Ideal Sets and Ideals of Compact Operators on Hilbert Space

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1. Introduction. In this paper we pursue the study of the two sided ideal structure of the ring $\mathcal{L}(\mathcal{H})$ of all bounded linear operators on a separable infinite dimensional complex Hilbert space $\mathcal{H}$, employing the same approach as in [1]. The fundamental tool of this approach (used implicitly throughout the present paper) is a theorem of Calkin [2, §1] which states the existence of a one to one lattice preserving correspondence between the two sided ideals of $\mathcal{L}(\mathcal{H})$ and certain subsets of the set of all bounded sequences of non-negative real numbers called ideal sets.

Definition. A set J of bounded sequences of non-negative numbers is called an ideal set if it satisfies the following conditions:

- i) if $\{\lambda_n\} \in J$, then $\{\lambda_{\pi(n)}\} \in J$ for every permutation π of the set Z^+ of positive integers;
- ii) if $\{\lambda_n\} \in J$ and $0 \leq \mu_n \leq \lambda_n$, for every $n \in Z^+$, then $\{\mu_n\} \in J$;
- iii) if $\{\mu_n\} \in J$ and $\{\lambda_n\} \in J$, then $\{\mu_n + \lambda_n\} \in J$.

We recall that, if $\mathcal{J}$ is a proper two sided ideal of $\mathcal{L}(\mathcal{H})$, then the ideal set J , which corresponds to $\mathcal{J}$ under the above mentioned one to one mapping is the set of all sequences of eigenvalues (counted according to multiplicity) of all positive operators of $\mathcal{J}$.

The simplest non trivial examples of ideal sets are the set C of all sequences of non-negative numbers tending to zero (corresponding to the ideal of all compact operators on $\mathcal{H}$) and the subset F of C consisting of all sequences such that only finitely many terms are different from zero (corresponding to the ideal of all finite rank operators on $\mathcal{H}$). C and F turn out to be the maximal and the minimal non trivial ideal sets, respectively [2]. From now on, an ideal set J will be called proper if the inclusions $F \subset J \subset C$ are proper. The classical examples of proper ideal sets are the ideal sets C_p of the p -Schatten ideals

$\mathcal{C}_p$ [6]; for $p > 0$, C_p is defined as

$$C_p = \left\{ \{\lambda_n\} \in C : \sum_{n=1}^{\infty} \lambda_n^p < \infty \right\}$$

(for other examples of proper ideal sets see [3, Chapter 3]).

As a natural generalization of the classes C_p Brown, Pearcy and the author introduced in [1] the ideal sets $S(f)$ and $D(f)$, where f is an admissible function. Following [1] we shall say that f is an admissible function if f is a non-negative non-decreasing function defined on the set of non-negative real numbers such that $f(x) > 0$, for $x > 0$ and $f(0) = 0 = \lim_{x \rightarrow 0} f(x)$. The ideal sets $S(f)$ and $D(f)$ are defined, respectively as

$$S(f) = \left\{ \{\lambda_n\} \in C : \sum_{n=1}^{\infty} f(\alpha \lambda_n) < \infty, \text{ for some } \alpha > 0 \right\},$$

$$D(f) = \left\{ \{\lambda_n\} \in C : \sum_{n=1}^{\infty} f(\alpha \lambda_n) < \infty, \text{ for every } \alpha > 0 \right\}.$$

Clearly, if $f_{(p)}$ is the admissible function defined by $f_{(p)}(x) = x^p$, $x \geq 0$, $p > 0$, then $S(f_{(p)}) = C_p = D(f_{(p)})$. It is shown in [1] that $S(f) = D(f)$ if and only if the admissible function f satisfies the following condition:

- (*) There exist $\alpha > 1$, $c > 0$, $\epsilon > 0$ such
that $f(\alpha x) \leq cf(x)$, for every $0 \leq x \leq \epsilon$.

In view of the above property admissible functions that satisfy condition (*) will be called mono-generating functions. There are several interesting questions concerning mono-generating functions and the ideal sets they generate. In particular, the following problems were posed in [1].

- I. Is the set of ideal sets generated by mono-generating functions linearly ordered under inclusion?
- II. Given any mono-generating function f , does there exist a continuous admissible function g that generates the same ideal set as f does?
- III. If f and g are admissible functions and $S(f) = D(g)$, does it follow that f and g are mono-generating functions?

In the same circle of ideas, other interesting questions are the following.

- IV. If f is a mono-generating function, does there exist $p > 0$ such that $S(f) \subset C_p$?
- V. Does there exist an admissible function f which is not mono-generating and satisfies $S(f) \subset \bigcap_{p>0} C_p$?
- VI. Is either $\bigcup_{p>0} C_p$ or $\bigcap_{p>0} C_p$ an ideal set of the form $S(f)$ or $D(f)$, for some admissible function f ?

The objective of the present paper is to clarify some points of the theory of ideal sets which might lead to a further understanding of the algebraic ideal

structure of $\mathcal{L}(\mathcal{H})$, via the lattice isomorphism established by Calkin's theorem; in particular we give a complete solution to the above mentioned questions I–VI. On the other hand, we extend the results of [1] defining, for any admissible function f and any ideal set J , the ideal sets

$$S(f; J) = \{ \{ \lambda_n \} \in C : \{ f(\alpha \lambda_n) \} \in J, \text{ for some } \alpha > 0 \},$$

$$D(f; J) = \{ \{ \lambda_n \} \in C : \{ f(\alpha \lambda_n) \} \in J, \text{ for every } \alpha > 0 \}.$$

Observe that, with this notation, $S(f) = S(f; C_1)$ and $D(f) = D(f; C_1)$. In Section 2, we introduce the concept of *geometric ideal set* (which includes as examples the classes C_p , for every $p > 0$), and we find (Sections 2 and 3) the answers to questions I–VI formulated in a more general fashion (in terms of the ideal sets $S(f; J)$ and $D(f; J)$ where J is a geometric ideal set). Finally, in Section 4 we make some remarks about joins and intersections of ideal sets. In particular we prove that the intersection of a countable family of proper ideal sets is also a proper ideal set; this answers negatively a question raised in [1].

In the next section we shall make use of [2, Lemmas 1.1 and 1.2] without explicit mention.

2. Geometric ideal sets. To begin with let us state the definition of ideal set in a more convenient way. This is expressed in the following elementary lemma whose proof will be omitted.

Lemma 2.1. *A subset J of C is an ideal set if and only if the following conditions hold:*

- a) *If $\{ \lambda_n \} \in J$, then $\{ \lambda_{\pi(n)} \} \in J$ for every bijection π of $\mathbb{Z}^+$,*
- b) *if $\{ \lambda_n \} \in J$ and $\{ \mu_n \} \in C$ such that $\mu_n \leq \lambda_n$, for every $n \in \mathbb{Z}^+$, then $\{ \mu_n \} \in J$,*
- c) *if $\{ \lambda_n \} \in J$ and $\{ \mu_n \} \in J$, then $\{ \lambda_n \vee \mu_n \} \in J$,*
- d) *if $\{ \lambda_n \} \in J$ and $\alpha > 0$, then $\{ \alpha \lambda_n \} \in J$.*

Throughout the rest of the paper the set of all admissible functions will be denoted by $\mathcal{A}$, and the subset of $\mathcal{A}$ consisting of all mono-generating functions by $\mathfrak{M}$.

Remark 2.2. Let $f \in \mathcal{A}$, and let J be an ideal set. We define the set

$$M(f; J) = \{ \{ \lambda_n \} \in C : \{ f(\lambda_n) \} \in J \}.$$

If $J = C_1$ we shall write $M(f)$ instead of $M(f; J)$ [1; Section 4]. It is easy to see that $M(f; J)$ satisfies conditions a), b) and c) of Lemma 2.1. For $\alpha > 0$, let $\alpha M(f; J)$ be the set $\{ \{ \mu_n \} \in C : \{ \mu_n \} = \{ \alpha \nu_n \} \text{ for some } \{ \nu_n \} \in M(f; J) \}$. Then it readily follows from Lemma 2.1 that the smallest ideal set containing $M(f; J)$ is given by $\bigcup_{\alpha > 0} \alpha M(f; J) = S(f; J)$, and the largest ideal set contained in $M(f; J)$ is $\bigcap_{\alpha > 0} \alpha M(f; J) = D(f; J)$. Let's observe that $S(f; J) = D(f; J)$ if and only if for every $\alpha > 0$, $\alpha M(f; J) = M(f; J)$ (if and only if for some $\alpha > 0$, $\alpha \neq 1$, $\alpha M(f; J) = M(f; J)$).

The proof of the following lemma is a straightforward verification.

Lemma 2.3. Let J be any ideal set,

a) if $f \in \mathcal{A}$, $g \in \mathcal{A}$ satisfy

(**) there exist $\epsilon > 0$, $c > 0$ such that $f(x) \leq cg(x)$ for every $0 \leq x \leq \epsilon$; then $M(g; J) \subset M(f; J)$;

b) if $h \in \mathfrak{M}$,

then $M(h; J)$ is an ideal set.

Remark 2.4. The converse of Lemma 2.3 is not valid in general. In fact, let $h \in \mathcal{A}$ be a continuous function such that $h(x) = 1/(-\ln x)$, for $0 < x < 1/2$, $\lim_{x \rightarrow \infty} h(x) = \infty$ and set $J = M(h)$. Since $h \in \mathfrak{M}$, $M(h)$ is an ideal set (Lemma 2.3). If f is the inverse function of h , then $f(x) = \exp(-1/x)$, $0 < x < 1/\ln 2$. Since $f \notin \mathfrak{M}$ and $M(f; J) = C_1$ the desired counterexample to the converse of Lemma 2.3, part b), is found. On the other hand, if g is the inverse function of $2h$ we see that $g(x) = \exp(-2/x)$, $0 < x < 2/\ln 2$. It follows that $M(g; J) = C_1$, and hence $M(g; J) \subset M(f; J)$. However f and g do not satisfy condition (**).

For many reasons it is very convenient to work with ideal sets for which the converse of Lemma 2.3 holds. As we shall see below there exists a wide class of ideal sets satisfying that property.

In what follows it will be convenient to adopt the following terminology: let $\{\lambda_n\} \in C$ and let $\{k_n\}$ be a sequence of positive integers; we shall denote by $\{\lambda_n; k_n\}$ the sequence in C obtained from $\{\lambda_n\}$ by repeating λ_1 , k_1 times, then repeating λ_2 , k_2 times, and so forth; i.e., $\{\lambda_n; k_n\}$ is the sequence

$$\underbrace{\lambda_1, \dots, \lambda_1}_{k_1}, \underbrace{\lambda_2, \dots, \lambda_2}_{k_2}, \dots, \underbrace{\lambda_n, \dots, \lambda_n}_{k_n}, \dots$$

Definition 2.5. We shall say that $\{\gamma_n\} \in C$ is a geometrically decreasing sequence if it is strictly decreasing, $\gamma_1 \geq 1$ and $\rho = \sup_{k \in \mathbb{Z}^+} \gamma_k / \gamma_{k+1} < \infty$. The number $1/\rho$ will be called the ratio of the sequence $\{\gamma_n\}$ (observe that $\gamma_n \geq 1/\rho^{n-1}$, $n = 1, 2, \dots$).

Definition 2.6. We shall say that J is a geometric ideal set (g.i.s.) if it is a proper ideal set and there exists a geometrically decreasing sequence $\{\gamma_n\}$ such that if $\{\lambda_n\} \in C$ and $\{k_n\}$ is any sequence of positive integers, then we have: $\{\lambda_n\} \in J$ if and only if $\{\gamma_{k_n} \lambda_n; k_n\} \in J$. The sequence $\{\gamma_n\}$ will be said to be associated with J .

Remark 2.7. C_p is a g.i.s., for every $p > 0$. A geometrically decreasing sequence associated with C_p is $\{n^{-1/p}\}$. On the other hand, there are proper ideal sets which are not g.i.s., as the example in Remark 2.4 and Corollary 2.9 show.

Theorem 2.8. Let $f, g \in \mathcal{A}$, and let J be a g.i.s. Then $M(g; J) \subset M(f; J)$ if and only if condition (**) holds.

Proof. Let $\{\lambda_n\} \in M(g; J)$ be strictly decreasing and let $\{c_n\}$ be a sequence of positive numbers such that $\{c_n g(\lambda_n)\} \notin J$. Suppose that condition (**) is not valid. Then there exist $\mu_n \in (0, \lambda_n)$ such that $c_n g(\mu_n) \leq f(\mu_n)$, $n = 1, 2, \dots$. On the other hand, let $\{\gamma_n\}$ be a geometrically decreasing sequence associated with J and let $1/\rho$ be its ratio. For each $n = 1, 2, \dots$ let k_n be the first positive integer such that $\gamma_{k_n} g(\lambda_n) \leq g(\mu_n)$. Then $c_n \gamma_{k_n} g(\lambda_n) \leq f(\mu_n)$. From the definition of the sequence $\{k_n\}$ it also follows that $g(\mu_n) \leq \rho \gamma_{k_n} g(\lambda_n)$, $n = 1, 2, \dots$. Therefore, from the last estimations and the definition of g.i.s. we conclude that $\{g(\mu_n); k_n\} \in J$ while $\{f(\mu_n); k_n\} \notin J$. Thus $M(g; J) \not\subset M(f; J)$. The other implication is provided by Lemma 2.3.

Corollary 2.9. *Let $f \in \mathfrak{A}$, and let J be a g.i.s. Then the following statements are equivalent.*

- a) $S(f; J) = D(f; J)$;
- b) $M(f; J)$ is an ideal set ;
- c) $f \in \mathfrak{M}$ (i.e., f satisfies condition (*));
- d) for every $\alpha > 0$ there exist $c > 0$, $\epsilon > 0$, such that $f(\alpha x) \leq cf(x)$, for every $0 \leq x \leq \epsilon$.

Proof. It is an immediate consequence of Remark 2.2, Lemma 2.3 and Theorem 2.8.

The next two theorems are concerned with the comparison between the ideal sets $S(f; J)$ and $D(g; J)$, where J is a g.i.s.

Theorem 2.10. *Let $f, g \in \mathfrak{A}$, and let J be a g.i.s. Then $S(g; J) \subset D(f; J)$ if and only if*

(***) for every $\alpha > 0$ there exist $c > 0$, $\epsilon > 0$, such that $f(\alpha x) \leq cg(x)$, $0 \leq x \leq \epsilon$.

Proof. Suppose that (***) is valid, and let $\{\lambda_n\} \in S(g; J)$. Then there exists $\xi > 0$ such that $\{g(\xi \lambda_n)\} \in J$. For every $\eta > 0$ let $\alpha = \eta/\xi$ and let $\epsilon > 0$, $c > 0$, be the numbers provided by (***). Then for sufficiently large n , $f(\eta \lambda_n) = f(\alpha \xi \lambda_n) \leq cg(\xi \lambda_n)$. Thus, $\{f(\eta \lambda_n)\} \in J$ for every $\eta > 0$, and hence $\{\lambda_n\} \in D(f; J)$. To prove the other implications, assume (**) does not hold, and let $\{\lambda_n\}$ be a strictly decreasing sequence in $M(g; J)$. Also, let $\{c_n\}$ be a sequence of positive numbers such that $\{c_n g(\lambda_n)\} \notin J$. Since (**) is not valid, there exist $\mu_n \in (0, \lambda_n)$ and $\alpha > 0$ such that $c_n g(\mu_n) \leq f(\alpha \mu_n)$, $n = 1, 2, \dots$. Let $\{\gamma_n\}$ be a geometrically decreasing sequence associated with J of ratio $1/\rho$. Now, for each $n = 1, 2, \dots$, let k_n be the first positive integer such that $\gamma_{k_n} g(\lambda_n) \leq g(\mu_n)$. Then $c_n \gamma_{k_n} g(\lambda_n) \leq f(\alpha \mu_n)$. On the other hand, $g(\mu_n) \leq \rho \gamma_{k_n} g(\lambda_n)$. From the last two inequalities we conclude that $\{g(\mu_n); k_n\} \in J$ and $\{f(\alpha \mu_n); k_n\} \notin J$. Therefore $S(g; J) \not\subset D(f; J)$ and the proof of the theorem is completed.

Remark 2.11. It is easy to see that the implication $(***) \Rightarrow S(g; J) \subset D(f; J)$ holds for every ideal set J . Also, it readily follows that (**) is equivalent to

each of the following conditions:

- a) for every $\beta > 0$, there exist $c > 0$, $\epsilon > 0$ such that $f(x) \leq cg(\beta x)$, $0 \leq x \leq \epsilon$;
- b) for every $\alpha > 0$, there exist $\beta > 0$, $c > 0$, $\epsilon > 0$ such that $f(\alpha x) \leq cg(\beta x)$, $0 \leq x \leq \epsilon$;
- c) for every $\beta > 0$, there exist $\alpha > 0$, $c > 0$, $\epsilon > 0$ such that $f(\alpha x) \leq cg(\beta x)$, $0 \leq x \leq \epsilon$.

Theorem 2.12. *Let $f, g \in \mathcal{A}$, and let J be a g.i.s. Then $D(g; J) \subset S(f; J)$ if and only if*

- (*v) *there exist positive constants α, β, c and ϵ such that $f(\alpha x) \leq cg(\beta x)$, $0 \leq x \leq \epsilon$.*

Proof. The proof of $(*v) \Rightarrow D(g; J) \subset S(f; J)$ follows the pattern of the first part of the proof of Theorem 2.10. Conversely, suppose that $(*v)$ is not valid, and let $\{\lambda_n\} \in C$ be a strictly decreasing sequence such that $\{n\lambda_n\} \in M(g; J)$. Also let $\{c_n\}$ be a sequence of positive numbers such that $\{c_n g(\lambda_n)\} \notin J$. By hypothesis, for each $n = 1, 2, \dots$, there exists $\mu_n \in (0, \lambda_n)$ such that $c_n g(n\mu_n) \leq f(\mu_n/n)$. Now, let $\{\gamma_n\}$ be a geometrically decreasing sequence associated with J of ratio $1/\rho$ and for each $n = 1, 2, \dots$, let k_n be the first positive integer such that $\gamma_{k_n} g(n\lambda_n) \leq g(n\mu_n)$. Then $c_n \gamma_{k_n} g(n\lambda_n) \leq f(\mu_n/n)$ and $g(n\mu_n) \leq \rho \gamma_{k_n} g(n\lambda_n)$, $n = 1, 2, \dots$. From these inequalities it follows that for every positive integer j , $\{g(j\mu_n); k_n\} \in J$ and $\{f(\mu_n/j); k_n\} \notin J$. Thus, $D(g; J) \subsetneq S(f; J)$ and the theorem follows.

Remark 2.13. If J is any ideal set, it is easily seen that $(*v)$ implies that $D(g; J) \subset S(f; J)$; moreover, it is equally easy to see that $(*v)$ also implies that $S(g; J) \subset S(f; J)$ and $D(g; J) \subset D(f; J)$. On the other hand, it is elementary to verify that $(*v)$ is equivalent to each of the following conditions:

- a) there exist positive constants α, c and ϵ such that $f(\alpha x) \leq cg(x)$, $0 \leq x \leq \epsilon$;
- b) there exist positive constants β, c and ϵ such that $f(x) \leq cg(\beta x)$, $0 \leq x \leq \epsilon$.

Theorem 2.14. *Let $f, g \in \mathcal{A}$, and let J be a g.i.s. Then $(*v)$ is equivalent to each of the following conditions:*

- a) $S(g; J) \subset S(f; J)$;
- b) $D(g; J) \subset D(f; J)$.

Proof. The proof of $(*v) \Rightarrow a)$ and $(*v) \Rightarrow b)$ is a consequence of Remark 2.13. On the other hand, $b) \Rightarrow (*v)$ and $c) \Rightarrow (*v)$ follow from Theorem 2.12.

Theorem 2.15. *Let $f, g \in \mathcal{A}$, and let J be a g.i.s. If $D(g; J) \subset S(f; J)$, then $S(g; J) \subset S(f; J)$ and $D(g; J) \subset D(f; J)$.*

Proof. It follows immediately from Theorems 2.12 and 2.14.

Corollary 2.16. *Let $f, g \in \mathcal{A}$, and let J be a g.i.s. If $D(f; J) = S(g; J)$, then $S(f; J) = D(f; J)$, $S(g; J) = D(g; J)$ and hence f and g are mono-generating functions.*

Proof. It is a direct consequence of Theorems 2.14 and 2.15.

In view of Theorem 2.14 it is appropriate to introduce the following order relation $\prec$ and equivalent relation $\sim$ on $\mathfrak{A}$.

Definition 2.17. We shall say that $f \prec g$ if condition $(*\nu)$ holds and $f \sim g$ whenever $f \prec g$ and $g \prec f$.

The equivalence class of $f \in \mathfrak{A}$ with respect to $\sim$ will be denoted by $\bar{f}$. The following two corollaries are easy consequences of Theorems 2.8 and 2.14. They can also be obtained directly from the corresponding definitions.

Corollary 2.18. If $f \in \mathfrak{M}$ and $g \in \bar{f}$, then $g \in \mathfrak{M}$.

Corollary 2.19. Let $f, g \in \mathfrak{A}$, and suppose that either f or g is in $\mathfrak{M}$. Then condition $(**)$ is equivalent to each of the following:

- a) $\bar{f} \prec g$;
- b) $\lim_{x \rightarrow 0} f(x)/g(x) < \infty$.

As in the Introduction, let us denote by $f_{(p)}$, $p > 0$, the admissible function $f_{(p)}(x) = x^p$, $x \geq 0$. If J is an ideal set and $p > 0$, we shall write J^p in place of $M(f_{(1/p)}; J)$. Since $f_{(1/p)} \in \mathfrak{M}$, J^p is an ideal set for every $p > 0$. In particular, $C_p = (C_1)^{1/p}$. As is observed in [4] this notation is especially meaningful.

Theorem 2.20. Let J be a g.i.s. Then for every $f \in \mathfrak{A}$,

- a) $\bigcup_{p>0} J^p \neq S(f; J)$;
- b) $\bigcup_{p>0} J^p \neq D(f; J)$;
- c) $\bigcap_{p>0} J^p \neq S(f; J)$; and
- d) $\bigcap_{p>0} J^p \neq D(f; J)$.

Proof. a) Assume there exists $f \in \mathfrak{A}$ such that $\bigcup_{p>0} J^p = S(f; J)$. Then, from Theorems 2.8 and 2.14 we conclude that $f \in \mathfrak{M}$. On the other hand, it also follows that $J^p \subset S(f^{1/2}; J) \subset S(f; J)$, for every $p > 0$. Therefore $\bigcup_{p>0} J^p = S(f^{1/2}; J)$ and hence $f^{1/2} \sim f$ which is a contradiction to the fact that $f \in \mathfrak{M}$.

b) Assume there exists $g \in \mathfrak{A}$ such that $\bigcup_{p>0} J^p = D(g; J)$ and let $h \in \mathfrak{A}$ be given by $h(x) = g(x^{1/2})$, $x \geq 0$. It follows, from the assumption, that $h \sim g$. This implies that there exist positive constants β , c and ϵ such that $h(x) \leq cg(\beta x)$, $0 \leq x \leq \epsilon$. Let $\epsilon_0 = \min(\epsilon/2, 1/4\beta^2)$. Then for every $0 \leq x \leq \epsilon_0$ we have $h(2x) \leq cg(2\beta x) \leq cg(2\beta(\epsilon_0 x)^{1/2}) \leq ch(x)$. Therefore $h \in \mathfrak{M}$ and hence $g \in \mathfrak{M}$. The same reasoning used in the proof of a) applies now to reach a contradiction.

The proofs of c) and d) require arguments similar to those given in a) and b) and will be left to the interested reader.

Remark 2.21. The last theorem leaves open the following question: Does

there exist a proper ideal set K such that $K \neq S(f; J)$ and $K \neq D(f; J)$, for every $f \in \mathcal{A}$ and every g.i.s. J ?

The above question is answered affirmatively when K is not required to be proper. This is shown in the following theorem which is a generalization of [1, Theorem 4.6].

Theorem 2.22. *J is a proper ideal set if and only if $S(f; J)$ and $D(f; J)$ are also proper ideal sets, for every $f \in \mathcal{A}$.*

Proof. Suppose that J is a proper ideal set. Since $D(f; J) \subset S(f; J)$, it suffices to show that $C \neq S(f; J)$ and $F \neq D(f; J)$. (Recall that C and F are the maximal and minimal ideal sets, respectively.) To prove the first of these assertions, let $\{\lambda_n\} \in M(f; J)$ such that $\{n\lambda_n\} \in C$ and $\{f(\lambda_n)\}$ is strictly decreasing. Then there exists a sequence of positive integers $\{k_n\}$ such that $\{f(\lambda_n); k_n\} \notin J$. It is clear now that for every positive integer j $\{n\lambda_n; k_n\} \notin jM(f; J)$, and hence $\{n\lambda_n; k_n\} \notin S(f; J)$. To prove the second assertion let $\{\mu_n\}$ be a strictly decreasing sequence in $M(f; J)$. Then $\{\mu_n/n\} \in C \setminus F$ and $\{\mu_n/n\} \in (1/j)M(f; J)$, for every positive integer j . Thus $\{\mu_n/n\} \in D(f; J)$ and we conclude that $D(f; J) \neq F$. Conversely, if J is not a proper ideal set, it is immediate to conclude that neither $S(f; J)$ nor $D(f; J)$ is proper.

Remark 2.23. There are several other interesting questions connected with the construction of ideal sets given in the present section. Some of them are similar to those raised in [1, §6]. In particular we mention the following one, which is related to the principal ideal structure of $\mathcal{L}(\mathcal{H})$. Given $\{\lambda_n\} \in C$, does there exist a g.i.s. J such that the smallest ideal set containing $\{\lambda_n\}$ coincides with $\bigcap_f S(f; J)$, where the intersection is taken over all $f \in \mathcal{A}$ such that $\{f(\lambda_n)\} \in J$?

3. Mono-generating functions. In this section we shall be concerned with some of those properties of admissible functions and mono-generating functions which arise naturally from the properties of the ideal sets they generate.

Remark 3.1. Let J be a g.i.s. It follows, from Theorem 2.14 and Definition 2.17, that if $f, g \in \mathcal{A}$, then $f < g$ if and only if $S(g; J) \subset S(f; J)$ and $D(g; J) \subset D(f; J)$; thus, $f \sim g$ if and only if $S(g; J) = S(f; J)$ and $D(g; J) = D(f; J)$. The order relation $<$ induces a partial ordering on the set of equivalence classes of $\mathcal{A}$ with respect to $\sim$; the mappings $\bar{f} \rightarrow S(f; J)$, $\bar{f} \rightarrow D(f; J)$ are injective, order preserving and they coincide on the equivalence classes of $\mathfrak{N}$. Furthermore, from Corollary 2.16, we see that $S(f; J) = D(g; J)$ if and only if $\bar{f} = \bar{g}$ and $f, g \in \mathfrak{N}$.

For the sake of brevity, we shall omit the translations of the results of the present section into the language of ideal sets of the form $S(f; J)$ and $D(f; J)$. However, this can be easily done by using the above remark and the results of the preceding section.

In the following definition we introduce various useful regularizations of an admissible function.

Definition 3.2. Let $f \in \mathcal{Q}$. We define the functions f_* , f^* and $\tilde{f}$, on the non-negative real line, as follows:

$$f_*(0) = f^*(0) = \tilde{f}(0) = 0,$$

and for $x > 0$,

$$f_*(x) = \lim_{t \rightarrow 0^+} f(x - t),$$

$$f^*(x) = \lim_{t \rightarrow 0^+} f(x + t),$$

$$\tilde{f}(x) = \frac{1}{x} \int_0^x f(t) dt.$$

The principal properties of the functions f_* , f^* and $\tilde{f}$ are shown in the following two theorems.

Theorem 3.3. Let $f \in \mathcal{Q}$. Then

- a) f_* and f^* are in $\mathcal{Q}$,
- b) f_* is a left continuous function, while f^* is right continuous,
- c) $f_* \sim f \sim f^*$.

Proof. Only the last assertion is not obvious. Since the proof of $f \sim f^*$ is analogous to the proof $f \sim f_*$ we shall concentrate our attention to proving $f \sim f^*$. To this end it suffices to prove $f^* \prec f$ (observe that $f(x) \leq f^*(x)$, for every $x \geq 0$). Let Δ be the set of points of discontinuity of f . Since the equivalence class $\tilde{f}$ depends only on the restriction of f to a neighborhood of zero we may assume that $\Delta \subset (0, 1)$. Therefore (notice that Δ is countable and $0 \notin \Delta$) there exists $\alpha \in (0, 1)$ such that $\Delta \cap \alpha\Delta = \emptyset$. Now, if $x \in \Delta$, then $\alpha x \notin \Delta$ and hence $f^*(\alpha x) = f(\alpha x) \leq f(x)$; if $x \notin \Delta$, then $f^*(\alpha x) \leq f^*(x) = f(x)$. Therefore for every $x \in (0, 1)$, $f^*(\alpha x) \leq f(x)$. This completes the proof of the theorem.

Theorem 3.4. Let $f \in \mathcal{Q}$.

- a) $\tilde{f}$ is a continuous admissible function, equivalent to f .
- b) If f is a convex function, then so is $\tilde{f}$.
- c) If g and h denote the functions given by $g(x) = \int_0^x f(t) dt$, $h(x) = xf(x)$, $x \geq 0$, then $h \sim g$, and hence $g \prec f$. In particular $f \in \mathfrak{M}$ if and only if $\tilde{f}$ (and hence g) is in $\mathfrak{M}$.

Proof. In order to prove part a) let us observe first that $\tilde{f}(x) = \int_0^1 f(tx) dt \leq f(x)$. From this remark it readily follows that $\tilde{f}$ is a continuous function in $\mathcal{Q}$. On the other hand, since $f(x) \leq (1/x) \int_x^{2x} f(t) dt \leq 2\tilde{f}(2x)$ we conclude that $f \sim \tilde{f}$. For the proof of b) let us recall that the function f is convex if and only if there exists a non-negative, non-decreasing function φ , defined on the non-negative real line, such that $f(x) = \int_0^x \varphi(t) dt$, $x \geq 0$. If such a function φ exists let ψ be the function given by $\psi(x) = \int_0^1 t\varphi(tx) dt$, $x \geq 0$. It is clear that ψ is a non-negative, non-decreasing function. Now we have $\tilde{f}(x) = \int_0^1 f(tx) dt = \int_0^1 \int_0^{tx} \varphi(s) ds dt = \int_0^1 \int_0^x t\varphi(st) ds dt = \int_0^x \int_0^1 t\varphi(ts) dt ds = \int_0^x \psi(s) ds$, $x \geq 0$.

The last chain of identities implies that $\tilde{f}$ is a convex function. Since part c) is a consequence of part a), the proof of the theorem is completed.

We shall say that $f \in \mathcal{A}$ is a convex function with respect to the origin if for each $x > 0$, the segment joining the origin $(0, 0)$, with the point $(x, f(x))$ is above the graph of the function $f|_{[0, x]}$. Clearly, every convex admissible function is convex with respect to the origin but the converse does not hold in general. It is easily seen that $f \in \mathcal{A}$ is a convex function with respect to the origin if and only if the function $f(x)/x$, $x > 0$ is non-decreasing. The following corollary is a consequence of Theorem 3.4.

Corollary 3.5. *If $f \in \mathcal{A}$ is a convex function with respect to the origin, then f is equivalent to a convex function.*

The following theorem indicates that in each equivalence class of $\mathcal{A}$ there are admissible functions with any degree of smoothness.

Theorem 3.6. *Let $f \in \mathcal{A}$. Then for every positive integer n there exists $g_n \in \mathcal{A}$ which is an n -times differentiable function on $\{x > 0\}$ such that $f \sim g_n$. Moreover, if f is a convex function with respect to the origin, the function g_n can be taken to be convex.*

Proof. Let $g_1 = \tilde{f}$. Having defined g_{n-1} , $n > 1$, let $g_n = \tilde{g}_{n-1}$. Now the theorem is a consequence of Theorem 3.4 and Corollary 3.5.

Remark 3.7. An admissible function f is a mono-generating function if and only if it is almost sub-additive, i.e., there exist $\epsilon > 0$, $c > 0$ such that for every $0 \leq x, y \leq \epsilon$ we have $f(x + y) \leq c[f(x) + f(y)]$. In particular, since concave functions are sub-additive, they are mono-generating functions.

Also, it is easy to see that if $f, g \in \mathfrak{M}$, then $f \wedge g \in \mathfrak{M}$ and $f \vee g \in \mathfrak{M}$. Therefore, the set of equivalence classes of $\mathfrak{M}$ with the partial ordering induced by $<$ form a lattice. The following theorem indicates that this lattice is not linearly ordered.

Theorem 3.8. *There exist two concave functions f and g which are not comparable with respect to $<$. Also, there are convex functions $p, q \in \mathfrak{M}$ which are not comparable, either.*

Proof. Let $\lambda_n = 1/(2n)!$, $\mu_n = 1/(2n + 1)!$, $n = 1, 2, \dots$, and let p_n and q_n be the points in the real plane with coordinates $p_n = (\lambda_n^2, \lambda_n)$, $q_n = (\mu_n^2, \mu_n)$. Since p_n and q_n lie on the curve (x^2, x) , $x \geq 0$, $n = 1, 2, \dots$, there exist concave functions f and g such that the graph of $f|_{[0, 1/4]}$ coincides with the polygonal joining consecutive points of $\{p_n\}$ and the graphs of $g|_{[0, 1/36]}$ coincides with the polygonal joining consecutive points of $\{q_n\}$. It is clear now that $f, g \in \mathfrak{M}$. On the other hand, calculations show that there exist constants $c > 0$, $d > 0$ such that $f(\mu_n^2) \leq c\lambda_{n+1}^2$, $g(\lambda_n^2) \leq d\mu_n$, $n = 1, 2, \dots$. Therefore

$$\begin{aligned} \lim_{x \rightarrow 0} f(x)/g(x) &\geq \lim_{n \rightarrow \infty} f(\lambda_n^2)/g(\lambda_n^2) \geq (1/d) \lim_{n \rightarrow \infty} \lambda_n/\mu_n = \infty, \\ \lim_{x \rightarrow 0} g(x)/f(x) &\geq \lim_{n \rightarrow \infty} g(\mu_n^2)/f(\mu_n^2) \geq (1/c) \lim_{n \rightarrow \infty} \mu_n/\lambda_{n+1} = \infty. \end{aligned}$$

From Corollary 2.19 we deduce that neither $f \prec g$ nor $g \prec f$ holds. Let $p, q \in \mathcal{A}$ be given by $p(x) = \int_0^x f(t) dt$, $q(x) = \int_0^x g(t) dt$. Then p and q are convex functions and from Theorems 3.4 $p, q \in \mathfrak{M}$. Since f, g are not comparable, it follows immediately, again from Theorem 3.4, that p and q are not comparable either.

The following result gives a necessary and sufficient condition for an admissible function to be mono-generating.

Theorem 3.9. *Let $f \in \mathcal{A}$. Then $f \in \mathfrak{M}$ if and only if there exist $\epsilon > 0$, $c > 1$, such that $xf(x) \leq c \int_0^x f(t) dt$.*

Proof. Suppose that $f \in \mathfrak{M}$. From Corollary 2.19 and Theorem 3.4 it follows that there exist $c > 0$ and $\epsilon > 0$ such that $f(x) \leq c\tilde{f}(x)$, $0 \leq x \leq \epsilon$. Since $\tilde{f}(x) \leq f(x)$, $x \geq 0$, we must have $c \geq 1$. If $c = 1$, then $\tilde{f}(x) = f(x)$, $0 \leq x \leq \epsilon$, and hence $x\tilde{f}(x) = \int_0^x \tilde{f}(t) dt$, $0 \leq x \leq \epsilon$. Taking derivatives on both sides of the last identity we conclude that $\tilde{f}(x) + x\tilde{f}'(x) = \tilde{f}(x)$, and hence $\tilde{f}'(x) = 0$, for every $x \in (0, \epsilon)$. This contradicts the fact that $\tilde{f} \in \mathcal{A}$, and the first part of the theorem is proved. Conversely, suppose there exist $c > 1$, $\epsilon > 0$ such that $f(x) \leq c\tilde{f}(x)$, $0 \leq x \leq \epsilon$. Then $\tilde{f}(x) \leq c\tilde{f}(x)$, $0 \leq x \leq \epsilon$. Let h be the function defined by $h(x) = x\tilde{f}(x)$, $x \geq 0$. It follows that the derivative $h'(x)$ of h at every $x > 0$ coincides with $\tilde{f}(x)$. Thus we have $h'(x)/h(x) \leq c/x$, $0 < x \leq \epsilon$. Integrating both sides of the last inequality between x and $2x$, $0 < x \leq \epsilon/2$, we obtain $h(2x) \leq 2^c h(x)$, $0 \leq x \leq \epsilon/2$, and hence $h \in \mathfrak{M}$. Another application of Theorem 3.4 completes the proof of the theorem.

Theorem 3.10. *Let $f \in \mathfrak{M}$. Then there exists $p > 0$ such that $f_{(p)} \prec f$.*

Proof. From Theorem 3.4 we can assume without loss of generality that f is continuous. Let $g \in \mathcal{A}$ be defined by $g(x) = \int_0^x f(t) dt$. From Theorem 3.9 there exist $\epsilon > 0$, $c > 1$ such that $f(x)/g(x) \leq c/x$, $0 < x \leq \epsilon$. An integration of the last inequality between x and ϵ , for each $x \in (0, \epsilon)$ yields $x^c \leq c'g(x)$, where c' is a positive constant. From Theorem 3.4 we deduce that there exist $d > 0$ and $\delta > 0$ such that $x^c \leq dx f(x)$, $0 \leq x \leq \delta$. Therefore, the desired conclusion is obtained by taking $p = c - 1$.

The next theorem shows that even a much weaker version than the converse of Theorem 3.10 is false.

Theorem 3.11. *There exists $f \in \mathcal{A}$ such that $f \notin \mathfrak{M}$ and $f_{(p)} \prec f$, for every $p > 0$. Moreover, for every $p > 0$, there exists $g \in \mathcal{A}$ such that $g \notin \mathfrak{M}$, $f_{(p)} \prec g \prec f_{(p)}$, for every $q > p$ and $g \sim f_{(p)}$.*

Proof. Let $\nu_n = 1/n^n$ and let h be a function defined on $[0, 1]$ as follows: if $x \in (\nu_{n+1}, \nu_n]$, then $h(x) = 1/n!$. Let $f \in \mathcal{A}$ such that $f|_{[0,1]} = h$. For each $n = 1, 2, \dots$ we see that $f(2\nu_{n+1}) = f(\nu_n) = (n+1)f(\nu_{n+1})$. Therefore $f \notin \mathfrak{M}$. In view of Corollary 2.19, to prove the first assertion, it suffices to show that $\lim_{x \rightarrow 0} f(x)/x^p = \infty$, for every $p > 0$. Thus, let $p > 0$ and let n_0 sufficiently large such that $n < pn^2$, for $n \geq n_0$. For every positive integer $n \geq n_0$, let $x \in (0, \nu_n]$. Then there exists $m \geq n$ such that $x \in (\nu_{m+1}, \nu_m]$ and we have

$$f(x)/x^p = f(\nu_m)/x^p \geq f(\nu_m)/\nu_m^p \geq m^m/m! \geq n.$$

Therefore $\lim_{x \rightarrow 0} f(x)/x^p = \infty$, as desired. To prove the last assertion of the theorem we define $g \in \mathcal{A}$ by $g(x) = x^p f(x)$, $x \geq 0$. It readily follows that the function g satisfies the required conditions.

4. Join and intersection of certain ideal sets. We begin this section with a remark about convex admissible functions.

Remark 4.1. In [5] it is shown that if f is a convex admissible function, then the ideal of compact operators $\mathcal{S}(f)$ corresponding to $S(f)$ is a norm ideal. Moreover, if $\mathcal{D}(f)$ is the ideal corresponding to $D(f)$, then $\mathcal{D}(f)$ is the unique separable norm ideal which is a closed subspace of $\mathcal{S}(f)$. Thus, $\mathcal{S}(f)$ is a separable norm ideal if and only if f is a mono-generating convex function. Also, from Corollary 3.5, we see that if $f \in \mathcal{A}$ is a convex function with respect to the origin, then $\mathcal{S}(f)$ is a norm ideal. On the other hand, as a consequence of Theorem 3.8, we deduce that the set of all separable norm ideals is not linearly ordered under inclusion. Since convex admissible functions produce norm ideals and concave admissible functions are the simplest examples of mono-generating functions, convexity or concavity are important properties to be enjoyed by an admissible function. The following result is an improvement of [1, Lemma 6.1].

Theorem 4.2.¹ *Let J be any proper ideal set. Then*

$$\text{a) } C = \bigvee_{\substack{f \in \mathcal{A} \\ f \text{ convex}}} D(f; J), \text{ and}$$

$$\text{b) } F = \bigcap_{\substack{g \in \mathcal{A} \\ g \text{ concave}}} D(g; J).$$

Proof. Let $\{\lambda_n\} \in C \setminus F$. To prove the theorem it is enough to exhibit a convex admissible function f and a concave admissible function g such that $\{f(k\lambda_n)\} \in J$, for every positive integer k and $\{g(\lambda_n)\} \notin J$. To this end, we can assume (without loss of generality) that $\lambda_n < 1$, for every $n = 1, 2, \dots$. Furthermore, we can find an increasing sequence $\{k_n\}$ of positive integers such that $k_n\lambda_n < 1$, $n = 1, 2, \dots$, $\lim_{n \rightarrow \infty} k_n = \infty$, and $\{k_n\lambda_n\} \in C$. Setting $\kappa_n = k_n\lambda_n$, $n = 1, 2, \dots$, let $\{\mu_n\}$ and $\{\nu_n\}$ be strictly decreasing sequences in C such that $0 < \nu_n \leq \lambda_n \leq \mu_n < 1$, $n = 1, 2, \dots$. Also, let $\{\alpha_n\}$ and $\{\beta_n\}$ be strictly decreasing sequences such that $\alpha_n, \beta_n \in (0, 1)$, $n = 1, 2, \dots$ and $\{\alpha_n\} \in J$ and $\{\beta_n\} \in C \setminus J$. Define the points in the plane $p_0 = (1, 1)$, $q_0 = (0, 0)$, $p_n = (\mu_n, \alpha_n)$, $q_n = (\nu_n, \beta_n)$, and let Γ be the convex hull of the set of points p_n and q_n , $n = 0, 1, 2, \dots$. Moreover, let p and q be the functions defined on $[0, 1]$ whose graphs coincide with the part of the boundary of Γ below the segment $[p_0, q_0]$ and above $[p_0, q_0]$, respectively. Now, the desired functions f and g are obtained by taking any convex admissible function f and any concave admissible function g such that $f|_{[0,1]} = p$ and $g|_{[0,1]} = q$. Observe that $g(\lambda_n) \geq g(\nu_n) > \beta_n$ and for

every positive integer k and n large enough $f(k\lambda_n) \leq f(\kappa_n) \leq f(\mu_n) \leq \alpha_n$. Therefore $\{g(\lambda_n)\} \notin J$ and $\{f(k\lambda_n)\} \in J, k = 1, 2, \dots$.

The following theorem tells us that the family of concave admissible functions on which the intersection is taken in part b) of the preceding theorem cannot be countable.

Theorem 4.3.² *The intersection of a countable family of proper ideal sets is a proper ideal set.*

Proof. If J is a proper ideal set, by definition, J contains a decreasing sequence $\{\mu_n\} \in C \setminus F$. Let $J_k, k = 1, 2, \dots$ be a countable family of proper ideal sets and let $\{\lambda_{n,k}\}$ be a decreasing sequence in $J_k \setminus F, k = 1, 2, \dots$. For each $n = 1, 2, \dots$ and $k = 1, 2, \dots$, define $\mu_{n,k} = \inf_{1 \leq i \leq k} \lambda_{n,i}$. Then $\{\mu_{n,k}\}, k = 1, 2, \dots$, are decreasing sequences in $C \setminus F$. Moreover, $\{\mu_{n,k}\} \in \bigcap_{i=1}^k J_i$, for each $k = 1, 2, \dots$. Set $\lambda_n = \mu_{n,n}$ (the diagonal sequence). We claim $\{\lambda_n\} \in \bigcap_{k=1}^{\infty} J_k$. Since $\{\lambda_n\} \notin F$ and the intersection of an arbitrary family of ideal sets is also an ideal set, the proof of the claim will also complete the proof of the theorem. Given any positive integer k , let $\alpha = \sup_{1 \leq n \leq k} \lambda_n, \beta = \inf_{1 \leq n \leq k} \mu_{n,k}$, and set $\gamma = \max(\alpha/\beta, 1)$. Then $\lambda_n \leq \gamma \mu_{n,k}, n = 1, 2, \dots$ and $\{\lambda_n\} \in J_k$.

Remark 4.4. The following question remains open in spite of its dual aspect with respect to the statement of Theorem 4.3. [1, Section 6].

Does there exist a countable family of proper ideal sets $J_k, k = 1, 2, \dots$ such that $C = \bigvee_{k=1}^{\infty} J_k$?

It is shown in [5] that if for each $k = 1, 2, \dots, J_k$ corresponds to a norm ideal of compact operators, then the above problem has a negative answer.

Remark 4.5. If $f, g \in \mathcal{A}$ and J is any ideal set it follows easily (using Lemma 2.1 and reasonings similar to those employed in [1, Section 5]) that $S(f \wedge g; J) = S(f; J) \vee S(g; J)$, and $S(f \vee g; J) = S(f; J) \cap S(g; J)$. It is worth noting that there are functions $f_i, g_i \in \mathcal{A}, i = 1, 2$, with the following properties:

$$\begin{aligned} f_i &\prec f_{(1)} \prec g_i, f_i \sim f_{(1)}, g_i \sim f_{(1)}, i = 1, 2, \text{ and} \\ f_1 \vee f_2 &= f_{(1)} = g_1 \wedge g_2. \end{aligned}$$

If J is a g.i.s., the above relations can be restated as:

$$S(f_1; J) \cap S(f_2; J) = J = S(g_1; J) \vee S(g_2; J);$$

$$S(g_i; J) \subset J \subset S(f_i; J), i = 1, 2, \text{ and both inclusions are proper.}$$

The functions f_i and g_i can be defined as follows. Let $\lambda_k = 1/(2k)!$, $\mu_k = 1/(2k+1)!, k = 1, 2, \dots$, take $f_i, g_i \in \mathcal{A}, i = 1, 2$, satisfying

$$\begin{aligned} g_1(x) &= x, x \in [\mu_k, \lambda_k), \\ g_1(x) &= \mu_k, x \in [\lambda_{k+1}, \mu_k), \\ g_2(x) &= \lambda_k, x \in [\mu_k, \lambda_k), \end{aligned}$$

$$\begin{aligned}
g_2(x) &= x, x \in [\lambda_{k+1}, \mu_k], \\
f_1(x) &= \mu_k, x \in (\mu_k, \lambda_k], \\
f_1(x) &= x, x \in (\lambda_{k+1}, \mu_k], \\
f_2(x) &= x, x \in (\mu_k, \lambda_k], \\
f_2(x) &= \lambda_{k+1}, x \in (\lambda_{k+1}, \mu_k].
\end{aligned}$$

Since $\lim_{k \rightarrow \infty} g_1(\lambda_k)/\lambda_k = \lim_{k \rightarrow \infty} g_2(\mu_k)/\mu_k = \infty$, and $\lim_{k \rightarrow \infty} f_1(\lambda_k)/\lambda_k = \lim_{k \rightarrow \infty} f_2(\mu_k)/\mu_k = 0$, it readily follows that f_i and g_i , $j = 1, 2$, satisfy the required properties.

In the above examples the functions f_i and g_i are not in $\mathfrak{M}$. On the other hand, it can be easily proved that if f, f' are convex admissible functions and g, g' are concave admissible functions, then $f \vee f' \sim f_{(1)}$ implies either $f \sim f_{(1)}$ or $f' \sim f_{(1)}$; likewise, $g \wedge g' \sim f_{(1)}$ implies either $g \sim f_{(1)}$ or $g' \sim f_{(1)}$. Thus, it seems natural to pose the following problem.

Let J be a g.i.s. Do there exist a sequence of concave admissible functions $\{g_n\}$ and a sequence of convex mono-generating functions $\{f_n\}$ such that $g_n \sim f_{(1)}$, $f_n \sim f_{(1)}$, $n = 1, 2, \dots$, and

$$\bigvee_{n=1}^{\infty} M(g_n; J) = J = \bigcap_{n=1}^{\infty} M(f_n; J)?$$

The case $J = C_1$ is particularly interesting. In this direction the following result indicates that if the families of functions in the above problem are not required to be countable, then the answer is affirmative.

Theorem 4.6. $C_1 = \bigvee_{\mathfrak{g}} M(g)$, where the join is taken over a family of concave admissible functions g such that $g \sim f_{(1)}$. Likewise, $C_1 = \bigcap_{\mathfrak{f}} M(f)$, where the intersection is taken over a family of convex mono-generating functions f such that $f \sim f_{(1)}$.

Proof. This proof is suggested by an elementary geometric construction. To prove the first assertion suppose that $\{\lambda_n\}$ is any sequence in C_1 arranged in decreasing order. Let $\{\mu_n\}$, $\{\alpha_n\}$ be sequences of positive numbers satisfying the following conditions: $\{\mu_n\}$ is strictly decreasing, $\lambda_n \leq \mu_n$, $n = 1, 2, \dots$, $\{\alpha_n\}$ is strictly increasing, $\lim_{n \rightarrow \infty} \alpha_n = \infty$, and $\{\alpha_n \mu_n\} \in C_1$. We define now another sequence of positive numbers $\{\beta_n\}$ as follows:

$$\beta_1 = \alpha_1, \beta_2 = \beta_1 \mu_1 / \mu_2, \beta_3 = \sup \gamma, \text{ over those } \gamma \leq \alpha_3,$$

$$\frac{\mu_2 \beta_2 - \gamma \mu_3}{\mu_2 - \mu_3} \geq \frac{\beta_1 \mu_1 - \beta_2 \mu_2}{\mu_1 - \mu_2}.$$

Having defined β_m , $2 \leq m \leq n$, we define $\beta_{n+1} = \sup \gamma$, over those

$$\gamma \leq \alpha_{n+1}, \frac{\mu_n \beta_n - \gamma \mu_{n+1}}{\mu_n - \mu_{n+1}} \geq \frac{\beta_{n-1} \mu_{n-1} - \beta_n \mu_n}{\mu_{n-1} - \mu_n}.$$

It can be easily checked that $\{\beta_n\}$ is a strictly increasing sequence and $\lim_{n \rightarrow \infty} \beta_n = \infty$. Let Q be the polygonal in the plane joining consecutive points of the sequence $\{(\mu_n, \beta_n \mu_n)\}$. By construction Q is the graph of a concave function on $(0, \mu_1)$. Let g be any concave admissible function such that the graph of $g|_{(0, \mu_1)}$ coincides with Q . Then,

$$\sum_{n=1}^{\infty} g(\lambda_n) \leq \sum_{n=1}^{\infty} g(\mu_n) = \sum_{n=1}^{\infty} \beta_n \mu_n \leq \sum_{n=1}^{\infty} \alpha_n \mu_n < \infty.$$

Since $g(x)/x$, $x > 0$, is non-increasing we also have

$$\lim_{x \rightarrow 0} g(x)/x \geq \lim_{n \rightarrow \infty} g(\mu_n)/\mu_n = \lim_{n \rightarrow \infty} \beta_n = \infty.$$

Therefore $\{\lambda_n\} \in M(g)$ and $g \sim f_{(1)}$. Since $\{\lambda_n\}$ is an arbitrary sequence in C_1 , the first part of the theorem follows. To prove the second assertion, let $\{\nu_n\} \in C \setminus C_1$ be arranged in decreasing order, and let $\{\delta_n\} \in C \setminus C_1$ be a geometrically decreasing sequence of ratio $1/\rho$. We define an increasing sequence of positive integers $\{k_n\}$ as follows:

$$k_1 = 1;$$

$$k_2 \text{ is the first positive integer greater than } k_1 \text{ such that } \sum_{i=k_1}^{k_2} \nu_i \geq 1;$$

$$k_3 \text{ is the first positive integer greater than } k_2 \text{ such that}$$

$$\sum_{i=k_2+1}^{k_3} \nu_i \geq 1, \quad 2\nu_{k_3} \leq \nu_{k_2} \quad \text{and} \quad \frac{\nu_{k_3} \delta_2 - \nu_{k_2} \delta_3}{\nu_{k_2} - \nu_{k_3}} \leq \frac{\nu_{k_1} \delta_1 - \nu_{k_2} \delta_2}{\nu_{k_1} - \nu_{k_2}}.$$

Having defined k_m , $2 \leq m \leq n$, we define

$$k_{n+1} \text{ is the first positive integer greater than } k_n \text{ such that}$$

$$\sum_{i=k_n+1}^{k_{n+1}} \nu_i \geq 1, \quad 2\nu_{k_{n+1}} \leq \nu_{k_n} \quad \text{and} \quad \frac{\nu_{k_n} \delta_n - \nu_{k_{n+1}} \delta_{n+1}}{\nu_{k_n} - \nu_{k_{n+1}}} \leq \frac{\nu_{k_{n-1}} \delta_{n-1} - \nu_{k_n} \delta_n}{\nu_{k_{n-1}} - \nu_{k_n}}.$$

Let P be the polygonal joining consecutive points of the sequence $\{(\nu_{k_n}, \delta_n \nu_{k_n})\}$. By construction P is the graph of a convex function on $(0, \nu_1)$. Let f be any convex admissible function such that the graph of $f|_{(0, \nu_1)}$ coincides with P . We claim that $f \in \mathfrak{N}$. In fact, let $x \in [0, \nu_{k_2})$ and let m be the first positive integer such that $\nu_{k_{m+1}} \leq x$. Then $x \in (\nu_{k_{m+1}}, \nu_{k_m})$ and $f(2x) \leq 2\delta_{m-1}x \leq 2\rho^2 \delta_{m+1}x \leq 2\rho^2 f(x)$, and hence $f \in \mathfrak{N}$, as asserted. We observe that, (since $f(x)/x$, $x > 0$ is non-decreasing)

$$\lim_{x \rightarrow 0} f(x)/x \leq \lim_{n \rightarrow \infty} f(\nu_{k_n})/\nu_{k_n} = \lim_{n \rightarrow \infty} \delta_n = 0.$$

Also,

$$\sum_{i=1}^{\infty} f(\nu_i) = \sum_{n=1}^{\infty} \sum_{i=k_n+1}^{k_{n+1}} f(\nu_i) \geq \sum_{n=1}^{\infty} \delta_{n+1} \sum_{i=k_n+1}^{k_{n+1}} \nu_i \geq \sum_{n=1}^{\infty} \delta_{n+1} = \infty.$$

Therefore $f \sim f_{(1)}$ and $\{\nu_n\} \notin M(f)$. The proof of the theorem is completed.

The last theorem of this paper is concerned with the lattice structure of the ideal set $D(f)$, $f \in \mathfrak{A}$, and it answers the question raised at the beginning of [1, Section 6].

Theorem 4.7. *For every $f, g \in \mathfrak{A}$,*

$$D(f \wedge g) = D(f) \vee D(g).$$

Proof. First we shall prove that, for $h \in \mathfrak{A}$, $\{\nu_n\} \in D(h)$ if and only if there exists an increasing sequence $\{k_n\}$ of positive integers such that $\lim_{n \rightarrow \infty} k_n = \infty$ and $\{k_n \nu_n\} \in M(h)$. In fact, the only if part is a simple verification. To show the other implication, let $\{\nu_n\} \in D(h)$ and let $\sigma = \sum_{n=1}^{\infty} h(\nu_n)$. We define, inductively, the sequence $\{m_i\}$ as follows: let m_1 be the first positive integer such that $\sum_{n=m_1}^{\infty} h(2\nu_n) \leq \sigma/2$; having defined m_i , $i \geq 1$, let m_{i+1} be the first positive integer greater than m_i such that $\sum_{n=m_{i+1}}^{\infty} h[(j+2)\nu_n] \leq \sigma/2^{i+1}$. Let $\{k_n\}$ denote the sequence

$$\underbrace{1, \dots, 1}_{m_1}, \underbrace{2, \dots, 2}_{m_2}, \dots, \underbrace{j, \dots, j}_{m_j}, \dots.$$

Then we have $\sum_{n=1}^{\infty} h(k_n \nu_n) \leq \sum_{n=1}^{\infty} \sigma/2^{n-1} = 2\sigma$. Therefore $\{k_n \nu_n\} \in M(h)$, as required. Now, we turn to the proof of the theorem. Let $f, g \in \mathfrak{A}$, and let $h = f \wedge g$. The inclusion $D(f) \vee D(g) \subset D(h)$ is easily established. To prove the other inclusion, let $\{\nu_n\} \in D(h)$. By the first part of the proof there exists an unbounded increasing sequence $\{k_n\}$ of positive integers such that $\{k_n \nu_n\} \in M(h)$. Define the sequences $\{\lambda_n\}, \{\mu_n\} \in C$ as follows: $\lambda_n = \nu_n$, $\mu_n = 0$ whenever $f(k_n \nu_n) = h(k_n \nu_n)$, and $\lambda_n = 0$, $\mu_n = \nu_n$, otherwise. Then we have $\nu_n = \lambda_n \vee \mu_n$, $f(k_n \lambda_n) \leq h(k_n \nu_n)$ and $g(k_n \mu_n) \leq h(k_n \nu_n)$, $n = 1, 2, \dots$. Therefore $\{k_n \lambda_n\} \in M(f)$, $\{k_n \mu_n\} \in M(g)$, and hence $\{\lambda_n\} \in D(f)$, $\{\mu_n\} \in D(g)$. From [1, Lemma 4.9] we conclude that $\{\nu_n\} \in D(f) \vee D(g)$.

It is not difficult to see that if J is any ideal set, then $D(f \vee g; J) = D(f; J) \cap D(g; J)$. Thus it is reasonable to ask: Do there exist functions $f, g \in \mathfrak{A}$ and a proper ideal set J such that $D(f \wedge g; J) \neq D(f; J) \vee D(g; J)$? From Remark 4.5 it follows that the mapping $\hat{f} \rightarrow S(\hat{f}; J)$ is lattice preserving for every ideal set J . It is worth noting that for those ideal sets J such that the preceding question has a negative answer, the mapping $\hat{f} \rightarrow D(\hat{f}; J)$ is also lattice preserving.

FOOTNOTES

1. The joint $J \vee K$ of the ideal sets J and K is the smallest ideal set containing J and K .
2. It was brought to our attention that this result was also obtained, independently, by R. Sneider.

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