Toeplitz Operators and Division by Inner Functions

M. RABINDRANATHAN

Communicated by the Editors

In this paper we study Toeplitz operators on certain weighted sequence spaces. We consider the Hilbert space of all sequences square summable with respect to a nondecreasing sequence of positive weights. It is shown that the complex conjugate of a bounded analytic function in the unit disk induces a bounded Toeplitz operator on this space. This implies the following result on an analytic function: If the sequence of Fourier coefficients of a function in the Hardy class H^2 is square summable with respect to a weight of the above type, then the sequence of Fourier coefficients of its outer part has the same property. From this theorem we can deduce a recent theorem of Korenbljum [6] on the division of analytic functions with derivatives in H^2 by inner functions. It is also possible to obtain a simple proof of a result on the division of analytic functions with finite Dirichlet integral by inner functions. This last result follows also from the representation formula for the Dirichlet integral due to Carleson [2].

Some operator theoretic problems concerning Toeplitz operators on weighted sequence spaces are studied in §2.

The author wishes to express his warm thanks to Professor Allen Shields for his valuable suggestions.

§1. Let H^p ($1 \leq p \leq \infty$) denote the Hardy spaces of analytic functions in the unit disk in the complex plane. L^p ($1 \leq p \leq \infty$) will denote the Lebesgue spaces of the unit circle with Haar measure. The norm in L^p will be denoted by $\|\cdot\|_p$. The correspondence that sends a function in H^p to its boundary value function is an isometric isomorphism of H^p onto the subspace of L^p consisting of those functions whose negative Fourier coefficients vanish. (For these and other results on H^p theory, we refer the reader to Hoffman's book [5].) We can thus regard H^p as a subspace of L^p . We shall use the same notation for a function in H^p and its boundary value function.

Let P be the orthogonal projection of L^2 onto H^2 . If φ is in L^∞ the Toeplitz operator T_φ is defined as the (bounded) operator on H^2 that sends f in H^2 to $P(\varphi f)$. Basic properties of these operators can be found in Halmos [4, Chapter 20].

We shall now consider some generalizations of these operators. Let $\{\rho(n)\}$, $n = 0, 1, 2, \dots$, be a nondecreasing sequence of positive numbers. An analytic function $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is said to be in the space $H^2(\rho)$ if

$$(1) \quad |||f|||_{\rho} = \left[\sum_{n=0}^{\infty} |a_n|^2 \rho(n) \right]^{1/2}$$

is finite. With the norm defined in (1) $H^2(\rho)$ is a Hilbert space isomorphic to the Hilbert space $\ell^2(\rho)$ of all one sided infinite sequences of complex numbers that are square summable with respect to ρ . (See [4, p. 48] for the construction of the space $\ell^2(\rho)$.) If $\rho(n) = 1$ for all n , the space $H^2(\rho)$ coincides with H^2 . Since the weight sequence ρ is positive and nondecreasing, every function in $H^2(\rho)$ automatically belongs to H^2 .

1.1. Theorem. *Let ρ be a nondecreasing sequence of positive numbers and let φ be an element of H^{∞} . Then for every f in $H^2(\rho)$,*

$$(2) \quad |||P(\bar{\varphi}f)|||_{\rho} \leq ||\varphi||_{\infty} |||f|||_{\rho}.$$

Thus the Toeplitz operator T_{φ} is a bounded operator of $H^2(\rho)$ into itself.

Proof. We shall first consider the case when φ is a polynomial; $\varphi(z) = \varphi_0 + \varphi_1 z + \dots + \varphi_k z^k$. Denote $P(\bar{\varphi}f)$ by g . Let $\{a_n\}$ and $\{b_n\}$, $n = 0, 1, 2, \dots$, be the sequences of Fourier coefficients of f and g respectively. An easy calculation shows that $b_n = \bar{\varphi}_0 a_n + \bar{\varphi}_1 a_{n+1} + \dots + \bar{\varphi}_k a_{n+k}$. Inequality (2) can be rewritten as follows:

$$(3) \quad \sum_{n=0}^{\infty} |b_n|^2 \rho(n) \leq ||\varphi||_{\infty}^2 \sum_{n=0}^{\infty} |a_n|^2 \rho(n).$$

Let B be the backward shift operator on the space $\ell^2(\rho)$; if $\alpha = (\alpha_0, \alpha_1, \dots)$, $B\alpha = (\alpha_1, \alpha_2, \dots)$. Since the weight sequence is nondecreasing, $||B|| \leq 1$. Let $\bar{\varphi}(z) = \bar{\varphi}_0 + \bar{\varphi}_1 z + \dots + \bar{\varphi}_k z^k$. If we denote the sequences $\{a_k\}$ and $\{b_k\}$ by a and b respectively,

$$\bar{\varphi}(B)a = b.$$

Let us denote the norm in $\ell^2(\rho)$ again by $|||\cdot|||_{\rho}$. By von Neumann's inequality for contractions on Hilbert space [4, p. 123, problem 180],

$$(4) \quad |||b|||_{\rho} \leq ||\bar{\varphi}||_{\infty} |||a|||_{\rho}$$

where $||\bar{\varphi}||_{\infty} = \sup_{|z| \leq 1} |\bar{\varphi}(z)|$. It is easily seen that $||\bar{\varphi}||_{\infty} = ||\varphi||_{\infty}$. Thus inequality (4) is nothing but inequality (3).

We shall now extend this result to an arbitrary φ in H^{∞} . Choose a sequence $\{\varphi_n\}$ of polynomials such that $||\varphi_n||_{\infty} \leq ||\varphi||_{\infty}$ and such that $\varphi_n(e^{i\theta}) \rightarrow \varphi(e^{i\theta})$ for almost every θ in $[0, 2\pi)$. (A subsequence of the Cesàro means of the partial sums of the Fourier series of φ would suffice [5, p. 17].) Let $g_n = P(\bar{\varphi}_n f)$. If z is in the open unit disk,

$$g_n(z) = \frac{1}{2\pi} \int_0^{2\pi} \bar{\varphi}_n(e^{i\theta}) f(e^{i\theta}) \frac{e^{i\theta}}{e^{i\theta} - z} d\theta.$$

By the dominated convergence theorem, $\{\bar{\varphi}_n f\}$ converges to $\bar{\varphi}f$ in L^1 . Hence $g_n(z) \rightarrow g(z)$ uniformly for z in any compact subset of the open unit disk.

Since $\{\varphi_n\}$ is a sequence of polynomials,

$$\begin{aligned} |||g_n|||_\rho &\leq ||\varphi_n||_\infty |||f|||_\rho \\ &\leq ||\varphi||_\infty |||f|||_\rho. \end{aligned}$$

$\{g_n\}$ being a bounded sequence is relatively compact in the weak topology of $H^2(\rho)$. A subsequence of $\{g_n\}$ converges weakly to a vector h in $H^2(\rho)$. By changing the notation if necessary, we can assume that $\{g_n\}$ itself converges to h . Since point evaluations are bounded linear functionals on $H^2(\rho)$, it follows that $g_n(z) \rightarrow h(z)$ for every z in the open unit disk. Thus $g(z) = h(z)$. Now

$$\begin{aligned} |||g|||_\rho &\leq \liminf |||g_n|||_\rho \\ &\leq ||\varphi||_\infty |||f|||_\rho. \quad \square \end{aligned}$$

Every nonzero function f in H^2 can be uniquely factored in the form $f = uF$ where u is *inner* and F is *outer* [5, Chapter 5]. u and F are called the inner and outer parts of f respectively. An inner function v is said to *divide* u if the quotient u/v is a bounded analytic function. In that case, $P(\bar{v}f) = f/v$. The next result follows from this observation and Theorem 1.1.

1.2. Corollary. *Let ρ be a nondecreasing sequence of positive numbers. If f is an analytic function in $H^2(\rho)$ and v is an inner function that divides the inner part of f , f/v is in $H^2(\rho)$. Moreover $|||f/v|||_\rho \leq |||f|||_\rho$. In particular, the outer part F of f is in $H^2(\rho)$ and $|||F|||_\rho \leq |||f|||_\rho$.*

We shall now apply Theorem 1.1 to analytic functions whose derivatives satisfy certain growth conditions. Theorem 1.3 below has been proved by Korenbljum [6] in case the multiplier φ is an inner function. His proof depends on the earlier work of Korenbljum and Korolevič [7]. Their method involves some explicit calculations.

In what follows we shall use the symbol $f^{(N)}$ to denote the N^{th} derivative of the function f analytic in the unit disk; *i.e.*,

$$f^{(N)}(z) = \frac{d^N}{dz^N} f(z), \quad N = 1, 2, \dots$$

1.3. Theorem. *If φ is a bounded analytic function in the unit disk and if f is an analytic function in the unit disk whose N^{th} derivative is in H^2 ,*

$$(5) \quad ||[P(\bar{\varphi}f)]^{(N)}||_2 \leq ||\varphi||_\infty ||f^{(N)}||_2, \quad N = 1, 2, \dots$$

Proof. It is obvious that f belongs to H^2 . As before, let $g = P(\bar{\varphi}f)$. Let $\{a_n\}$ and $\{b_n\}$ denote the Fourier coefficients of f and g respectively. Using Parseval's formula we can rewrite inequality (5) as

$$(6) \quad \sum_{n=0}^{\infty} n^2(n-1)^2 \cdots (n-N+1)^2 |b_n|^2 \leq \|\varphi\|_{\infty}^2 \sum_{n=0}^{\infty} n^2(n-1)^2 \cdots (n-N+1)^2 |a_n|^2.$$

Theorem 1.1 is not applicable directly, since the first N weights are 0. Let ϵ be an arbitrary number, $0 < \epsilon < 1$. Let $\rho(n) = \epsilon$ for $0 \leq n < N$ and $\rho(n) = n^2(n-1)^2 \cdots (n-N+1)^2$ for $n \geq N$. Now apply Theorem 1.1 and let ϵ tend to zero to obtain (6). $\square$

1.4. Corollary. Let f be an analytic function in the unit disk whose N^{th} derivative is in H^2 . If v is any inner function that divides the inner part of f ,

$$\|(f/v)^{(N)}\|_2 \leq \|f^{(N)}\|_2.$$

If $f(z) = \sum_{n=0}^{\infty} a_n z^n$ is an analytic function in the open unit disk, its Dirichlet integral is defined by

$$D(f) = \iint |f^{(1)}(x+iy)|^2 dx dy$$

where the integral is over the open unit disk. It is easy to verify that

$$D(f) = \pi \sum_{n=1}^{\infty} n |a_n|^2.$$

If a function has finite Dirichlet integral, it belongs to H^2 . The following theorem can be proved by the method used to prove Theorem 1.3. A result analogous to Corollary 1.4 can be proved as well.

1.5. Theorem. Let f be an analytic function in the unit disk with finite Dirichlet integral. If φ is a bounded analytic function in the unit disk,

$$D(P(\bar{\varphi}f)) \leq \|\varphi\|_{\infty}^2 D(f).$$

§2. This section contains a discussion of some operator theoretic properties of Toeplitz operators on weighted sequence spaces. First, we consider the co-analytic ones and prove a generalization of a theorem of Wintner [8, p. 279] on the spectrum of a co-analytic Toeplitz operator on H^2 . The proof is identical with that given by Wintner.

2.1. Theorem. Let ρ be a nondecreasing sequence of positive numbers for which $\limsup_{n \rightarrow \infty} \rho(n)^{1/n} \leq 1$. Let φ be in H^{∞} . Then the spectrum of the operator T_{φ} acting on $H^2(\rho)$ is the closure of the set $\{\varphi(\lambda): |\lambda| < 1\}$ in the plane. Consequently, its norm equals $\|\varphi\|_{\infty}$.

Proof. First observe that $T_{\varphi\psi} = T_{\varphi}T_{\psi}$ for φ, ψ in H^{∞} . So if φ is invertible in H^{∞} , T_{φ} is invertible. Let $R(\bar{\varphi})$ denote the closure of the set $\{\bar{\varphi}(\lambda): |\lambda| < 1\}$. If μ does not belong to $R(\bar{\varphi})$, $\varphi - \bar{\mu}$ is invertible in H^{∞} . Hence the spectrum of T_{φ} is contained in $R(\bar{\varphi})$. To prove the opposite inclusion, consider the functions k_{λ} ($|\lambda| < 1$) defined by

$$k_{\lambda}(z) = \frac{1}{1 - \bar{\lambda}z} \quad (|z| < 1).$$

Since $\limsup \rho(n)^{1/n} \leq 1$, k_λ belongs to $H^2(\rho)$. An easy calculation shows that

$$T_{\bar{\varphi}} k_\lambda = \overline{\varphi(\lambda)} k_\lambda.$$

The assertion about the spectrum follows from this.

Since the spectrum of $T_{\bar{\varphi}}$ is $R(\bar{\varphi})$, $\|T_{\bar{\varphi}}\| \geq \|\varphi\|_\infty$. The opposite inequality follows from Theorem 1.1. Thus $\|T_{\bar{\varphi}}\| = \|\varphi\|_\infty$. $\square$

For an arbitrary φ in L^∞ , the Toeplitz operator T_φ need not be a bounded operator on $H^2(\rho)$. For the space of functions with a derivative in H^2 there is a natural class of multipliers. This is the class of functions with a derivative in L^2 . The remainder of this paper is devoted to the Fredholm theory of Toeplitz operators induced by these functions.

Let $\mathcal{H}$ denote the Hilbert space of all functions f analytic in the open unit disk with the first derivative $f^{(1)}$ in H^2 . The norm on $\mathcal{H}$ is given by

$$\|f\|_{\mathcal{H}}^2 = \|f\|_2^2 + \|f^{(1)}\|_2^2.$$

Note that $\mathcal{H} = H^2(\rho)$ with $\rho(n) = 1 + n^2$.

Let $\mathcal{A}$ denote the space of all continuous complex valued functions φ on the circle such that φ is differentiable a.e. and the derivative $\varphi^{(1)}$ is in L^2 . Define a norm on $\mathcal{A}$ by

$$\|\varphi\|_{\mathcal{A}} = \|\varphi\|_\infty + \|\varphi^{(1)}\|_2.$$

With the usual algebraic operations and the above norm, $\mathcal{A}$ is a Banach algebra. The class of trigonometric polynomials is dense in $\mathcal{A}$.

Every function f in $\mathcal{H}$ has an absolutely convergent Fourier series. Let $\{a_n\}$ denote the sequence of Fourier coefficients of f . Then

$$\begin{aligned} \|f\|_\infty^2 &\leq \left(\sum_{n=0}^{\infty} |a_n| \right)^2 \\ &\leq 2 \left[|a_0|^2 + \left(\sum_{n=1}^{\infty} |a_n| \right)^2 \right] \\ &\leq 2 \left[|a_0|^2 + \left(\sum_{n=1}^{\infty} n^2 |a_n|^2 \right) \left(\sum_{n=1}^{\infty} \frac{1}{n^2} \right) \right] \\ &\leq k \|f\|_{\mathcal{H}}^2 \end{aligned}$$

where k is constant greater than one. For every φ in $\mathcal{A}$ and f in $\mathcal{H}$,

$$\begin{aligned} \|(P\varphi f)^{(1)}\|_2^2 &\leq \|(\varphi f)^{(1)}\|_2^2 \\ &\leq (\|\varphi\|_\infty \|f^{(1)}\|_2 + \|\varphi^{(1)}\|_2 \|f\|_\infty)^2 \\ &\leq 2(\|\varphi\|_\infty + \|\varphi^{(1)}\|_2)^2 (\|f^{(1)}\|_2^2 + \|f\|_\infty^2) \\ &\leq 2k(\|\varphi\|_\infty + \|\varphi^{(1)}\|_2)^2 \|f\|_{\mathcal{H}}^2. \end{aligned}$$

So there exists a constant K such that

$$\|T_\varphi f\|_{\mathcal{H}} \leq K \|\varphi\|_{\mathcal{A}} \|f\|_{\mathcal{H}}.$$

We see that the mapping $\varphi \mapsto T_\varphi$ is a continuous linear mapping of $\mathfrak{A}$ into the Banach algebra of all operators on $\mathfrak{H}$.

The next result is an analog of an index theorem for Toeplitz operators on H^2 ([1], [3]). The proof given in the case of H^2 requires only minor modifications. So we shall give only a sketch of the proof.

2.2. Theorem. *Let φ be a function in the algebra $\mathfrak{A}$. Suppose that φ does not vanish on the circle. Then the operator T_φ is a Fredholm operator on $\mathfrak{H}$. Its index is $-w(\varphi)$, where $w(\varphi)$ is the winding number of φ around the origin.*

Proof. If φ and ψ are trigonometric polynomials, the operator $T_\varphi T_\psi - T_\psi T_\varphi$ has finite dimensional range. Using the density of trigonometric polynomials in $\mathfrak{A}$ and the continuity of the map $\varphi \mapsto T_\varphi$ it can be shown that the Toeplitz operator induced by a nonvanishing function φ is a Fredholm operator [1, p. 112].

To compute the index we use the argument in [3, p. 434]. It suffices to prove the following: If φ and ψ are two nonvanishing functions in $\mathfrak{A}$ that are homotopic, we can find a homotopy h connecting φ and ψ such that for a fixed t the function $h(t, \cdot)$ is a nonvanishing function in $\mathfrak{A}$, $0 \leq t \leq 1$, and such that the map $t \mapsto h(t, \cdot)$ is continuous in the norm of $\mathfrak{A}$. This can be done by a standard construction using the path lifting theorem. $\square$

2.3. Corollary. *If φ is a nonvanishing function in $\mathfrak{A}$ and the winding number of φ around the origin is zero, the operator T_φ on $\mathfrak{H}$ is invertible.*

Proof. By Theorem 2.2, T_φ is a Fredholm operator of index zero. But the Toeplitz operator T_φ on H^2 is invertible [3]. So the kernel of T_φ on H^2 is trivial. This implies that the kernel of T_φ on $\mathfrak{H}$ is also trivial. Hence T_φ is invertible. $\square$

2.4. Remark. Suppose φ is a function in $\mathfrak{A}$. Then the operator T_φ on H^2 leaves the set of functions in H^2 with a derivative in H^2 invariant. Corollary 2.3 shows that in case T_φ is invertible on H^2 , T_φ^{-1} also leaves this set invariant.

2.5. Remark. We have not been able to decide whether the invertibility of T_φ on $\mathfrak{H}$ implies that of φ in $\mathfrak{A}$. For H^2 the result is true and is the spectral inclusion theorem of Hartman and Wintner [4, p. 139, problem 196]. If it is true also for T_φ acting on $\mathfrak{H}$, it will have two interesting consequences.

- (i) T_φ is invertible on $\mathfrak{H}$ if and only if it is invertible on H^2 .
- (ii) T_φ is Fredholm on $\mathfrak{H}$ if and only if φ is nonvanishing.

REFERENCES

1. M. F. ATIYAH, *Algebraic topology and operators in Hilbert space*, Lectures in modern analysis and applications I, Lecture notes in Mathematics No. 103, Springer-Verlag, New York, 1969.
2. L. CARLSON, *A representation formula for the Dirichlet integral*, Math. Z. **73** (1960), 190–196.
3. R. G. DOUGLAS, *On the spectrum of a class of Toeplitz operators*, J. Math. Mech. **18** (1968), 433–436.

4. P. R. HALMOS, *A Hilbert space problem book*, D. Van Nostrand Co., Princeton, N. J. 1967.
5. K. HOFFMAN, *Banach spaces of analytic functions*, Prentice-Hall, Englewood Cliffs, N. J., 1962.
6. B. I. KORENBLJUM, *On an extremal property of outer functions*, Mat. Zametki 10 (1971), 53–56 (in Russian).
7. B. I. KORENBLJUM & V. S. KOROLEVIČ, *On analytic functions regular in a circle and smooth on its boundary*, Mat. Zametki 7 (1970), 165–172 (in Russian).
8. A. WINTNER, *Zur theorie der beschränkten bilinearformen*, Math. Z. 30 (1929), 228–282.

University of Michigan

Date communicated: APRIL 21, 1972