Lie Subalgebras of Finite Codimension in $\Gamma(TX)$

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The study of the infinite dimensional Lie group structure of $\text{Diff}^\infty(X)$, where X is a compact C^∞ -manifold was begun in [3]. The Lie algebra of this Lie group is $\Gamma^\infty(TX)$ and the investigation of $\text{Diff}^\infty(X)$ requires the study of this infinite dimensional Lie algebra. The new interest is with the infinite dimensional subalgebras. In this paper we show that every subalgebra of finite codimension which is closed in the C^∞ -topology is contained in the finite intersection of Lie subalgebras, each of which is the set of all vector fields vanishing at some fixed point.

In developing these ideas it is advantageous to drop the requirement that X be compact but work with $\Gamma_c^\infty(TX)$, the Lie algebra of smooth vector fields of compact support. X will always denote a 2nd countable, T_2 , C^∞ -manifold. Such an X is metrizable and we choose a metric, d , for X .

We first establish some local results. N will denote the set of nonnegative integers, and if $p = (p_1, \dots, p_n)$ is in N^n we define $|p| = p_1 + p_2 + \dots + p_n$. D_p will denote the partial derivative operator $\partial^{|p|}/\partial x_1^{p_1} \partial x_2^{p_2} \dots \partial x_n^{p_n}$. If V is an open subset of R^n , $\mathfrak{D}^\infty(V, R^m)$ will denote the C^∞ -mappings of V into R^m which have compact support. When $n = m$ this is a Lie algebra with $[\xi, \eta](x) = \xi'(x)\eta(x) - \eta'(x)\xi(x)$. Here $\xi'(x)$ denotes the derivative at x of the mapping $\xi: V \rightarrow R^n$, considered as a linear transformation of R^n into R^n ; similarly for η' .¹ $\pi_j: R^m \rightarrow R$ will denote the j^{th} projection so that we get $\mathfrak{D}^\infty(V, R^m) = \sum_1^m \mathfrak{D}^\infty(V, R)$ by $f = (f_1, f_2, \dots, f_m)$ where $f_i = \pi_i f$.

Lemma 1. (E. Borel). Suppose that $(c_p)_{p \in N^n}$ is a sequence of numbers indexed by N^n and that $x \in V$. Then there is an $f \in \mathfrak{D}^\infty(V, R)$ so that $(D_p f)(x) = c_p$ for all p in N^n .

Lemma 2. For $1 \leq k$, $1 \leq n$ let $S_{nk} = \{(r_1, r_2, \dots, r_n) \mid \text{each } r_i \text{ is in } N \text{ and } r_1 + r_2 + \dots + r_n = k\}$. Suppose that $L: S_{nk} \rightarrow R$ and that $(x_1, \dots, x_n)$ is an n -tuple of numbers which satisfy: $(s_1, \dots, s_n)$ in S_{nk+1} implies $\sum \{L(s_1, \dots,$

$s_i - 1, \dots, s_n)x_i \mid 1 \leq j \leq n, 1 \leq s_i\} = 0$. Then either $L = 0$ or $(x_1, \dots, x_n) = 0$.

Proof. Suppose that L and $(x_1, \dots, x_n)$ satisfy the condition and that $(x_1, \dots, x_n) \neq 0$. We can assume that $x_1 \neq 0$. Let S_{nk} have the lexicographic order starting from the right. S_{nk} is linearly ordered by this order and we will prove that $L = 0$ by induction with respect to this order. The smallest element of S_{nk} is $(k, 0, 0, \dots, 0)$. We apply the given condition with $(s_1, s_2, \dots, s_n) = (k+1, 0, 0, \dots, 0)$, and obtain the equation $L(k, 0, 0, \dots, 0)x_1 = 0$. Thus $L(k, 0, 0, \dots, 0) = 0$. Assume that $L(t_1, \dots, t_n) = 0$ when $(t_1, \dots, t_n) < (r_1, \dots, r_n)$. Apply the given condition with $(s_1, s_2, \dots, s_n) = (r_1+1, r_2, \dots, r_n)$. We get $L(r_1, r_2, \dots, r_n)x_1 + \sum \{L(r_1+1, r_2, \dots, r_i-1, \dots, r_n)x_i \mid 1 \leq j \leq n, 1 \leq r_i\} = 0$. But each term in the sum is zero, since each $(r_1+1, r_2, \dots, r_i-1, \dots, r_n)$ precedes $(r_1, r_2, \dots, r_n)$ in the order. Thus we get $L(r_1, \dots, r_n) = 0$.

Lemma 3. Suppose that $\{x_1, x_2, \dots, x_r\}$ is a set of r distinct points in V which is open in R^n and let $D_{pa}(g) = D_p(g)(x_a)$ for $1 \leq a \leq r$, $p \in N^n$, and $g \in \mathfrak{D}^\infty(V, R)$. Suppose that $F_i : \mathfrak{D}^\infty(V, R^n) \rightarrow R$, $1 \leq i \leq m$, are linear and satisfy

$$1) F_i = \sum_{a=1}^r \sum_{j=1}^n \sum_{|p| \leq k} L_{ip}^{aj} D_{pa} \pi_j.$$

2) $\mathfrak{L} = \cap \{\text{Ker } F_i \mid 1 \leq i \leq m\}$ is a Lie subalgebra of $\mathfrak{D}^\infty(V, R^n)$.

3) For each a , $1 \leq a \leq r$, there is i, j , and p so that $L_{ip}^{aj} \neq 0$.

Then

$$\mathfrak{L} \subset \bigcap_{a=1}^r \{\xi \mid \xi(x_a) = 0\}.$$

Proof. If $\xi = (\xi_1, \dots, \xi_n)$ and $\eta = (\eta_1, \dots, \eta_n)$ are in $\mathfrak{D}^\infty(V, R^n)$, then

$$[\xi, \eta]_i = \sum_{j=1}^n \left(\frac{\partial \xi_i}{\partial x_j} \eta_j - \frac{\partial \eta_i}{\partial x_j} \xi_j \right).$$

Suppose that ξ and η are in $\mathfrak{L}$. For each b , $1 \leq b \leq r$, let $k_b = \max \{|p| : L_{ip}^{bj} \neq 0, 1 \leq j \leq n, 1 \leq i \leq m, |p| \leq k\}$. Now let a be fixed, $1 \leq a \leq r$. Choose i and j so that there is p which satisfies $|p| = k_a$ and $L_{ip}^{aj} \neq 0$. We now work with F_i . We have

$$F_i([\xi, \eta]) = \sum_{b=1}^r \sum_{c=1}^n \sum_{|p| \leq k_b} L_{ip}^{bc} D_{pb}([\xi, \eta]_c)$$

and

$$D_{pb}([\xi, \eta]_c) = \sum_{h=1}^n D_{pb} \left(\frac{\partial \xi_c}{\partial x_h} \right) \eta_h(x_b) - D_{pb} \left(\frac{\partial \eta_c}{\partial x_h} \right) \xi_h(x_b) + G_{pb,c}$$

where $G_{pb,c}$ involves only derivatives of ξ and η of order less than or equal to k_b and the sum over h is present only if $|p| = k_b$. Thus we have

$$\begin{aligned}
 0 = F_i([\xi, \eta]) &= \sum_{b=1}^r \sum_{c=1}^n \sum_{|p|=k_b} \sum_{h=1}^n \left\{ L_{ip}^{bc} D_{pb} \left(\frac{\partial \xi_c}{\partial x_h} \right) \eta_h(x_b) \right. \\
 &\quad \left. - L_{ip}^{bc} D_{pb} \left(\frac{\partial \eta_c}{\partial x_h} \right) \xi_h(x_b) \right\} + \sum_{b,c,p} L_{ip}^{bc} G_{pbce} .
 \end{aligned}$$

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By Lemma 1 we can choose $\bar{\xi}$ and $\bar{\eta}$ so that

$$D_{pb}(\pi_c \bar{\xi}) = D_{pb}(\pi_c \xi), \quad D_{pb}(\pi_c \bar{\eta}) = D_{pb}(\pi_c \eta) \quad \text{for } |p| \leq k_b \quad \text{and} \quad 1 \leq c \leq n,$$

$$D_{pb}(\pi_c \bar{\xi}) = 0 = D_{pb}(\pi_c \bar{\eta}) \quad \text{for } |p| > k_b \quad \text{and} \quad 1 \leq c \leq n.$$

Then $\bar{\xi}$ and $\bar{\eta}$ are in $\mathfrak{L}$ and replacing ξ by $\bar{\xi}$ and η by $\bar{\eta}$ in * gives the equation $\sum_{b,c,p} L_{ip}^{bc} G_{pbce} = 0$. Now leave $\bar{\eta}$ as described but choose $\bar{\xi}$ so that $D_{pb}(\pi_c \bar{\xi}) = D_{pb}(\pi_c \xi)$ for $|p| \leq k_b$, $1 \leq c \leq n$, $D_{pb}(\pi_c \bar{\xi}) = 0$ for $|p| > k_b$, $b \neq a$, $1 \leq c \leq n$, and $D_{pa}(\pi_c \bar{\xi}) = 0$ for $|p| > k_a$, $c \neq j$. Now using $\bar{\xi}$, $\bar{\eta}$ * becomes

$$\sum_{h=1}^n \sum_{p=k_a} L_{ip}^{aj} D_{pa} \left(\frac{\partial \bar{\xi}_j}{\partial x_h} \right) \eta_h(x_a) = 0.$$

This can be rewritten as

$$\sum_{q=k_a+1} D_q \bar{\xi}_j(x_a) \left(\sum_{\substack{1 \leq h \leq n \\ 1 \leq q_h}} L_{i(q_1, \dots, q_{h-1}, \dots, q_n)}^{aj} \eta_h(x_a) \right) = 0.$$

For each $q = (q_1, \dots, q_n)$ in S_{nk_a+1} we can further restrict $\bar{\xi}$ so that $D_{q'} \bar{\xi}_j(x_a) = 0$ for $q' \neq q$ but $D_q \bar{\xi}_j(x_a) = 1$. Thus we see that $q_1 + \dots + q_n = k_a + 1$ implies

$$\sum_{\substack{1 \leq h \leq n \\ 1 \leq q_h}} L_{i(q_1, \dots, q_{h-1}, \dots, q_n)}^{aj} \eta_h(x_a) = 0.$$

Since there is a p for which $|p| = k_a$ and $L_{ip}^{aj} \neq 0$ Lemma 2 gives that $\eta(x_a) = 0$. Thus $\mathfrak{L} \subset \{\eta \mid \eta(x_a) = 0\}$ for $1 \leq a \leq r$.

If $F: \Gamma_c^\infty(TX) \rightarrow R$ is linear we define the support of F by $\text{supp } F = \{x \mid \text{for every open set } V \text{ containing } x \text{ there is a vector field } \xi \text{ with compact support so that } \text{supp } \xi \subset V \text{ and } F(\xi) = 1\}$. Clearly $X \setminus \text{supp } F$ is an open set so $\text{supp } F$ is closed.

Lemma 4. *If $F: \Gamma_c^\infty(TX) \rightarrow R$ is linear, ξ has compact support, and $\text{supp } \xi \cap \text{supp } F = \emptyset$, then $F(\xi) = 0$. Thus $F \neq 0$ implies $\text{supp } F \neq \emptyset$.*

Proof. Since $\text{supp } \xi \subset X \setminus \text{supp } F$, for each x in $\text{supp } \xi$ there is an open set V_x which contains x and is such that $\text{supp } \eta \subset V_x$ implies $F(\eta) = 0$. Now $\text{supp } \xi$ is compact so there is a finite set $\{x_1, \dots, x_n\}$ which gives $\text{supp } \xi \subset \bigcup \{V_{x_i} \mid 1 \leq i \leq n\}$. Let $\{p_i \mid 1 \leq i \leq n\} \cup \{p\}$ be a partition of unity subordinate to the covering $\{V_{x_i} \mid 1 \leq i \leq n\} \cup \{X \setminus \text{supp } \xi\}$. Now

$$F(\xi) = F\left(\sum_{i=1}^n p_i \xi + p \xi\right) = \sum_{i=1}^n F(p_i \xi) + F(0) = 0.$$

Lemma 4 has a generalization to finite sets of vector fields and linear independence.

Lemma 5. Suppose that $F_1, F_2, \dots, F_n : \Gamma_c^\infty(TX) \rightarrow R$ is a set of n linearly independent linear functionals. Then there is a finite set $F \subset X$ so that $\text{card}(F) \leq n$ and for each $r > 0$ there are vector fields $\xi_1, \dots, \xi_n$ which satisfy

- 1) $\text{supp } \xi_i \subset \{x \mid d(x, F) < r\}$ for $1 \leq i \leq n$.
- 2) $F_1, F_2, \dots, F_n$ are linearly independent on the finite dimensional vector space $\text{span}(\xi_1, \dots, \xi_n)$.

Proof. The proof is by induction on n . For $n = 1$ the result follows immediately from the fact that the support of a nonzero linear functional is nonempty. Assume the result is true for $1, 2, \dots, n-1$. Given $F_1, F_2, \dots, F_n$ a linearly independent set we can choose a finite set $G \subset X$ which has the desired property with respect to the set $F_1, F_2, \dots, F_{n-1}$ and $\text{card}(G) \leq n-1$. Now choose $\eta_1, \eta_2, \dots, \eta_n$ from $\Gamma_c^\infty(TX)$ so that $F_1, F_2, \dots, F_n$ are linearly independent on $\text{span}(\eta_1, \dots, \eta_n)$, let $N = \bigcup \{\text{supp } \eta_i \mid 1 \leq i \leq n\}$, and choose an open set W so that $G \cup N \subset W \subset \bar{W}$ and $\bar{W}$ is compact. There are two cases:

1) $(\exists r > 0)(\exists (a_1, \dots, a_{n-1}))(\text{supp } \xi \subset \{x \mid d(x, G) < r \text{ implies } F_n(\xi) = \sum_{i=1}^{n-1} a_i F_i(\xi))$. Choose V open so that $G \subset V \subset \bar{V} \subset \{x \mid d(x, G) < r\} \cap W$, and let $K = \bar{W} \setminus V$. Suppose that $(\forall x \in K)(\exists r_x > 0)(\text{supp } \xi \subset \{y \mid d(x, y) < r_x\} \text{ implies } F_n(\xi) = \sum_{i=1}^{n-1} a_i F_i(\xi))$. Using the compactness of K and partitions of unity one could then show that $F_n = \sum_{i=1}^{n-1} a_i F_i$ on the vector space $\{\xi \mid \text{supp } \xi \text{ is a compact subset of } W\}$. But this is a contradiction since $\eta_1, \eta_2, \dots, \eta_n$ are in this set. Therefore we have that $(\exists x \in K)(\forall s > 0)(\exists \xi)(\text{supp } \xi \subset \{y \mid d(x, y) < s\} \text{ and } F_n(\xi) \neq \sum_{i=1}^{n-1} a_i F_i(\xi))$. Let $F = G \cup \{x\}$. Then $\text{card}(F) \leq n$. Let $s > 0$. Choose $\xi_1, \dots, \xi_{n-1}$ so that $\text{supp } \xi_i \subset \{x \mid d(x, G) < r \text{ and } d(x, G) < s\}$ and $F_1, F_2, \dots, F_{n-1}$ are linearly independent on $\text{span}(\xi_1, \dots, \xi_{n-1})$. By the choice of x we can choose ξ_n so that $\text{supp } \xi_n \subset \{y \mid d(x, y) < s\}$ and $F_n(\xi_n) \neq \sum_{i=1}^{n-1} a_i F_i(\xi_n)$. It follows that $F_1, \dots, F_n$ are linearly independent on $\text{span}(\xi_1, \dots, \xi_n)$.

2) $(\forall r > 0)(\forall (a_1, \dots, a_{n-1}))(\exists \xi)(\text{supp } \xi \subset \{x \mid d(x, G) < r\} \text{ and } F_n(\xi) \neq \sum_{i=1}^{n-1} a_i F_i(\xi))$. In this case we can take $F = G$. Let $r > 0$. Choose $\xi_1, \dots, \xi_{n-1}$ so that $\text{supp } \xi_i \subset \{x \mid d(x, G) < r\}$ and $F_1, \dots, F_{n-1}$ are linearly independent on $\text{span}(\xi_1, \dots, \xi_{n-1})$. Now there is $(a_1, \dots, a_{n-1})$ so that $F_n = \sum_{i=1}^{n-1} a_i F_i$ on $\text{span}(\xi_1, \dots, \xi_{n-1})$. Choose ξ_n so that $\text{supp } \xi_n \subset \{x \mid d(x, G) < r\}$ and $F_n(\xi_n) \neq \sum_{i=1}^{n-1} a_i F_i(\xi_n)$. Then $F_1, \dots, F_n$ are linearly independent on $\text{span}(\xi_1, \dots, \xi_n)$.

Lemma 6. Suppose that V is open in R^n and $F: \mathcal{D}^\infty(V, R^n) \rightarrow R$ is linear and satisfies $F([\xi, \eta]) = 0$ for all ξ, η in $\mathcal{D}^\infty(V, R^n)$. Then $F = 0$.

Proof. We will use $\mathcal{E}^\infty(V, R^n)$ to denote the space of all C^∞ -mappings of V into R^n . If $\xi \in \mathcal{D}^\infty(V, R^n)$ and $\eta \in \mathcal{E}^\infty(V, R^n)$ there is $\eta_1 \in \mathcal{D}^\infty(V, R^n)$ so that $\eta_1(x) = \eta(x)$ for x in a neighborhood of $\text{supp } \xi$. Then $F([\xi, \eta]) = F([\xi, \eta_1]) = 0$ so that we have $F([\xi, \eta]) = 0$ for all ξ in $\mathcal{D}^\infty(V, R^n)$ and η in $\mathcal{D}^\infty(V, R^n)$. Now, with obvious notation, we have $F([\xi, \eta]) = \sum_{i=1}^n F_i([\xi, \eta]_i)$ where

$$[\xi, \eta]_i = \sum_{j=1}^n \left(\frac{\partial \xi_j}{\partial x_i} \eta_j - \frac{\partial \eta_j}{\partial x_i} \xi_j \right).$$

Let i be fixed, and take $\xi = (0, 0, \dots, \xi_i, \dots, 0)$, $\eta = (0, 0, \dots, \eta_i, \dots, 0)$. Then

$$F([\xi, \eta]) = F_i \left(\frac{\partial \xi_i}{\partial x_i} \eta_i - \frac{\partial \eta_i}{\partial x_i} \xi_i \right) = 0.$$

- 1) Taking $\eta_i = 1$ gives $F_i(\partial \xi_i / \partial x_i) = 0$ so we get $F_i(\xi_i + x_i(\partial \xi_i / \partial x_i)) = F_i(\partial / \partial x_i)(x_i \xi_i) = 0$ for all ξ_i in $\mathfrak{D}^\infty(V, R)$.
 - 2) Taking $\eta_i = x_i$ gives $0 = F_i((\partial \xi_i / \partial x_i)x_i - \xi_i)$ for all ξ_i in $\mathfrak{D}^\infty(V, R)$.
 - 3) Subtracting 2) from 1) gives $0 = 2F_i(\xi_i)$ for all ξ_i in $\mathfrak{D}^\infty(V, R)$.
- Thus $F_i = 0$ for $1 \leq i \leq n$. Thus $F = 0$.

Corollary 1. Suppose that $\mathfrak{L} \subset \Gamma_c^\infty(TX)$ is a vector subspace of codimension 1. Then there are ξ and η so that $[\xi, \eta] \notin \mathfrak{L}$.

Proof. There is $F: \Gamma_c^\infty(TX) \rightarrow R$ so that $\mathfrak{L} = \text{Ker } F$. By Lemma 4 there is $x \in \text{supp } F$. Let (V, a) be a coordinate chart and let $\mathfrak{L}_V = \{\xi \mid \xi \in \Gamma_c^\infty(TX) \text{ and } \text{supp } \xi \subset V\}$. There is a natural identification $\mathfrak{L}_V = \mathfrak{D}^\infty(a(V), R^n)$. Let $\bar{F} = F|_{\mathfrak{L}_V}$. Then $\bar{F}: \mathfrak{D}^\infty(a(V), R^n) \rightarrow R$, $\mathfrak{L} \cap \mathfrak{L}_V = \text{Ker } F \cap \mathfrak{L}_V = \text{Ker } \bar{F}$, and since $x \in \text{supp } F$ we know that $\bar{F} \neq 0$. Lemma 6 shows that there is ξ and η in $\mathfrak{L}_V$ so that $F([\xi, \eta]) = \bar{F}([\xi, \eta]) \neq 0$.

Lemma 7. Suppose that $\mathfrak{L} \subset \Gamma_c^\infty(TX)$ is a Lie subalgebra of finite codimension, and that $F_1, F_2, \dots, F_n: \Gamma_c^\infty(TX) \rightarrow R$ is a set of linearly independent linear functionals so that $\mathfrak{L} = \bigcap \{\text{Ker } F_i \mid 1 \leq i \leq n\}$. Then $\bigcup \{\text{supp } F_i \mid 1 \leq i \leq n\}$ is finite.

Proof. The proof is by induction on n , the codimension of $\mathfrak{L}$ (note that this is not assumed to be related to the dimension of X).

1) $n = 1$. Choose x in $\text{supp } F_1$. Let $r > 0$. Choose ξ so that $\text{supp } \xi \subset \{y \mid d(x, y) < r\}$ and $F_1(\xi) = 1$. Suppose that $\text{supp } \eta \subseteq \{y \mid d(x, y) > r\}$. If γ is in $\mathfrak{L}$ construct γ_1 so that γ_1 agrees with γ on a neighborhood of $\text{supp } \eta$ and $\text{supp } \gamma_1 \subset \{y \mid d(x, y) > r\}$. Now $[\gamma_1, \eta] - F_1(\gamma_1)[\xi, \eta] - F_1(\eta)[\gamma_1, \xi] = [\gamma_1 - F_1(\gamma_1)\xi, \eta - F_1(\eta)\xi] \in \mathfrak{L}$ so we get that $[\gamma, \eta] = [\gamma_1, \eta] \in \mathfrak{L}$, the last two terms of the sum being zero by support considerations. Assume that $F_1(\eta) \neq 0$. Now we can write $\Gamma_c^\infty(TX) = \mathfrak{L} \oplus R\eta$. By Corollary 1 there are ξ_0 and ξ_1 so that $[\xi_0, \xi_1] \notin \mathfrak{L}$. Writing $\xi_0 = \gamma_0 + r_0\eta$ and $\xi_1 = \gamma_1 + r_1\eta$ gives $[\xi_0, \xi_1] = [\gamma_0, \gamma_1] + r_0[\eta, \gamma_1] + r_1[\gamma_0, \eta] \in \mathfrak{L}$, a contradiction. Thus we must have $F_1(\eta) = 0$. We have shown that $\{y \mid d(x, y) > r\} \subset X \setminus \text{supp } F_1$. Since r was arbitrary we get that $\text{supp } F_1 = \{x\}$.

2) the inductive step. Assume the result is true for $1, 2, \dots, n-1$. Let M_1 be a finite subset of X which has the property with respect to $F_1, F_2, \dots, F_n$ given by Lemma 5. Let $V_1 = X \setminus M_1$ and $\mathfrak{L}_1 = \{\xi \mid \xi \in \Gamma_c^\infty(TX) \text{ and } \text{supp } \xi \subset V_1\}$. If $\mathfrak{L} \cap \mathfrak{L}_1 = \mathfrak{L}_1$, then $\bigcup \{\text{supp } F_i \mid 1 \leq i \leq n\} = M_1$ and we are done. Otherwise, let $\bar{F}_i = F_i|_{\mathfrak{L}_1}$. Then $\mathfrak{L} \cap \mathfrak{L}_1 = \bigcap \{\text{Ker } \bar{F}_i \mid 1 \leq i \leq n\}$. If $\bar{F}_1, \dots, \bar{F}_n$ are dependent, then by the induction hypothesis there is a finite set $G_1 \subset V_1$

so that $\cup \{\text{supp } \bar{F}_i \mid 1 \leq i \leq n\} = G_1$. Then $\cup \{\text{supp } F_i \mid 1 \leq i \leq n\} \subset M_1 \cup G_1$. If $\bar{F}_1, \dots, \bar{F}_n$ are independent we use Lemma 5 to choose a finite set $M_2 \subset X \setminus M_1$ and repeat the process. Either we obtain the desired finite set or we get $M_1, \dots, M_n$, n finite disjoint sets each having the property described in Lemma 5 with respect to $F_1, \dots, F_n$. In this latter case let $V = X \setminus \cup \{M_i \mid 1 \leq i \leq n\}$ and $\mathcal{L}_V = \{\xi \mid \xi \in \Gamma_c^\infty(TX) \text{ and } \text{supp } \xi \subset V\}$. Suppose that η_1 and η_2 are in $\mathcal{L}_V$. Let $r = \frac{1}{2}$ minimum $(\{d(M_i, M_j) \mid i \neq j\} \cup \{d(M_i, \text{supp } \eta_1) \mid 1 \leq i \leq n\} \cup \{d(M_i, \text{supp } \eta_2) \mid 1 \leq i \leq n\})$. We can choose $\xi_1, \dots, \xi_n$ so that $\text{supp } \xi_i \subset \{x \mid d(x, M_i) < r\}$ and $F_i(\xi_i) = \delta_{ii}$. Then $\eta_1 - \sum_{i=1}^n F_i(\eta_1)\xi_i$ and $\eta_2 - \sum_{i=1}^n F_i(\eta_2)\xi_i$ are in $\mathcal{L}$ so that

$$[\eta_1, \eta_2] - \sum_{i=1}^n F_i(\eta_1)[\xi_i, \eta_2] - \sum_{i=1}^n F_i(\eta_2)[\eta_1, \xi_i] + \sum_{i,j} F_i(\eta_1)F_j(\eta_2)[\xi_i, \xi_j]$$

is in $\mathcal{L}$. But all of these terms are zero except $[\eta_1, \eta_2]$ which is therefore in $\mathcal{L}$. We have shown that $\mathcal{L} \cap \mathcal{L}_V$ is a vector subspace of $\mathcal{L}_V = \Gamma_c^\infty(TV)$ so that $[\eta_1, \eta_2]$ is in $\mathcal{L} \cap \mathcal{L}_V$ for all η_1 and η_2 in $\mathcal{L}_V$. Corollary 1 now gives that $\mathcal{L} \cap \mathcal{L}_V = \mathcal{L}_V$. Thus $\Gamma_V \subset \text{Ker } F_i$ for each i so that we get $\text{supp } F_i \subset \cup \{M_j \mid 1 \leq j \leq n\}$ for each i .

We now describe the r -topologies on $\Gamma_c^\infty(TX)$. For each (V, a, W, r) where (V, a) is a coordinate chart for X , W is an open set so that $\bar{W} \subset V$ and $\bar{W}$ is compact, and r is a nonnegative integer we define a seminorm by $p(V, a, W, r) = \sum_{i=0}^r \text{lub } \{||\bar{\xi}^{(i)}(x)|| \mid x \in a(\bar{W})\}$ where $\bar{\xi}$ is the principal part of the local representation of ξ using the coordinate system (V, a) . The topology T_r on $\Gamma_c^\infty(TX)$ is the locally convex topology generated by the family of seminorms $\{p(V, a, W, r) \mid (V, a), W\}$ and the topology T_∞ is generated by $\{p(V, a, W, r) \mid (V, a), W, r\}$. We have that $T_1 \leq T_2 \leq T_3 \leq \dots \leq T_r \leq \dots \leq T_\infty$ and $(\Gamma_c^\infty(TX), T_\infty)' = \cup \{(\Gamma_c^\infty(TX), T_r)' \mid 0 \leq r < \infty\}$. For each $r \geq 0$ let $j_r(TX)$ denote the bundle of r -jets of sections of TX and let $j_r : \Gamma^\infty(TX) \rightarrow \Gamma(j_r(TX))$ be the natural mapping (see [1], 19–21). A local result concerning distributions ([2], 342) is true on manifolds. Its proof can be obtained from Theorem 2 page 342 of [2] and the use of coordinate charts with partitions of unity on X .

Lemma 8. Suppose that $F: \Gamma_c^\infty(TX) \rightarrow R$ is linear and continuous with respect to T_r . If ξ is in $\Gamma_c^\infty(TX)$ and is such that $j_r(\xi) \mid \text{supp } F = 0$, then $F(\xi) = 0$.

Theorem 1. Suppose that $\mathcal{L} \subset \Gamma_c^\infty(TX)$ is a Lie subalgebra of finite codimension which is closed with respect to T_∞ . Let $F_1, F_2, \dots, F_n: \Gamma_c^\infty(TX) \rightarrow R$ be a linearly independent set of T_∞ continuous linear functionals so that $\mathcal{L} = \cap \{\text{Ker } F_i \mid 1 \leq i \leq n\}$. Then $M = \cup \{\text{supp } F_i \mid 1 \leq i \leq n\}$ is finite and

$$\mathcal{L} \subset \bigcap_{x \in M} \{\xi \mid \xi(x) = 0\}.$$

Proof. By Lemma 7 we know that M is finite, say $M = \{x_1, x_2, \dots, x_s\}$. Choose coordinate charts (V_i, a_i) around x_i so that $V_i \cap V_j = \emptyset$ for $i \neq j$, $a_i(V_i) \cap a_j(V_j) = \emptyset$ for $i \neq j$, and $\bar{V}_i$ is compact for $1 \leq i \leq s$. Let $V =$

$\cup \{V_i \mid 1 \leq i \leq s\}$ and define $a: V \rightarrow R^m$ by $a(x) = a_i(x)$ if $x \in V_i$. Let $\mathcal{L}_V = \{\xi \mid \xi \in \Gamma_c^\infty(TX) \text{ and } \text{supp } \xi \subset V\}$. There is a natural identification $\mathcal{L}_V = \mathcal{D}^\infty(a(V), R^m)$ using a . Now there is an r so that each F_k is continuous with respect to T_r . Let $\bar{F}_k = F_k \mid \mathcal{L}_V$. Lemma 8 shows that each $\bar{F}_k$ is of the form described in the hypothesis of Lemma 3. Thus, by Lemma 3 we have $\mathcal{L} \cap \mathcal{L}_V = \cap \{\text{Ker } \bar{F}_k \mid 1 \leq k \leq n\} \subset \{\xi \mid \xi \in \mathcal{L}_V \text{ and } \xi(x) = 0 \text{ for each } x \text{ in } M\}$. For each i , $1 \leq i \leq s$, choose W_i open so that $x_i \in W_i \subset \bar{W}_i \subset V_i$ and let $p: X \rightarrow [0, 1]$ be C^∞ so that $\cup W_i \subset p^{-1}(1)$ and $\text{supp } p \subset V$. Then for all $\xi \in \Gamma_c^\infty(TX)$ we have that $\xi \in \mathcal{L}$ if and only if $p\xi \in \mathcal{L}$. Thus $\mathcal{L} \subset \{\xi \mid \xi(x_i) = 0, 1 \leq i \leq s\}$.

We have shown that every C^∞ -closed Lie subalgebra of finite codimension in $\Gamma_c^\infty(TX)$ is contained in the finite intersection of subalgebras each of which is the set of vector fields whose 0-jet vanishes at a certain point. It might be thought that all closed Lie subalgebras of finite codimension can be obtained as the finite intersection, $\cap \mathcal{L}_i$, where there are points x_i and integers r_i so that $\mathcal{L}_i = \{\xi \mid j_{r_i}(\xi)(x_i) = 0\}$. The following example shows that this is not true. Take $X = S^1$, the circle. There are identifications $\Gamma^\infty(S^1) = \{f: [-\pi, \pi] \rightarrow \mathbb{R} \mid f^{(r)}(\pi) = f^{(r)}(-\pi) \text{ for all } r\}$ and $[f, g](x) = f'(x)g(x) - f(x)g'(x)$. Let $\mathcal{L} = \{f \mid f(0) = 0 \text{ and } f'(0) = f''(0)\}$.

FOOTNOTE

1. We should remark that it is more common to take $[\xi, \eta](x) = \eta'(x)\xi(x) - \xi'(x)\eta(x)$. Clearly both definitions give the same Lie Subalgebras and for the considerations of this paper it does not matter which definition we use.

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