold for weighted shifts. Halmos has asked in [4], Problem 2, whether every part of a weighted shift is similar to a weighted shift (a part being a restriction to an invariant subspace). In Section I it is shown that this possible generalization of 1) is false, and also that there is a weighted shift with an invariant subspace not the closure of the range of any operator commuting with the shift, thus disposing of a weak possible generalization of 2).

The main tool in finding these counterexamples is a complete description of the lattice of finite co-dimensional shift invariant subspaces. Information is extracted from [3], Section 9, which describes all such subspaces for two-sided shifts in Banach space, but does not make the lattice structure explicit. The absence of a proof will signify that the proof is left as an easy exercise.

In Section II it is shown that an operator commuting with an invertible bilateral weighted shift is the strong limit of polynomials in the shift and its inverse, that the commutant is naturally isomorphic to a ring of bounded measurable functions on the spectrum of the shift, and that the invariant subspaces with invariant complements form a lattice isomorphic with a lattice of measurable subsets of a circle invariant under rotation.

I. The lattice of finite co-dimensional invariant subspaces.

2. Representation. Let $H^2(w)$ be the Hilbert space of all formal complex power series $f = \sum_0^\infty b_n z^n$ such that

$$||f||^2 = \sum_0^\infty |b_n|^2 |w_n|^2 < \infty,$$

where $w = \{w_n\}_0^\infty$ is a never-zero complex sequence with $w_0 = 1$. Let the linear operator of multiplication by z in $H^2(w)$ be denoted by T .

Lemma 1. *If $\{e_n\}_0^\infty$ is an orthonormal basis in a Hilbert space the isomorphism defined by*

$$(2.1) \quad Ue_n = w_n^{-1} z^n$$

induces a unitary equivalence between T and the weighted shift (1.1) if $a_n = w_n w_{n-1}^{-1}$, $w_n = \prod_{i=1}^n a_i$ for $n > 0$ and $w_0 = 1$.

T is assumed bounded or equivalently $\sup |a_i| < \infty$. Representing a weighted shift as multiplication by z in $H^2(w)$ appears to be the correct and fruitful generalization of the trick of representing the unweighted shift as multiplication by z in the function space H^2 .

3. Circular symmetry.

Lemma 2. *$H^2(w)$ is invariant under the "rotation" mappings*

$$\sum_0^\infty b_n z^n \rightarrow \sum_0^\infty b_n c^n z^n$$

where c is a complex number of modulus 1.

In fact these mappings are unitary and induce a unitary equivalence between T and cT . We describe this behavior by saying that a weighted shift has *circular symmetry*.

4. Finite co-dimensional invariant subspaces. We henceforth abbreviate "Closure of the range" by "Clorange".

Proposition 1. *Suppose an operator A has a cyclic vector. Then*

- 1) *for each A -invariant subspace M of co-dimension n there is a unique n th-degree monic polynomial p with $M = \text{clorange } p(A)$;*
- 2) *for any polynomial p , co-dimension clorange $p(A) \leq \text{degree } p$.*

Proof. These facts are easily established using the proof of [3, Lemma 1] as an outline. For part 2) one must note that if $M = \text{clorange } p(A)$, then $p(A_M) = 0$ (using the notation of the cited lemma).

Remark. The hypothesis above could be weakened to " A^* has only simple eigenvalues".

Lemma 3. a) For the weighted shift T of multiplication by z in $H^2(w)$ and for the k th-degree polynomial $p(z) = \Pi(z - \lambda_i)^{n_i}$, $N = \text{Clorange } p(T)$ is of co-dimension k iff the mappings which send a series in $H^2(w)$ to the evaluation at $\lambda = \lambda_i$ of its formal m th derivative for $m < n_i$ are all bounded linear functionals.

b) If the conditions of a) are satisfied, N is the intersection of the null spaces of these functionals.

c) The above described mappings are all bounded iff

$$\sum_{n=0}^{\infty} |w_n^{-1} n^m|^2 |\lambda|^{2n} < \infty \quad (m = n_i - 1, \lambda = \lambda_i).$$

Proof. a) Suppose degree p is n . First assume that the n functionals mentioned are bounded. These functionals are linearly independent (even when restricted to the subspace P_n of polynomials of degree $< n$) and they vanish at the elements

$$(4.1) \quad \{z^n p\} = \{p(T)z^n\}_0^\infty$$

which span $\text{clorange } p(T)$. Hence, co-dimension ($\text{clorange } p(T)$) $\geq n$.

Conversely it is clear that the union of N and the subspace P_n spans $H^2(w)$. Thus if co-dimension $N = n$, then P_n is an n -dimensional complement of N . Each of the mappings mentioned in a) can then be written as one of the bounded functionals

$$F(f) = \left. \frac{d^m q}{dz^m} \right|_{z=\lambda_i} \quad (\text{all } f \in H^2(w))$$

where

$$(4.2) \quad f = q + g, \quad q \in P_n, \quad g \in N.$$

To see that this is the mapping in question it suffices to note that if $z^n = q + pr$ (r a polynomial), then the n th-derivative of pr at λ_i is 0.

b) $f \in N$ iff $q = 0$ in (4.2) iff $0 = F(q) = F(f)$ for all the functionals in question. (The second equivalence follows from the linear independence of these n functionals on the n -dimensional subspace P_n .)

c) After being composed with the mapping U (2.1), such a mapping will send

$$e_n \text{ to } w_n^{-1} \lambda_i^n \text{ for } m = 0$$

or to

$$w_n^{-1} n(n-1) \cdots (n-m+1) \lambda_i^{n-m} \text{ for } n \geq m > 0.$$

Thus all the mappings are bounded iff these complex sequences in n are square summable for all $m \leq n_i - 1$ which is equivalent to the given condition.

Denote by $\mathcal{P}_T$ the set of monic complex polynomials p such that $\text{clorange } p(T) = N$ is of co-dimension equal to degree p . The set $\mathcal{P}_T$ can be defined for any operator in Hilbert space and is clearly a similarity invariant. Partially order $\mathcal{P}_T$ by the rule " $p_1 \leq p_2$ iff p_1 divides p_2 ".

Define $I_T = \varliminf w_n^{1/n}$ and let j be the smallest of 0, the positive integers or ∞ satisfying

$$\sum_{n=0}^{\infty} |w_n^{-1} n^j|^2 |I_T|^{2n} = \infty.$$

Denote by $\Lambda_{I_T, j}$ the lattice of all finitely non-zero functions $n : \lambda \rightarrow n_\lambda$ from the complex disc $\{|\lambda| \leq I_T\}$ to the non-negative integers satisfying $n_\lambda \leq j$ for $|\lambda| = I_T$.

Theorem 1. *For a weighted shift T , $\Lambda_T \stackrel{\text{def}}{=} \Lambda_{I_T, j}$ is order isomorphic to $\mathcal{O}_T$ and both Λ_T and $\mathcal{O}_T$ are order anti-isomorphic to the lattice of finite co-dimensional T -invariant subspaces under the mappings*

$$n \rightarrow \prod_{|\lambda| \leq I_T} (z - \lambda)^{n_\lambda} \rightarrow \text{clorange} \prod_{|\lambda| \leq I_T} (T - \lambda I)^{n_\lambda}.$$

Proof. After noting that the requirements of 3c) (for a polynomial $p = \prod (z - \lambda_i)^{n_i}$ to lie in $\mathcal{O}_T$) is the convergence at $\lambda = \lambda_i$ of a certain power series in $|\lambda|$ with radius of convergence I_T , it follows without difficulty that the first mapping is a lattice isomorphism.

As a consequence of this first isomorphism if p_1 and p_2 are in $\mathcal{O}_T$, so is their least common multiple p . It follows, using 3b), that if $N_1 = \text{clorange } p_1(T)$ and $N_2 = \text{clorange } p_2(T)$, then $N_1 \cap N_2 = \text{clorange } p(T)$. This shows that the second mapping is an order anti-isomorphism.

If N is an invariant subspace let $T|N$ denote T restricted to N .

Corollary 1. *Let N be the finite co-dimensional invariant subspace corresponding under the mapping of Theorem 1 to the function n in Λ_T . Then*

a) *the lattice of finite co-dimensional $T|N$ -invariant subspaces of N is order anti-isomorphic to the sublattice $\Lambda_{T|N}$ of those functions $m \in \Lambda_T$ with $m_\lambda \geq n_\lambda$ (all λ), under the mapping*

$$m \rightarrow \text{clorange} \prod_{|\lambda| \leq I_T} ((T|N) - \lambda I)^{m_\lambda}$$

where $q_\lambda = m_\lambda - n_\lambda$.

b) $\mathcal{O}_{T|N} = \{ \prod_{|\lambda| \leq I_T} (z - \lambda)^{q_\lambda} : q_\lambda + n_\lambda \leq j \text{ if } |\lambda| = I_T \}.$

Proof. N_1 is a finite co-dimensional $T|N$ -invariant subspace of N iff N_1 is a finite co-dimensional T -invariant subspace contained in N iff $p_1 = qp$ for some polynomial q , where p and p_1 are the polynomials in $\mathcal{O}_T$ corresponding to N and N_1 . Degree $q = \text{degree } p_1 - \text{degree } p = (\text{co-dimension } N_1 \text{ in } H) - (\text{co-dimension } N \text{ in } H) = \text{co-dimension } N_1 \text{ in } N$. Clorange $q(T|N) = N_1$ because both these sets are the closure of $\{p_1(T)f = q(T)[p(T)f] : f \in H\}$. We know $T|N$ is cyclic with cyclic vector $p = p(T)1$ (4.1) and thus q is the *unique* polynomial in $\mathcal{O}_{T|N}$ satisfying clorange $q(T|N) = N_1$. The remaining details now follow easily.

Example 1. Let N be the subspace corresponding to the function n .

a) If $0 < n_\mu \leq j < \infty$ for some μ with $|\mu| = I_T$, then $\mathcal{O}_{T|N}$ differs from $\mathcal{O}_{T_1}$ for any weighted shift T_1 . Hence $T|N$ is not similar to a weighted shift.

b) If $0 < n_\mu < j < \infty$ for some μ with $|\mu| = I_T$, then $\Lambda_{T|N}$ is not lattice isomorphic to Λ_{T_1} for any weighted shift T_1 . Hence T_1 and $T|N$ have different lattices of finite co-dimensional invariant subspaces so $T|N$ is not similar to any weighted shift T_1 .

Proof. What $T|N$ lacks in both cases is circular symmetry.

a) If $(z - \lambda)^n \in \mathcal{O}_{T_1}$ and $|\mu| = |\lambda|$, then $(z - \mu)^n \in \mathcal{O}_{T_1}$ also. But $(z + \mu)^j \in \mathcal{O}_{T|N}$ and $(z - \mu)^j \notin \mathcal{O}_{T|N}$.

b) All finite maximal linearly ordered subintervals (if any) of Λ_{T_1} are of the same length. But $\Lambda_{T|N}$ has such subintervals of length $j + 1$ and has one of length $j - n_\mu + 1$ (the latter will be the subinterval $\{n \in \Lambda_{T|N} : n_\lambda = 0 \text{ for } \lambda \neq \mu\}$).

5. Remark. Even when n_λ is not finitely non-zero, the set of series vanishing to order at least n_λ at λ ($|\lambda| \leq I_T$, $n_\lambda \leq j$ if $|\lambda| = I_T$) forms an invariant subspace of T . Of course this subspace will contain only 0 if there is no non-zero f in $H^2(w)$ vanishing in just this way. A description of the zeros of functions in $H^2(w)$ is a minimum prerequisite to describing all invariant subspaces of weighted shifts.

6. An invariant subspace which is neither a commuting clorange nor a commuting kernel. If A is a linear transformation on a Hilbert space H , call a subspace N a *commuting clorange* (for A) in case there exists a linear transformation B on H such that B commutes with A and such that the clorange of B is equal to N . Similarly call M a *commuting kernel* in case there exists a commuting C such that the kernel of C is equal to M . These definitions are (more or less) lifted from Halmos [5]. Commuting cloranges and kernels are A -invariant subspaces.

As we have already noted, for a weighted shift T every finite co-dimensional subspace is a commuting clorange. Additional information can be derived from a description of the commutant of T . The following is a short summary of the facts about the commutant of a weighted shift [see 2] together with some remarks and unsolved problems.

For two formal series $f = \sum_0^\infty b_n z^n$ and $g = \sum_0^\infty c_n z^n$ denote by $f \cdot g$ the formal product $\sum_{n=0}^\infty (\sum_{i=0}^n b_i c_{n-i}) z^n$.

Define $H^\infty(w) = \{f \in H^2(w) : f \cdot g \in H^2(w) \text{ for all } g \in H^2(w)\}$. For $f \in H^\infty(w)$ define the linear operator $f(T)$ on $H^\infty(w)$ by

$$f(T)g = f \cdot g,$$

$\{f(T) \mid f \in H^\infty(w)\} = \{T\}'$, the ring of bounded operators commuting with T . Define a norm on $H^\infty(w)$ by $\|f\|_{\infty(w)} = \text{operator norm of } f(T)$. With this norm

and the multiplication $f \cdot g$, $H^\infty(w)$ is a commutative Banach algebra. The mapping $f \rightarrow f(T)$ provides a Banach algebra isomorphism with $\{T\}'$.

Since $f \cdot g = 0$ iff $f = 0$ or $g = 0$, the only commuting kernels of T are $\{0\}$ and the whole space. Since non-trivial idempotents ($P^2 = P$) have non-trivial kernels, T commutes with no non-trivial idempotents. Because the existence of a non-trivial commuting idempotent is a similarity invariant for an operator, an operator similar to a weighted shift commutes with no non-trivial idempotents, so *a fortiori* is irreducible. Essentially this proof was first given by Kelley [8]. The set of operators commuting with no non-trivial self-adjoint idempotents has been studied (see [4], Problem 8). However, the structure of the set described by removing the word "self-adjoint" above appears to be completely unknown, aside from the fact that its complement is dense [11].

6.1. Every series $f = \sum_0^\infty b_n z^n$ in $H^\infty(w)$ converges to a bounded analytic function $f(\lambda)$ on the disc $\{|\lambda| < R\}$ where $R = \lim_{n \rightarrow \infty} (\sup_{m > 0} |w_{m+n} w_n^{-1}|)^{1/n}$, the spectral radius of T .

6.2. Spectrum $f(T) \supseteq \{f(\lambda) : |\lambda| < R\}$.

6.3. $\|f(T)\| \geq \sup_{|\lambda| < R} |f(\lambda)|$

6.4. $\sum_0^\infty |b_n| \|T^n\| \geq \|f(T)\| \geq \sup_n |b_n| \|T^n\|$.

6.5. If $f = \sum_0^\infty b_n z^n \in H^\infty(w)$ and $f_r = \sum_0^\infty b_n r^n z^n$, then $\|f_r\|_{\infty(w)} \leq \|f\|_{\infty(w)}$ for $0 < r < 1$ and $\lim f_r(T) = f(T)$ in the strong operator topology.

6.5 is derived from work of Shields and Wallen [9] and will be generalized to two-sided shifts in Part II of this paper, and to shifts in Banach space in [12].

Properties 6.1–6.5 above are intended to convince the reader that representing the commutant of a weighted shift as the Banach algebra $H^\infty(w)$ is the correct generalization of the trick of representing the commutant of the unweighted shift as H^∞ . Of course far less is known about $H^\infty(w)$ than about H^∞ . For example, "If $|\lambda| < R$, is $\lambda - z$ a member of only one maximal ideal in $H^\infty(w)$?" is an open question. The answer for H^∞ is "yes, the ideal $\{f : f(\lambda) = 0\}$ ".

Example 2. If $R > I_T = 1 \geq w_n$ for all n , then T has invariant subspaces which are neither commuting ranges nor commuting kernels. The sequence $w = 1, \frac{1}{2}, 1, \frac{1}{4}, \frac{1}{2}, 1, \frac{1}{8}, \frac{1}{4}, \frac{1}{2}, 1, \frac{1}{16}, \dots$ is a particular example.

Proof. As noted, T has no non-trivial commuting kernels. Let $f(T)$ be an operator commuting with T . Then by 6.1, $f = \sum b_n z^n$ has radius of convergence greater than $Q_T \stackrel{\text{def}}{=} \overline{\lim} w_n^{1/n}$. Therefore [3, Note 4] the sequence $T^n f$ spans a finite co-dimensional subspace. Since z^n is dense in H and $T^n f = f(T) z^n$, this subspace is closure $f(T)$. To complete the example it is now only necessary to find a non-trivial invariant subspace of infinite co-dimension. By 5) it suffices to find $g \in H^2(w)$ with $g(\lambda)$ vanishing infinitely often in $|\lambda| < I_T$. Such a g can be found in H^2 and since $0 < w_n < 1$, $H^2 \subseteq H^2(w)$.

II. Invariant subspaces with invariant complements.

7. Functional representation of the commutant. The theory of two-sided weighted shifts parallels that outlined for one-sided shifts in Part I [see 2].

$$Ue_n = w_n^{-1}z^n$$

($w_n = \prod_{i=1}^n a_i$ for $n > 0$, $w_0 = 1$, $w_n = (\prod_{i=-n+1}^0 a_i)^{-1}$ for $n < 0$) induces a unitary equivalence between the weighted shift

$$e_n \rightarrow a_{n+1}e_{n+1}, \quad -\infty < n < \infty,$$

($\{e_n\}_m^{-1}$ a two-sided orthonormal basis) and the operator T of multiplication by z on the space $L^2(w)$ of series $f = \sum_{-\infty}^{\infty} b_n z^n$ with $\|f\|^2 = \sum_{-\infty}^{\infty} |b_n|^2 |w_n|^2 < \infty$. The commutant of T is the Banach algebra $L^\infty(w)$ of multiplying series in $L^2(w)$. The spectrum of T is the set $\{R_2 \leq |\lambda| \leq R_1\}$ where $R_1 = \lim_{n \rightarrow \infty} (\sup_m |w_{m+n} w_m^{-1}|)^{1/n}$ and $R_2 = \lim_{n \rightarrow \infty} (\inf_m |w_{m+n} w_m^{-1}|)^{1/n}$. For $f = \sum_{-\infty}^{\infty} b_n z^n \in L^\infty(w)$, the series $\sum_0^\infty b_n z^n$ and $\sum_{-\infty}^{-1} b_n z^n$ converge on $\{|z| < R_1\}$ and $\{|z| > R_2\}$ respectively.

Define $f_r = \sum_{-\infty}^{\infty} b_n r^{|n|} z^n$.

Theorem 2. *If $f \in L^\infty(w)$, then $f_r \in L^\infty(w)$ for $0 < r < 1$. $\|f_r(T)\| \leq 4 \|f(T)\|$. $\lim_{r \rightarrow 1^-} f_r(T) = f(T)$ in the strong operator topology.*

Proof. This generalization of Shield's and Wallen's result [9] for the one-sided case is proven by using the same methods as in their paper. The matrix of $f(T)$ with respect to the orthonormal basis $\dots, e_{-2}, e_{-1}, e_0, e_1, \dots$ is $(b_{i-j} w_i w_j^{-1})$. The matrix of $f_r(T)$ is $(b_{i-j} r^{|i-j|} w_i w_j^{-1})$. The whole space is the direct sum of the subspaces H^+ and H^- spanned by $e_0, e_1, \dots$ and $e_{-1}, e_{-2}, \dots$. This decomposition splits each matrix into 4 submatrices. We shall show that a submatrix of $f_r(T)$ has (operator) norm less than the corresponding submatrix of $f(T)$. Then since the norm of $f_r(T)$ is $\leq$ the sum of the norms of its submatrices and the norm of $f(T)$ is $\geq$ the norm of each of its submatrices, we shall have obtained the first two assertions of the Theorem.

Now the 4 submatrices of $f_r(T)$ are obtained from those of $f(T)$ by entry-by-entry multiplication with one of the Toeplitz matrices $M_1 = (r^{|i-j|})$, $i, j \geq 0$, or $M_2 = (r^{|i-j|})$, $i, j < 0$, or one of the Hankel matrices $M_3 = (r^{|i-j|})$, $i \geq 0$, $j < 0$, or $M_4 = (r^{|i-j|})$, $i < 0$, $j \geq 0$. We use this theorem of Schur quoted by Shields and Wallen:

If $A = (a_{ij})$ is a bounded operator on ℓ^2 and if $B = (b_{ij})$ is a positive semi-definite matrix with bounded diagonal entries,

$$|b_{ii}| \leq M \quad (i = 0, 1, 2, \dots),$$

then the matrix $C = (a_{ij} b_{ij})$ is a bounded operator on ℓ^2 and

$$\|C\| \leq M \|A\|.$$

We need only verify that M_1, M_2, M_3 and M_4 are positive semi-definite. But $M_3 = (r^i r^{i|j|})$ and $M_4 = (r^{|i|} r^j)$ so these matrices are clearly positive semi-

definite. Also the periodic function associated with the Toeplitz matrices M_1 and M_2 is $\sum_{-\infty}^{\infty} r^{|n|} e^{in\theta} = 1 + 2 \operatorname{Re} (1/(1 - \operatorname{Re} e^{i\theta}))$ which takes only positive real values, which implies that M_1 and M_2 are positive definite (see [6], Chapter 20).

To prove the strong operator convergence part of the theorem, note first that the series $\sum_{-\infty}^{\infty} b_n r^{|n|} z^{n+m} = z^m f_r = f_r(T) z^m$ and $\sum_{-\infty}^{\infty} b_n z^{n+m} = z^m f = f(T) z^m$ lie in $L^2(w)$ and then (by straightforward computation of norms) that $\lim z^m f_r = z^m f$ in the $L^2(w)$ norm. In other words $f_r(T)$ converges strongly to $f(T)$ on the complete set z^m and being uniformly bounded must converge strongly to $f(T)$ on the whole space.

Corollary 2. *When T is invertible the commutant of T is the strong limit of polynomials in T and T^{-1} .*

Proof. For $f \in L^\infty(w)$ and fixed $r < 1$, f_r converges as a Laurent series uniformly on a neighborhood of spectrum T . Hence $f_r(T)$ is the norm limit of the partial sums of $\sum_{-\infty}^{\infty} b_n r^{|n|} T^n$ [13, p. 226].

Comment. Turner [10] has shown that when T is not invertible the commutant is the strong limit of polynomials in T alone.

8. $L^\infty(w)$ as a space of functions. In [2, Theorem 4] it was shown that if $R_2 < R_1$ and $f \in L^\infty(w)$, then $f = \sum b_n z^n$ is the Laurent series of a bounded analytic function on the interior of the spectrum of T . Thus $\{T\}'$ can be represented as a subring of the ring " H^∞ of the annulus $\{R_2 < |\lambda| < R_1\}$ ".

Theorem 3. *Suppose $0 < R_2 = R_1$.*

a) *If $f \in L^\infty(w)$, then $f = \sum b_n z^n$ is the "Fourier series" of a bounded measurable function $f(\lambda)$ on $\{|\lambda| = R_1\}$ (in the sense that for $\lambda = R_1 e^{i\theta}$, $f(\lambda) \sim \sum_{-\infty}^{\infty} (b_n R_1^{|n|}) e^{in\theta}$).*

b) *$L^\infty(w)$ (which is already identified with $\{T\}'$, the commutant of T) can be represented as a subring of the ring of bounded Lebesgue measurable functions (identified when agreeing a.e.) on $\{|\lambda| = R_1\}$. If $f(\lambda)$ is in this subring so is $f(c\lambda)$ for any c of modulus 1.*

c) *If $f \in L^\infty(w)$, spectrum $f(T) \supseteq$ closure of essential range of $f(\lambda)$ ($|\lambda| = R_1$).*

d)
$$\|f(T)\| \geq \operatorname{ess\,sup}_{|\lambda|=R_1} |f(\lambda)|.$$

Proof. a) If $0 < R_2 = R_1$, $f \in L^\infty(w)$ and $0 < r < 1$, then f_r is analytic on a neighborhood of the circle $\{|\lambda| = R_1\}$, which is the spectrum of T so $\frac{1}{4} \|f(T)\| \geq \|f_r(T)\| \geq \text{spectral radius } f_r(T) \geq |f_r(\lambda)|$ ($|\lambda| = R_1$) = $|\sum_{-\infty}^{\infty} b_n r^{|n|} \lambda^n| = |\sum_{-\infty}^{\infty} b_n R^{|n|} e^{in\theta}|$, where $R = rR_1$ and $e^{i\theta} = \lambda R_1^{-1}$. This expression inside the last absolute value sign thus defines a bounded harmonic function $g(u)$ ($u = \operatorname{Re} e^{i\theta}$) on $|u| < R_1$. By Fatou's lemma the radial limit $\lim_{R \rightarrow R_1} \sum_{-\infty}^{\infty} b_n R^{|n|} e^{in\theta}$ exists for a.a. θ and is bounded. This limit clearly has Fourier series $\sum_{-\infty}^{\infty} b_n R_1^{|n|} e^{in\theta}$.

b) Associate $f = \sum b_n z^n \in L^\infty(w)$ (and simultaneously $f(T) \in \{T\}'$) with the

function $f(\lambda) \sim \sum_{-\infty}^{\infty} b_n R_1^n e^{in\theta}$. The fact that Fourier series of bounded functions add and multiply formally proves isomorphism. The second claim follows from the circular symmetry of T .

- c) $f(T)$ can't be invertible in $\{T\}'$ unless $f(\lambda)$ has an inverse bounded a.e.
- d) follows from c).

9. Decomposing subspaces. Subspaces M and N are called complements in Hilbert space H if $M \cap N = \{0\}$ and $M + N = H$. M and N are complements iff there is a bounded idempotent P on H with kernel $P = N$ and range $P = M$. P is uniquely determined by the pair M, N . M and N are both invariant subspaces for an operator T iff $PT = TP$. We shall call M a *decomposing subspace* for T if it is T -invariant and has a T -invariant complement. (A reducing subspace is a decomposing subspace whose orthogonal complement is invariant.)

A one-sided weighted shift commutes with no idempotents and so has no decomposing subspaces. A two-sided shift may commute with idempotents though. In fact the unweighted two-sided shift S is unitary and thus has reducing subspaces. The decomposing subspaces of S (which are the same as reducing subspaces for any normal operator) form a lattice isomorphic to the set of measurable subsets of the unit circle, sets differing by a Lebesgue null set being identified (see [7], Theorem 2). We shall show that the decomposing subspaces of a weighted shift can also be identified with subsets of a circle.

Proposition 2. *Among those operators T whose decomposing subspaces form a sub-lattice of their lattice of invariant subspaces are all normal operators but not all isometries.*

Proof. Normal operators: If $PT = TP$, then by Fuglede's Theorem (see [6, Problem 152]) $PT^* = T^*P$, so decomposing subspaces = subspaces invariant under T and under T^* = reducing subspaces. The middle property is preserved by the lattice operations.

Isometries: Consider multiplication by z on $H^2 \oplus H^2$. The subspaces $N = \{(f, \frac{1}{2}zf): f \text{ in } H^2\}$ and $N' = \{(\frac{1}{2}zf, f): f \text{ in } H^2\}$ are complementary invariant subspaces. But $N \vee H^2 \oplus \{0\} = H^2 \oplus zH^2$ which has no invariant complement.

Theorem 4. *The decomposing subspaces of a weighted shift are a Boolean sub-lattice of the lattice of invariant subspaces and are Boolean isomorphic to an algebra of measurable subsets of a circle which is invariant under rotation.*

Proof. We must assume $0 < R_2 = R_1$ or else there are no idempotents commuting with T [2, Cor. 2]. Let M be a decomposing subspace. Then M has only one decomposing complement because if P and P' are idempotents with range M commuting with T , then P and P' commute and thus must be equal. Such an idempotent P is associated through the isomorphism of Theorem 3b) with a bounded function $p(\lambda)$ ($|\lambda| = R_1$) satisfying $(p(\lambda))^2 = p(\lambda)$ a.e. That is, $p(\lambda)$ is the characteristic function of a measurable subset E_M of $\{|\lambda| = R_1\}$. The unique complement of M is clorange $(I - P)$ and $I - P$ is associated with

the function $1 - p(\lambda)$, the characteristic function of the complement of E_M . Since all the idempotents involved mutually commute $M_1 \cap M_2 = \text{Range}(p_1(T)p_2(T))$ and $p_1(T)p_2(T)$ is associated with the function $p_1(\lambda)p_2(\lambda)$, the characteristic function of $E_{M_1} \cap E_{M_2}$. Thus the mapping

$$M \rightarrow E_M$$

is a Boolean isomorphism. Circular symmetry of T implies invariance under rotation.

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