Functions Whose Fourier Series Converge Uniformly for Every Change of Variable

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1. Let f be a continuous function of period 2π . We will say that $f \in GW$ if the Fourier series of $f \circ g$ converges everywhere for every homeomorphism g of $[-\pi, \pi]$ with itself, and that $f \in UGW$ if the series of $f \circ g$ always converges uniformly. Goffman and Waterman [4] determined a necessary and sufficient condition for a function to belong to GW . In this note we give a necessary and sufficient condition for a function to belong to UGW . We also give an example of a function which is in GW but not in UGW .

Since bounded variation is conserved under change of variable, it follows from Jordan's theorem that UGW must contain all continuous functions of bounded variation. More generally, it must contain every continuous function of Φ -bounded variation in the sense of L. C. Young for which the complementary function Ψ satisfies the Salem condition $\sum \Psi(i^{-1}) < \infty$ [5], as well as the continuous functions with logarithmically integrable Banach indicatrix considered by Garsia and Sawyer [3]. Recently Waterman [6] introduced a change of variable invariant Banach space, the functions of "harmonic bounded variation," which contains all these Φ -BV classes, but whose continuous members still have uniformly convergent Fourier series. In the other direction, when $\sum \Psi(i^{-1}) = \infty$, Baernstein [1] has constructed a continuous function of Φ -bounded variation whose Fourier series diverges at a point. Thus Φ -BV $\subset UGW$ if and only if $\sum \Psi(i^{-1}) = \infty$, and also $HBV \subset UGW$.

2. As in [4], our condition may be phrased in terms of "systems of intervals" at a point. For each n , let I_{nm} , $m = 1, 2, \dots, k_n$, be disjoint closed intervals such that $I_{n,m-1}$ is to the left of I_{nm} . Let there be a real x such that for every $\delta > 0$ there is an N such that $I_{nm} \subset (x, x + \delta)$ whenever $n > N$. Then the collection $\mathcal{I} = \{I_{nm}, n = 1, 2, \dots; m = 1, 2, \dots, k_n\}$ is called a right system of intervals at x . A left system is defined similarly. We remark that Goffman and Waterman assume in their definition of "system" that $\lim_{n \rightarrow \infty} k_n = \infty$ and $\lim_{n \rightarrow \infty} n^{-1}k_n = 0$. In fact, the validity of (1) below for all systems is in-

dependent of the choice of definition. To see this, assume that (1) holds for all systems in the sense of [4]. Given a system $\{I_{nm}\}$ in the present sense, choose an increasing sequence $\{h_n\}$ of positive integers such that $k_n/h_n \rightarrow 0$. Let $J_{pm} = I_{nm}$ for $h_n \leq p < h_{n+1}$, $1 \leq m \leq k_n$, and, when $k_n < n$, let J_{pm} be a suitably chosen single point for $k_n < m \leq n$. Then $\{J_{pm}\}$ is a system in the sense of [4]; hence (1) holds for $\{J_{pm}\}$ and, therefore, for $\{I_{nm}\}$ as well.

If $I = [a, b]$, write $f(I) = f(b) - f(a)$. Goffman and Waterman's characterization [4] is that $f \in GW$ if and only if

$$(1) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} \frac{1}{i} f(I_{ni}) = 0$$

for every system $\mathcal{I}$. For UGW we have the following result.

Theorem. *Let f be a continuous function of period 2π . Then $f \circ g$ has a uniformly convergent Fourier series for every homeomorphism g of $[-\pi, \pi]$ with itself if and only if both (1) and*

$$(2) \quad \lim_{n \rightarrow \infty} \sum_{i=1}^{k_n} \frac{1}{k_n + 1 - i} f(I_{ni}) = 0$$

hold for every system $\mathcal{I}$.

These conditions can be phrased in terms of ϵ 's and δ 's instead of systems. This is the content of the following

Proposition. *Let f be a continuous periodic function. Then*

A. (1) holds for every system $\mathcal{I}$ if and only if given $\epsilon > 0$ and x there exists $\delta > 0$ such that for every finite sequence $\{I_i\}$ of non-overlapping intervals indexed from left to right (right to left) with $\cup I_i \subset (x, x + \delta)$ ($\cup I_i \subset (x - \delta, x)$) we have

$$(3) \quad \sum i^{-1} |f(I_i)| < \epsilon.$$

B. (1) and (2) hold for every system $\mathcal{I}$ if and only if given $\epsilon > 0$ there exists $\delta > 0$ such that (3) holds whenever $\{I_i\}$ is a finite sequence of non-overlapping intervals indexed from right to left or left to right with

$$\text{diam}(\cup I_i) < \delta.$$

The condition in *B* is a uniform version of the one in *A*. This provides an illustration of how UGW functions are those that belong "uniformly" to GW . Note also that there is no restriction on the number of intervals in $\{I_i\}$.

3. We now prove part *B* of the proposition. Part *A* will not be used in the sequel, but the reader will perhaps notice that our proof of *B* in fact includes one of *A*.

One direction in the proof is simple. If (1) fails for some right system, then

$$\limsup_{n \rightarrow \infty} \sum_i i^{-1} (|f(I_{ni})|) \geq \limsup_{n \rightarrow \infty} |\sum_i i^{-1} f(I_{ni})| = 2\epsilon > 0.$$

Since $\text{diam} (\bigcup_i I_{n_i}) \rightarrow 0$ as $n \rightarrow \infty$, (3) fails for interval sets of arbitrarily small diameter indexed from left to right. If (2) fails for some right system, then by considering for each n the sequence

$$\{I_i^n\} = \{I_{n, k_n - i + 1}\},$$

we again violate (3), this time with interval sets indexed from right to left. The argument if (1) or (2) fails for some left system is similar.

For the other direction, assume that (3) fails for some $\epsilon > 0$ and some sequence of interval sets $\{I_i^n\}$, with $\text{diam} (\bigcup_i I_i^n) \rightarrow 0$. We can assume that $\{I_i^n\}$ is indexed from left to right for each n . For fixed n , by deleting some of the I_i^n and relabelling others we can assume all the $f(I_i^n)$ have the same sign, say $f(I_i^n) > 0$, and still

$$(4) \quad \sum_i i^{-1} f(I_i^n) > \epsilon/2.$$

Let a_n be the left endpoint of $I_{n,1}$. We can find a subsequence $\{a_{n_i}\}$ of $\{a_n\}$ such that a_{n_i} converges monotonically to a point x . If all $a_{n_i} > x$, then the interval sets $\{I_{i,i}\} = \{I_i^{n_i}\}$ form a right system at x for which (1) is violated. If $a_{n_i} = x$ for all large j , then, since f is continuous, we can move $I_1^{n_i}$ slightly to the right so that (4) still holds and $\{I_i^{n_i}\}$ again forms a right system at x for which (1) is violated.

Now assume $a_{n_i} < x$ for all j . Let $m_j (\geq 1)$ be the greatest i such that the left endpoint of $I_i^{n_j}$ is $\leq x$. Then $f(I_{m_j}^{n_j}) \rightarrow 0$, by continuity of f , so for each sufficiently large j we have either

$$(5) \quad \sum_{i > m_j} i^{-1} f(I_i^{n_j}) > \epsilon/5$$

or

$$(6) \quad \sum_{i < m_j} i^{-1} f(I_i^{n_j}) > \epsilon/5.$$

If (5) holds for infinitely many j , we have a right system violating (1). If (6) holds for infinitely many j , then

$$\{I_{i,i}\} = \{I_{m_j - i}^{n_j}\}_{i=1}^{m_j - 1}$$

is a left system violating (2). This completes the proof of the proposition.

4. As in [4], the proof of our theorem uses well-known results of Salem [5, chap. 6] on uniform convergence, although our procedure is different from that in [4]. Define

$$(7) \quad \begin{aligned} Q_n^R(x) &= \sum_{i=1}^{n-2} i^{-1} \int_0^\pi \left[f\left(x + \frac{t + i\pi}{n}\right) - f\left(x + \frac{t + (i+1)\pi}{n}\right) \right] \sin t \, dt, \\ Q_n^L(x) &= \sum_{i=1}^{n-2} i^{-1} \int_0^\pi \left[f\left(x - \frac{t + i\pi}{n}\right) - f\left(x - \frac{t + (i+1)\pi}{n}\right) \right] \sin t \, dt, \end{aligned}$$

where n is odd and $\sum'$ means sum over odd indices only. From the discussion

in [2, p. 305] it is clear that Salem's method leads to

$$(8) \quad f(x) - S_n(x) = \pi^{-2}(Q_n^R(x) + Q_n^L(x)) + o(1) \quad (n \rightarrow \infty),$$

uniformly in x , whenever f is continuous and periodic.

Assume that f satisfies the conditions of part B of the proposition. Fix n and x . Define

$$I_i(t) = \left[x + \frac{t + (2i-1)\pi}{n}, x + \frac{t + 2i\pi}{n} \right],$$

for $0 \leq t \leq \pi$, $1 \leq i \leq \frac{1}{2}(n-1)$. Let $\epsilon > 0$ be given. Choose δ as in part B of the proposition. Let m be the integer such that

$$\pi^{-1}n\delta - 1 \leq 2m < \pi^{-1}n\delta + 1.$$

Then $\text{diam}(\bigcup_{i=1}^m I_i(t)) = (2m-1)\pi n^{-1} < \delta$ for every t , so from (3) it follows that, for all x , we have

$$\begin{aligned} |Q_n^R(x)| &\leq \int_0^\pi \sum_{i=1}^{(n-1)/2} (2i-1)^{-1} |f(I_i(t))| \sin t \, dt \\ &\leq 2\epsilon + 2\omega(f, \pi/n) \sum_{i=m+1}^{(n-1)/2} i^{-1}. \end{aligned}$$

Here we have also used $(2i-1)^{-1} < i^{-1}$ and $\int_0^\pi \sin t \, dt = 2$. Since $(m+1) > (2\pi)^{-1}n\delta$, it follows that

$$\sum_{i=m+1}^{(n-1)/2} i^{-1} = O(1) \quad (n \rightarrow \infty).$$

So, for all sufficiently large n ,

$$\sup_x |Q_n^R(x)| < 3\epsilon.$$

A similar analysis is valid for $Q_n^L(x)$, so it follows from (8) that the Fourier series of f converges uniformly. Since the conditions of the proposition are clearly preserved under change of variable, we have proven our theorem in one direction.

5. To prove the theorem in the other direction, we note that if (1) fails for some system, then Goffman and Waterman [4] have already constructed a homeomorphism g such that the Fourier series of $f \circ g$ diverges at a point, so it suffices to consider the case when (2) fails. We may assume it fails for a right system at $x = 0$, from which we deduce the existence of $\alpha > 0$, positive even integers k_p , and points $x_{p,i}$ ($0 \leq i \leq k_p$, $p = 1, 2, \dots$) with

$$0 < x_{p+1,0} < x_{p,k_p} \leq x_{p,i+1} < x_{p,i} \leq x_{1,0} < \pi,$$

such that $x_{p,0} \searrow 0$, and

$$(9) \quad \sum'_{i=1}^{k_p-1} i^{-1} [f(x_{p,i+1}) - f(x_{p,i})] > \alpha$$

(recall $\sum'$ means sum over odd indices only), and also

$$(10) \quad |f(x) - f(y)| < \alpha \cdot 4^{-p} \quad (0 \leq x \leq y \leq x_{p,0}).$$

Choose an increasing sequence of odd integers $\{n_p\}$ such that

$$(11) \quad n_p > 2^{p+2}k_p \quad \text{and} \quad \lim_{p \rightarrow \infty} \omega(\pi/n_p, f) \log p = 0.$$

Let $v_{p,i}$ and $y_{p,i}$ ($1 \leq i \leq k_p$) be points satisfying

$$\frac{\pi}{2^p} - \frac{(i+1)\pi}{n_p} < v_{p,i} < \frac{\pi}{2^p} - \frac{i\pi}{n_p}, \quad x_{p,i+1} < y_{p,i} < x_{p,i},$$

for $1 \leq i \leq k_p - 1$, and

$$\frac{\pi}{2^{p+1}} < v_{p,k_p} < \frac{\pi}{2^p} - \frac{k_p\pi}{n_p}, \quad x_{p+1,0} < y_{p,k_p} < x_{p,k_p}.$$

Let r_p and q_p ($p \geq 2$) be the unique odd integers such that

$$(12) \quad r_p < 2^{-p-1}n_p \leq r_p + 2,$$

$$(13) \quad q_p < 2^{-p}n_p - k_{p-1} \leq q_p + 2.$$

Let $x = g(u)$ be a strictly increasing continuous function on $[-\pi, \pi]$ such that

$$g(u) = u \quad (-\pi \leq u \leq 0), \quad g(\pi) = \pi,$$

$$g\left(\frac{\pi}{2^p} - \frac{\pi i}{n_p}\right) = x_{p,i}, \quad g(v_{p,i}) = y_{p,i} \quad (0 \leq i \leq k_p, p = 1, 2, \dots).$$

Extend g to $(-\infty, \infty)$ by periodicity, and put $F = f \circ g$. By choosing $v_{p,i}$ close enough to $2^{-p}\pi - n_p^{-1}(i+1)\pi$ and $y_{p,i}$ close enough to $x_{p,i}$ ($1 \leq i \leq k_p - 1$) we render

$$(14) \quad \left| \int_0^\pi F\left(2^{-p}\pi - \frac{t+i\pi}{n_p}\right) \sin t \, dt - 2f(x_{p,i}) \right| < 1/p \cdot 2^{i+1}.$$

(Recall that $\int_0^\pi \sin t \, dt = 2$). We assume further that

$$v_{p,k_p} \leq 2^{-p-1}\pi + n_{p+1}^{-1}\pi.$$

By choosing y_{p,k_p} close enough to x_{p,k_p} we cause (14) to hold when $i = k_p$ and also cause, for $0 \leq t \leq \pi$,

$$(15) \quad \left| F\left(2^{-p}\pi - \frac{t+i\pi}{n_p}\right) - F\left(2^{-p}\pi - \frac{t+(i+1)\pi}{n_p}\right) \right| < (pr_p)^{-1},$$

$$(k_p \leq i \leq r_p - 2),$$

and

$$(16) \quad \left| F\left(2^{-p-1}\pi + \frac{t+i\pi}{n_{p+1}}\right) - F\left(2^{-p-1}\pi + \frac{t+(i+1)\pi}{n_{p+1}}\right) \right| < (pq_{p+1})^{-1},$$

$$(1 \leq i \leq q_{p+1} - 2).$$

Here r_p and q_p are defined by (12) and (13). We claim that, with the notation of (7) and F in place of f ,

$$(17) \quad Q_{n_p}^L(2^{-p}\pi) \geq 2\alpha + o(1) \quad (p \rightarrow \infty),$$

and

$$(18) \quad |Q_{n_p}^R(2^{-p}\pi)| \leq \alpha + o(1) \quad (p \rightarrow \infty).$$

Here α is as in (9). In view of (8), once (17) and (18) are established, it will follow that the Fourier series of $F' = f \circ g$ does not converge uniformly, and the proof of the theorem will be complete.

Split the sum defining $Q_{n_p}^L(2^{-p}\pi)$ as follows:

$$\begin{aligned} Q_{n_p}^L(2^{-p}\pi) &= \sum_{i=1}^{k_p-1} + \sum_{i=k_p+1}^{r_p-2} + \sum_{i=r_p}^{r_p^*-2} + \sum_{i=r_p^*}^{r_p^*} + \sum_{i=r_p^*+2}^{n_p-2} \\ &= I + II + III + IV + V. \end{aligned}$$

Here r_p is defined by (12), and r_p^* is the unique odd integer such that $r_p^* < 2^{-p}n_p \leq r_p^* + 2$. It follows from (14) and (9) that

$$I > \sum_{i=1}^{k_p-1} i^{-1}[(2f(x_{p,i}) - 2f(x_{p,i+1})) - p^{-1} \cdot 2^{-i}] > 2\alpha - p^{-1}.$$

In II there are less than r_p terms, so (15) implies that $|II| < p^{-1}$. From (10) it follows that each integral in III has absolute value less than $2\alpha \cdot 4^{-p}$. Since $(r_p^* - 2)/r_p < 3$, it follows that $III = O(4^{-p})$ as $p \rightarrow \infty$. Since F is continuous and $n_p \rightarrow \infty$, it follows that the single term $IV = o(1)$. Since $n_p/(2 + r_p^*) < 2^p$ and $g(u) = u$ on $[-\pi, 0]$, we can replace F by f in each of the integrals of V . Using, in addition, the second statement in (11), we have

$$V = O(\omega(\pi/n_p, f) \log p) = o(1) \quad (p \rightarrow \infty).$$

The validity of (17) follows from the above considerations.

For the proof of (18), define, for fixed p , odd integers β_j ($j = 1, \dots, p-1$) by $\beta_j < 2^{-p}n_p(2^{p-j} - 1) \leq \beta_j + 2$. Split the sum defining $Q_{n_p}^R(2^{-p}\pi)$ as follows:

$$\begin{aligned} Q_{n_p}^R(2^{-p}\pi) &= \sum_{i=1}^{q_p-2} + \sum_{i=q_p}^{(\beta_{p-1})-2} + \sum_{j=2}^{p-1} \sum_{i=\beta_{p-j}+1}^{(\beta_{p-j})-2} + \sum_{i=\beta_1}^{n_p-2} \\ &= Z_0 + Z_1 + \sum_{j=2}^{p-1} Z_j + Z_p \end{aligned}$$

Here q_p is defined by (13).

From (16), with p in place of $p + 1$, it follows that $Z_0 = o(1)$ as $p \rightarrow \infty$. From (10) it follows that

$$(19) \quad |Z_j| < 2\alpha \cdot 4^{-(p-j)} \sum' i^{-1} \quad (j = 1, \dots, p-1)$$

where the limits on $\sum'$ are those used in defining Z_j . For $j \geq 2$ we have $(\beta_{p-j} - 2)/\beta_{p-j+1} < 4$. It is also true that $(\beta_{p-1} - 2)/q_p < 3$. The last inequality follows from the inequality in (11) with $p - 1$ in place of p and the fact that $n_{p-1} < n_p$. Thus each $\sum'$ in (19) is less than $\log 4$, and we have

$$\sum_{j=1}^{p-1} |Z_j| < 2\alpha(\log 4) \sum_{j=1}^{p-1} 4^{-(p-j)} < \frac{2}{3}\alpha(\log 4) < \alpha.$$

Each integral in Z_p has absolute value less than $2\omega(\pi/n_p, F)$. Since $\beta_1 + 2 \geq \frac{3}{8}n_p$ ($p \geq 3$), it follows that

$$Z_p = O(\omega(\pi/n_p, F)) = o(1) \quad (p \rightarrow \infty).$$

We have shown that (18) holds, and thus the theorem is proven.

6. We give an example of a function which is in GW but not in UGW . Let $\{a_n\}$ and $\{b_n\}$ be sequences in $(0, \pi)$ with $a_{n+1} < b_{n+1} < a_n < b_1$ and $a_n \searrow 0$. Let $\{\lambda_n\}$ be a decreasing positive sequence with $\sum \lambda_n < \infty$. Choose a sequence $\{k_n\}$ such that

$$(20) \quad \lambda_n \sum_{i=1}^{k_n} [i \log(i+1)]^{-1} \geq 1.$$

Let $x_{n,i} = a_n + [(b_n - a_n)i/2k_n]$, $0 \leq i \leq 2k_n$. For $n = 1, 2, \dots$ define,

$$f(x_{n,i}) = \lambda_n [\log(x_n - \frac{1}{2}(i-3))]^{-1} \quad (i \text{ odd}, 1 \leq i \leq 2k_n - 1),$$

$$f(x_{n,i}) = 0 \quad (i \text{ even}, 0 \leq i \leq 2k_n).$$

Let f be linear on each interval $[x_{n,i}, x_{n,i+1}]$, and define $f = 0$ elsewhere in $[-\pi, \pi]$. Then f is continuous. Extend it to $(-\infty, \infty)$ by periodicity.

Because of (20), the sets of intervals $[x_{n,2i}, x_{n,2i+1}]$, $0 \leq i \leq k_n - 1$, is a right system at $x = 0$ for which (2) fails. Hence $f \notin UGW$.

On the other hand, f is of bounded variation in some neighborhood of each $x \neq 0$. Since f is continuous, it follows that (1) is satisfied for every system at each $x \neq 0$. Also, since $f = 0$ on $[-\pi, 0]$, (1) is satisfied for every left system at $x = 0$.

Suppose now that $\{I_i\}$ is a finite set of non-overlapping intervals indexed from left to right with $\cup I_i \subset (0, a_m)$. Then by a rather tedious argument with which we will not tax the reader we deduce that

$$(21) \quad \sum j^{-1} |f(I_j)| < 2 \sum_{n=m+1}^{\infty} \lambda_n \sum_{i=1}^{r_n} [i \log(r_n + 2 - i)]^{-1}$$

for some sequence of integers $\{r_n\}$. Let q_n be the integer satisfying

$$\frac{1}{2}r_n \leq q_n \leq \frac{1}{2}(r_n + 1).$$

Then

$$\begin{aligned} \sum_{i=1}^{r_n} [i \log (r_n + 2 - i)]^{-1} &= \sum_{i=1}^{q_n} + \sum_{i=q_n+1}^{r_n} \\ &\leq [\log (r_n + 2 - q_n)]^{-1} \sum_{i=1}^{q_n} i^{-1} + (\log 2)^{-1} \sum_{i=q_n+1}^{r_n} i^{-1} \\ &\leq [\log (q_n + 1)]^{-1}(1 + \log q_n) + (\log 2)^{-1} \log \left(\frac{r_n}{q_n} \right) \\ &< 2 + 1. \end{aligned}$$

Since $\sum \lambda_n < \infty$, the right hand side of (21) is $o(1)$ as $m \rightarrow \infty$, that is, as $a_m \rightarrow 0$. Hence $f \in GW$.

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