Hamiltonians for Singular Potentials

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1. Introduction. Quantum theory requires one to consider in $L^2(\mathbb{R}^3)$ operators of the form

$$(1.1) \quad L = -\Delta + V(x),$$

where Δ is the Laplacian and $V(x)$ is a function which is allowed to have singularities. We can define Lu for functions $u \in L^2$ such that Δu is in L^2 (in the sense of distributions) and Vu is also in L^2 . The theory requires that L thus defined have a “unique” (strictly) self-adjoint extension H , called the Hamiltonian. The best known result in this connection is that of Kato [13], who showed that L itself is (strictly) self-adjoint if V is real valued and in $L^2 + L^\infty$.

If L is not self-adjoint, the approach has been to find criteria for L to have a self-adjoint closure (we then call L essentially self-adjoint). Many authors have given criteria for this (see [7] for a partial list). The reason for the approach is that the closure is uniquely determined by L and consequently conforms with the requirements of the theory. However, one must require that the domain of L (i.e., the set of functions for which Lu is defined) is dense in L^2 . To accomplish this, it is usually assumed that V is locally in L^2 .

However, there are potentials V of interest in quantum mechanics which have such strong singularities that the domain of L is very small; indeed it may consist only of the function $u \equiv 0$. For such functions, L cannot have a self-adjoint closure. Hence there may be several self-adjoint extensions. The requirement that H depend uniquely on V causes a problem for such potentials.

Recently, Kato [7, p. 346], Schechter [4, 5], Faris [14] and Simon [12] have used a method of bilinear forms for defining the Hamiltonian (*cf.* Section 3). The basic idea is simple. Although Lu may not be in L^2 for many functions $u \in L^2$, the bilinear form

$$h(u, v) = (\nabla u, \nabla v) + (Vu, v)$$

is densely defined for V locally integrable. And under fairly mild conditions on V one can show that h determines a self-adjoint operator H which is an

extension of L and depends uniquely on V . We call H the "forms" extension of L because it is obtained by means of bilinear forms.

Kato assumed that $V = V_1 + V_2 + V_3$, where $V_1 \geq 0$ and for some $\epsilon > 0$

$$\int_{|x-y|<1} V_1(y) |x-y|^{-1-\epsilon} d^3y$$

is locally bounded. Also

$$\sup_x \int_{|x-y|<1} |V_2(y)| |x-y|^{-1-\epsilon} d^3y < \infty,$$

and

$$V_3(x) = \sum_{k=1}^N e_k |x - a_k|^{-2}$$

with

$$\sum_{k=1}^N \min [e_k, 0] > -\frac{1}{4}.$$

Schechter assumed that

$$\sup_x \int_{|x-y|<1} |V_{(-)}(y)| |x-y|^{-1} d^3y < \infty,$$

where

$$V_{(-)}(y) = \min [V(y), 0].$$

Simon assumed that $V = V_1 + V_2$, where $V_1 \in L^\infty$ and $V_2 \in R$, where R is the set of functions such that

$$\|W\|_R^2 = \int |W(x)W(y)| |x-y|^{-2} d^3x d^3y < \infty.$$

Each of these authors showed that under the assumptions given by him, L has a forms extension. None of these results contains either of the others. Faris [14] assumed that $V \in L^{3/2}$.

The purpose of the present paper is to extend these results to much wider classes of potentials. Essentially our results completely characterize the potentials V for which L has a forms extension. The new classes of potentials include:

1. The set of functions $V(x)$ such that

$$\inf_{\lambda \geq 0} \sup_{p, p'} \int |\hat{V}(p)| (\lambda + |p - p'|^2)^{-1} d^3p < 1,$$

where $\hat{V}$ denotes the Fourier transform of V .

2. The completion of R with respect to the norm given by

$$||V||_s^2 = \iint V(x)V(y)e^{-|x-y|} |x-y|^{-2} d^3x d^3y$$

(we show that this is a norm).

3. The completion of R with respect to the norm given by

$$(1.2) \quad ||V||_g^2 = \iint V(x)V(y) |x-y|^{-2} d^3x d^3y$$

(this gives a norm as well).

Also included are all potentials with singularities weaker than r^{-2} . In particular, $V = -(-\ln r)^{-\alpha}r^{-2}$ is included for all $\alpha > 0$. Even $V = -1/4r^2$ is included.

It follows from our results that L has a forms extension for potentials V in these classes (and for sums of functions in these classes). Our general theorem (Theorem 4.1) establishes the existence of a Hamiltonian for potentials in general classes containing those mentioned above. It also gives sufficient conditions for the essential spectrum of H to be contained in $[0, \infty)$ and for H to have at most a finite number of negative eigenvalues.

We also give an estimate for the number of negative eigenvalues which improves estimates of Birman [15], Schwinger [10] and Ghirardi-Rimini [11]. Their estimates are

$$||V_{(-)}||_R^2/(4\pi)^2$$

and

$$||V||_g^2/(4\pi)^2,$$

respectively. We obtain as an estimate

$$\frac{||V||_g^2}{(4\pi)^2} - \frac{\{\max [(V, V_{(+)})_g, 0]\}^2}{||V_{(+)}||_g^2},$$

where $(V, W)_g$ is the scalar product corresponding to (1.2) and $V_{(+)}(x) = \max [V(x), 0]$. This estimate is never greater than either of the others and is strictly less in many cases (see Section 2).

All of our results hold in $\mathbf{R}^n$, n arbitrary. Some of our theorems are of interest in themselves. For instance, we give necessary and sufficient conditions for an estimate of the form

$$||qu|| \leq C ||(\lambda - \Delta)^s u||$$

to hold (Theorem 5.2). We also give criteria for $(\lambda - \Delta)^{-s}q(\lambda - \Delta)^{-t}$ to be a Hilbert-Schmidt operator, a compact operator or a bounded operator on L^2 (see Theorems 5.3, 5.8, 5.9 and 5.10).

We have restricted our attention to real valued potentials $V(x)$ and self-adjoint extensions. However, our methods work *mutatis mutandis* for complex valued functions and regularly accretive extensions (cf. [4, 5]).

2. Potentials in three dimensions. We now consider several classes of functions and distributions on $\mathbf{R}^3$.

a. We let R be the set of measurable functions $V(x)$ such that

$$(2.1) \quad \|V\|_R^2 = \iint |V(x)V(y)| |x-y|^{-2} d^3x d^3y$$

is finite. For $V \in R$ the integral

$$(2.2) \quad \|V\|_G^2 = \iint V(x)\overline{V(y)} |x-y|^{-2} d^3x d^3y$$

exists. We shall prove

Lemma 2.1. $\|V\|_G$ is a norm on R .

Using this fact we

b. Let G be the completion of R with respect to the norm given by (2.2).

We let $G_{s,E}(x)$ denote the Green's function for the operator $(E - \Delta)^{s/2}$ (see Section 4). We define several classes of functions in terms of this function.

c. For $E \geq 0$, let M_E denote the set of measurable functions $V(x)$ such that

$$(2.3) \quad M_E(V) = \sup_x \int |V(y)| G_{2,E}(x-y) d^3y$$

is finite.

d. Let S be the set of tempered distributions $V(x)$ such that

$$(2.4) \quad \|V\|_S^2 = \int \left| \int V(x) G_{1/2}(x-y) d^3x \right|^2 d^3y$$

is finite, where $G_s(x) = G_{s,1}(x)$.

e. Let T denote the set of tempered distributions $V(x)$ such that

$$(2.5) \quad \|V\|_T^2 = \int |\hat{V}(p)|^2 (1 + |p|^2)^{-1/2} d^3p$$

is finite, where $\hat{V}$ is the Fourier transform of V (see Section 4).

Since $G_1(x) \leq \text{const. } |x|^{-2}$, the integral

$$(2.6) \quad \|V\|_U^2 = \iint V(x)\overline{V(y)} G_1(x-y) d^3x d^3y$$

exists for $V \in R$. We shall prove

Lemma 2.2. $\|V\|_U$ is a norm on R .

As in the case of G , we make the following definitions:

f. Let U be the completion of R with respect to the norm given by (2.6).

g. For $E \geq 0$ we let W_E denote the set of tempered distributions V such that

$$(2.7) \quad W_E(V) = \sup_{p'} \int |\hat{V}(p)| (E + |p - p'|^2)^{-1} d^3p$$

is finite.

h. Let B_E denote the set of measurable functions $V(x)$ such that

$$(2.8) \quad B_E(V) = \inf_{\Psi(x) \geq 0} \sup_x \frac{1}{\Psi(x)} \int |V(y)| G_{2,E}(x - y) \Psi(y) d^3y$$

is finite. We shall see in Section 4 that it makes no difference whether we take the infimum in (2.8) over all non-negative functions on $\mathbf{R}^3$ or only over those non-negative functions which are in $L^2(\mathbf{R}^3)$.

Concerning the spaces defined in a.-h. we shall prove the following in Section 4.

Lemma 2.3. *G, S, T and U are Hilbert spaces. For $E \geq 0$, M_E and W_E are Banach spaces.*

Theorem 2.4. *$S = T = U$ with equal norms.*

Theorem 2.5. *$R \subset G \subset S$ with continuous inclusions.*

We shall also prove

Theorem 2.6. *Suppose $V(x)$ is a real valued locally integrable function and there is an $E > 0$ such that $V^\bullet = V_1 + V_2 + V_3 + V_4$, where $V_1 \geq 0$, $V_2 \in S$ and*

$$B_E(V_3) + W_E(V_4) < 1.$$

Then $-\Delta + V$ has a self-adjoint extension H such that $\sigma_e(H) \subset [-E, \infty)$ and there are at most a finite number of eigenvalues of H below $-E$.

In Section 4 we put this theorem in a stronger and more precise form. In particular we show that H is “canonical” in a sense prescribed in the next section. In particular, if $-\Delta + V$ is essentially self-adjoint, then H is its closure. We also have

Theorem 2.7. *Assume $V(x)$ is real valued, locally integrable and $V = V_1 + V_2 + V_3 + V_4$, where $V_1 \geq 0$, $V_2 \in G$ and*

$$\theta = B_0(V_3) + W_0(V_4) < 1.$$

Then $-\Delta + V$ has a self-adjoint extension H such that

$$(2.9) \quad \sigma_e(H) \subset [0, \infty),$$

and H has no more than

$$(2.10) \quad ||V_2||_G^2 / (4\pi)^2 (1 - \theta)^2$$

negative eigenvalues.

As a corollary to this last result we obtain the following improvement of estimates due to Birman [15], Schwinger [10] and Ghirardi–Rimini [11].

Theorem 2.8. Assume $U(x)$, $V(x)$ are real valued functions with $V(x) \leq U(x)$ and $V \in G$. Then $-\Delta + U(x)$ has a self-adjoint extension H having no more than

$$(2.11) \quad N(V) = \frac{\|V\|_G^2}{(4\pi)^2} - \frac{\{\max [(V, V_{(+)})_G, 0]\}^2}{\|V_{(+)}\|_G^2}$$

negative eigenvalues, where $(V, W)_G$ is the scalar product in G and $V_{(+)}(x) = \max [V(x), 0]$.

The bound given by Birman and Schwinger is

$$(2.12) \quad \|V_{(-)}\|_R^2 / (4\pi)^2,$$

and that of Ghirardi–Rimini is

$$(2.13) \quad \|V\|_G^2 / (4\pi)^2,$$

where $V_{(-)}(x) = \min [V(x), 0]$. Note that $N(V)$ given by (2.11) is never greater than either of these bounds. This is obvious for (2.13). To show that this is also true for (2.12), note that

$$\|V_{(-)}\|_R^2 = \|V_{(-)}\|_G^2 = \|V - V_{(+)}\|_G^2 = \|V\|_G^2 + \|V_{(+)}\|_G^2 - 2(V, V_{(+)})_G.$$

If $(V, V_{(+)})_G \leq 0$, we have

$$\|V\|_G \leq \|V_{(-)}\|_R,$$

and consequently $N(V)$ is not greater than (2.12). Otherwise we have

$$\|V_{(-)}\|_R^2 = \|V\|_G^2 + \left[\|V_{(+)}\|_G - \frac{(V, V_{(+)})_G}{\|V_{(+)}\|_G} \right]^2 - (V, V_{(+)})_G^2 / \|V_{(+)}\|_G^2,$$

which also shows that $N(V)$ is not greater than Schwinger's bound. Note that $N(V)$ is strictly less than either of the other bounds unless $(V, V_{(+)})_G \leq 0$ (when it equals (2.13)) or $(V_{(-)}, V_{(+)})_G = 0$ (when it equals (2.12)).

Theorems 2.6–2.8 will be proved in Section 5.

3. Bilinear forms. In this section we present a method for defining the Hamiltonian for singular potentials. We employ the concept of regularly accretive operators introduced by Kato [3]. We follow the development of Schechter [4, 5].

For the definition and properties of bilinear forms we refer to Schechter [6]. Let H be a complex Hilbert space and let $a(u, v)$ be a bilinear form with domain $D(a)$ contained in H . We call a closed if $\{u_n\} \subset D(a)$, $u_n \rightarrow u$ in H and $a(u_n - u_m) \rightarrow 0$ implies that $u \in D(a)$ and $a(u_n - u) \rightarrow 0$. It is called closable if $\{u_n\} \subset D(a)$, $u_n \rightarrow 0$ in H and $a(u_n - u_m) \rightarrow 0$ implies $a(u_n) \rightarrow 0$. In [7, p. 315] and [6, p. 283] the following was proved

Lemma 3.1. Let $a(u, v)$ be a closable bilinear form satisfying

$$(3.1) \quad |a(u)| \leq C [\operatorname{Re} a(u) + M \|u\|^2], \quad u \in D(a),$$

for some constants $C > 0$ and M , where $a(u) = a(u, u)$. Then $a(u, v)$ has an extension which is closed, satisfies (3.1) and is the smallest closed extension of a .

If $a(u, v)$ is a densely defined bilinear form, we define the linear operator A associated with $a(u, v)$ as follows. We say that $u \in D(A)$ if $u \in D(a)$ and there is an $f \in H$ such that

$$(3.2) \quad a(u, v) = (f, v), \quad v \in D(a).$$

We then define Au to be f (cf. [6, p. 273]). We call an operator A *regularly accretive* if there exists a densely defined closed bilinear form $a(u, v)$ satisfying (3.1) with A as the associated operator.

Let $b(u, v)$, $c(u, v)$ be bilinear forms such that $D(a) \cap D(b)$ is dense in H , and let B, C be their associated operators, respectively. Set $a(u, v) = b(u, v) + c(u, v)$, with $D(a) = D(b) \cap D(c)$. If a satisfies (3.1) and is closable, then its closure also satisfies (3.1) (Lemma 3.1). The operator A associated with its closure is regularly accretive by definition. Clearly A is an extension of $B + C$. We call it the *forms extension*. It depends uniquely on b and c .

Theorem 3.2. *Let $a(u, v)$ be a closable bilinear form satisfying (3.1). Suppose $b(u, v)$ is a bilinear form satisfying*

$$(3.3) \quad \operatorname{Re} b(u) + \theta (\operatorname{Re} a(u) + M \|u\|^2) + K_0 \|u\|^2 \geq 0, \quad u \in D(a) \cap D(b),$$

$$(3.4) \quad |\operatorname{Im} b(u)| \leq K_1 (|a(u)| + \|u\|^2), \quad u \in D(a) \cap D(b),$$

for constants $\theta < 1$, $K_0, K_1 > 0$. Then the bilinear form $h = a + b$ is closable, and there are positive constants K_2, K_3 such that

$$(3.5) \quad |h(u)| \leq K_2 (\operatorname{Re} h(u) + K_3 \|u\|^2), \quad u \in D(h).$$

Proof. If $u \in D(h)$, we have by (3.3)

$$\operatorname{Re} a(u) + M \|u\|^2 \leq \operatorname{Re} h(u) + M \|u\|^2 + \theta (\operatorname{Re} a(u) + M \|u\|^2) + K_0 \|u\|^2.$$

Thus

$$(3.6) \quad (1 - \theta) (\operatorname{Re} a(u) + M \|u\|^2) \leq \operatorname{Re} h(u) + (M + K_0) \|u\|^2.$$

Since $b = h - a$, we have by (3.1)

$$\operatorname{Re} b(u) \leq \operatorname{Re} h(u) + M \|u\|^2.$$

Moreover by (3.3)

$$\begin{aligned} -\operatorname{Re} b(u) &\leq \theta (\operatorname{Re} a(u) + M \|u\|^2) + K_0 \|u\|^2 \\ &\leq \frac{\theta}{1 - \theta} (\operatorname{Re} h(u) + (M + K_0) \|u\|^2) + K_0 \|u\|^2. \end{aligned}$$

Thus there are positive constants K_4, K_5 such that

$$(3.7) \quad |\operatorname{Re} b(u)| \leq K_4 (\operatorname{Re} h(u) + K_5 \|u\|^2).$$

If we combine (3.1) (3.4) (3.6) and (3.7) we obtain positive constants K_2 and K_3 such that

$$(3.8) \quad |a(u)| + |b(u)| \leq K_2 (\operatorname{Re} h(u) + K_3 \|u\|^2), \quad u \in D(h).$$

This gives (3.5). To show that h is closable, suppose $\{u_k\} \subset D(h)$, $u_k \rightarrow 0$, and $h(u_k - u_k) \rightarrow 0$. Then by (3.8), $a(u_k - u_k) \rightarrow 0$. Since a is closable, $a(u_k) \rightarrow 0$. By (3.3), $b(u_k) \rightarrow 0$. Hence $h(u_k) \rightarrow 0$. Thus h is closable, and the proof is complete.

Corollary 3.3. *Let a, b be bilinear forms which satisfy the hypotheses of Theorem 3.2. Assume that $D(a) \cap D(b)$ is dense, and let A, B be the associated operators, respectively. Then $A + B$ has a forms extension.*

Lemma 3.4. *Let a be a closable bilinear form satisfying (3.1). Assume that b is a bilinear form such that $\{u_k\} \subset D(a) \cap D(b)$, $u_k \rightarrow 0$, $|\operatorname{Re} a(u_k)| \leq C$ imply $b(u_k) \rightarrow 0$. Then for each $\epsilon > 0$ there is a constant K such that*

$$(3.9) \quad |b(u)| \leq \epsilon [\operatorname{Re} a(u) + M \|u\|^2] + K \|u\|^2, \quad u \in D(a) \cap D(b).$$

Proof. We first show that there is a constant N such that

$$(3.10) \quad |b(u)| \leq N [\operatorname{Re} a(u) + (M + 1) \|u\|^2], \quad u \in D(a) \cap D(b).$$

If (3.10) did not hold, there would be a sequence $\{v_k\}$ in $D(a) \cap D(b)$ such that

$$|b(v_k)| = 1, \quad \operatorname{Re} a(v_k) + (M + 1) \|v_k\|^2 \rightarrow 0.$$

Thus $v_k \rightarrow 0$ and $|\operatorname{Re} a(v_k)| \leq \text{constant}$. By hypothesis this implies $b(v_k) \rightarrow 0$, providing the contradiction establishing (3.6). Now suppose (3.9) did not hold. Then there would exist an $\epsilon > 0$ and a sequence $\{u_k\} \subset D(a) \cap D(b)$ such that

$$\operatorname{Re} a(u_k) + (M + 1) \|u_k\|^2 = 1 \quad \text{and} \quad |b(u_k)| > \epsilon + k \|u_k\|^2.$$

By (3.10), $|b(u_k)| \leq N$. Hence $u_k \rightarrow 0$. By hypothesis this implies $b(u_k) \rightarrow 0$. But we have $|b(u_k)| > \epsilon$ for each k . This contradiction completes the proof.

A bilinear form $a(u, v)$ is called symmetric if $a(u)$ is real for all $u \in D(a)$ (cf. [6]).

Lemma 3.5. *Let a, h be closed, densely defined, symmetric bilinear forms bounded from below. Assume that $D(h) \subset D(a)$ and that*

$$(3.11) \quad a(u) \leq h(u), \quad u \in D(h).$$

Let A, H be the associated operators. They are self-adjoint, and the number of eigenvalues (counting multiplicities) of H in an interval of the form $(-\infty, \lambda]$ is not greater than the number of eigenvalues (counting multiplicities) of A in the same interval, provided the interval contains no points in the essential spectrum of A .

Proof. That A and H are self-adjoint was proved in [5, p. 33]. Let $\lambda_1 \leq \lambda_2 \leq \dots$ and $\mu_1 \leq \mu_2 \leq \dots$ be the eigenvalues of A and H in the interval,

respectively. Then

$$\begin{aligned}\lambda_n &= \max_{u_1, \dots, u_{n-1}} \left[\min_{\substack{v \in D(a), \|v\|=1 \\ v \perp u_1, \dots, u_{n-1}}} a(v) \right] \\ &\leq \max_{u_1, \dots, u_{n-1}} \left[\min_{\substack{v \in D(h), \|v\|=1 \\ v \perp u_1, \dots, u_{n-1}}} h(v) \right] = \mu_n.\end{aligned}$$

For the definition and properties of the essential spectrum we refer to [5, 6].

Theorem 3.6. *Let a, b, c be symmetric bilinear forms such that $D(a) \cap D(b) \subset D(c)$ and $D(a) \cap D(b)$ is dense. Assume that a is closable and satisfies (3.1). Assume that $\{u_k\} \subset D(a) \cap D(b)$, $|a(u_k)|$ bounded and $u_k \rightarrow 0$ imply $c(u_k) \rightarrow 0$. Finally assume that there are real numbers $\lambda < 1$ and K such that*

$$(3.13) \quad c(u) \leq b(u) + \lambda[a(u) + M \|u\|^2] + K \|u\|^2, \quad u \in D(a) \cap D(b).$$

Then $a + b$ is closable and the operator H , associated with its closure, is self-adjoint and regularly accretive.

Proof. By Lemma 3.4 for each $\epsilon > 0$ there is a constant K_ϵ such that

$$|c(u)| \leq \epsilon[a(u) + M \|u\|^2] + K_\epsilon \|u\|^2.$$

Hence there is a constant K'_ϵ such that

$$(3.14) \quad b(u) \geq -(\lambda + \epsilon)[a(u) + M \|u\|^2] - K'_\epsilon \|u\|^2.$$

If we take ϵ so small that $\lambda + \epsilon < 1$, we see that the hypotheses of Theorem 3.2 are satisfied. Thus $a + b$ is closable and bounded from below. Hence the operator H associated with its closure is self-adjoint and regularly accretive (cf. [5, p. 33]). This completes the proof.

Theorem 3.7. *Under the hypotheses of Theorem 3.6, assume that $\{u_k\} \subset D(a) \cap D(b)$, $|a(u_k)|$ bounded, $u_k \rightarrow 0$ imply that $c(u_k) \rightarrow 0$. Then*

$$(3.15) \quad \sigma_e(H) \subset [-M - K, \infty).$$

Proof. Let h denote the closure of $a + b$. If μ is a point in the essential spectrum of H , then there is $\{v_k\} \subset D(H)$ such that

$$\|v_k\| = 1, \quad v_k \rightarrow 0, \quad (H - \mu)v_k \rightarrow 0.$$

Thus

$$h(v_k) \rightarrow \mu.$$

Now for each k there is a $u_k \in D(a) \cap D(b)$ such that

$$|h(u_k - v_k)| < 1/k, \quad \|u_k - v_k\| < 1/k.$$

(cf. [6, p. 283]). Thus

$$\|u_k\| \rightarrow 1, \quad u_k \rightarrow 0, \quad h(u_k) \rightarrow \mu.$$

By (3.14), $|a(u_k)|$ is bounded. Thus $c(u_k) \rightarrow 0$. Since

$$h(u_k) \geq (1 - \lambda)[a(u_k) + M \|u_k\|^2] - (M + K) \|u_k\|^2 + c(u_k),$$

we have

$$\liminf_{k \rightarrow \infty} h(u_k) \geq -M - K.$$

Thus $\mu \geq -M - K$. This proves the theorem.

Theorem 3.8. *Let A be a closable linear operator and let b be a symmetric bilinear form such that $D(A) \cap D(b)$ is dense. Assume that there is a symmetric compact operator K and a number $\lambda < 1$ such that*

$$(3.16) \quad (KAu, Au) \leq b(u) + \lambda \|Au\|^2, \quad u \in D(A) \cap D(b).$$

Let B be the operator associated with b . Then $A^*A + B$ has a self-adjoint forms extension H such that

$$(3.17) \quad \sigma_*(H) \subset [0, \infty).$$

Moreover, the number N of negative eigenvalues (counting multiplicities) of H is finite and does not exceed the number of eigenvalues (counting multiplicities) of K in the interval $[-\|K\|, \lambda - 1]$.

Proof. Set $a(u, v) = (Au, Av)$, with $D(a) = D(A)$. Then a is closable. For if $\{u_n\} \subset D(a)$, $u_n \rightarrow 0$ and $a(u_n - u_m) \rightarrow 0$, then $Au_n \rightarrow w$ in H . Since A is closable, we must have $w = 0$. Thus $a(u_n) \rightarrow 0$.

Next set $c(u, v) = (KAu, Av)$, with $D(c) = D(A)$. Assume that $u_k \in D(a) \cap D(b)$, $|a(u_k)| \leq C$ and $u_k \rightarrow 0$. Then there is a subsequence (also denoted by $\{u_k\}$) such that $Au_k \rightarrow w$. Now $D(A^*)$ is dense (cf. [6, p. 177]). But if $v \in D(A^*)$ we have

$$(w, v) = \lim (Au_k, v) = \lim (u_k, A^*v) = 0.$$

Thus $w = 0$. By compactness $(KAu_k, Au_k) \rightarrow 0$. Thus the hypotheses of Theorem 3.7 are satisfied with $M = K = 0$. This proves (3.17).

Next let h be the closure of $a + b$ and let $\bar{A}$ be the closure of A . Set

$$(3.18) \quad g(u, v) = ([K + 1 - \lambda]\bar{A}u, \bar{A}v).$$

Clearly g is closed, $D(h) \subset D(g)$ and

$$(3.19) \quad g(u) \leq h(u), \quad u \in D(h).$$

Let G denote the operator associated with g . Theorem 3.7 applied to g shows that G has no negative points in its essential spectrum. Suppose it has negative eigenvalues $\lambda_1, \dots, \lambda_n$. Let $u_1, \dots, u_n$ be orthonormal eigenelements corresponding to them. If

$$u = \alpha_1 u_1 + \dots + \alpha_n u_n,$$

then

$$(3.20) \quad \begin{aligned} g(u) &= \sum \alpha_i \bar{\alpha}_k g(u_i, u_k) \\ &= \sum \alpha_i \bar{\alpha}_k \lambda_k (u_i, u_k) = \sum \lambda_k |\alpha_k|^2 \leq 0. \end{aligned}$$

Since

$$Gu = \sum \lambda_k \alpha_k u_k,$$

the λ_k do not vanish, and $\bar{A}u = 0$ implies $Gu = 0$, we see that all the α_k vanish when $\bar{A}u = 0$. This means that the elements $\bar{A}u_k$ are linearly independent. By (3.18) and (3.20) we see that there is an n -dimensional subspace W such that

$$([K + 1 - \lambda]v, v) \leq 0, \quad v \in W.$$

This implies that $K + 1 - \lambda$ has at least n non-positive eigenvalues (counting multiplicities). Now apply Lemma 3.5.

4. Potentials in $\mathbf{R}^n$. In this section we generalize some of the ideas formulated in Section 2 and give further results in terms of the concepts formulated in Section 3.

Let s be a real number, and let $C_0^\infty = C_0^\infty(\mathbf{R}^n)$ denote the set of test functions on $\mathbf{R}^n$ (i.e., the set of complex valued infinitely differentiable functions with compact supports). For $u \in C_0^\infty$ we set

$$(4.1) \quad |u|_*^2 = \int |\hat{u}(\xi)|^2 (1 + |\xi|^2)^s d\xi,$$

where $\hat{u}$ is the Fourier transform of u given by

$$(4.2) \quad \hat{u}(\xi) = (2\pi)^{-n/2} \int e^{-i\xi x} u(x) dx,$$

$\xi = (\xi_1, \dots, \xi_n)$, $\xi x = \xi_1 x_1 + \dots + \xi_n x_n$, $dx = d^n x$, etc. When T is a tempered distribution (cf. [8]), $\hat{T}$ is given by

$$(4.3) \quad (\hat{T}, \phi) = (T, \varphi), \quad \varphi \in C_0^\infty.$$

We define $H^s = H^s(\mathbf{R}^n)$ as the set of tempered distributions T such that $(1 + |\xi|^2)^{s/2} \hat{T}$ is in $L^2(\mathbf{R}^n)$ (i.e., that (4.1) is finite for $u = T$). It is not difficult to show that H^s is a Hilbert space and the C_0^∞ is dense in H^s for each real s (cf. [8, p. 330]).

For $s > 0$ and $\lambda \geq 0$ we let $G_{s,\lambda}(x)$ denote the inverse Fourier transform of $(2\pi)^{-n/2}(\lambda + |\xi|^2)^{-s/2}$. It is not difficult to show that

$$(4.4) \quad \begin{aligned} G_{s,\lambda}(x) &= (4\pi)^{-n/2} \Gamma(s/2)^{-1} \int_0^\infty \exp\left\{-\frac{|x|^2}{4t} - \lambda t\right\} t^{s/2-n/2-1} dt, \quad \lambda > 0, \\ G_{s,0}(x) &= 2^{-s} \pi^{-n/2} \Gamma\left(\frac{n-s}{2}\right) |x|^{s-n} \Gamma\left(\frac{s}{2}\right), \quad s < n, \end{aligned}$$

(cf. [9]). We note the following properties of these functions

$$(4.5) \quad G_{s,\lambda}(x) < G_{s,\mu}(x) \quad \text{for } \lambda > \mu,$$

$$(4.6) \quad 0 < G_{s,\lambda} \leq C |x|^{s-n} e^{-a|x|}, \quad \lambda > 0,$$

where C and a are positive constants depending only on s , n and λ .

For a function u set

$$(4.7) \quad G_{s,\lambda} u = G_{s,\lambda} * u = \int G_{s,\lambda}(x-y) u(y) dy.$$

Then one checks readily by means of Fourier transforms

$$(4.8) \quad G_{s,\lambda} G_{t,\lambda} u = G_{s+t,\lambda} u.$$

In three dimensions we have the particularly simple expression

$$(4.9) \quad G_{2,\lambda}(x) = (4\pi |x|)^{-1} \exp \{-\lambda^{1/2} |x|\}.$$

For real $s > 0$ we define the operator $(\lambda - \Delta)^s$ by means of the Fourier transform as

$$(4.10) \quad [(\lambda - \Delta)^s u]^\wedge = (\lambda + |\xi|^2)^s \hat{u}$$

Thus

$$(4.11) \quad (\lambda - \Delta)^s G_{2s,\lambda} u = u, \quad u \in L^2, \quad \lambda > 0,$$

$$(4.12) \quad G_{2s,\lambda} (\lambda - \Delta)^s u = u, \quad u \in H^{2s}.$$

For $s = 1$, (4.10) reduces to the usual definition of the operator $\lambda - \Delta$.

We now introduce some spaces related to those of Section 2. We assume $s > 0$ and $\lambda \geq 0$.

A. Let $M_{s,\lambda}$ denote the set of measurable functions $q(x)$ such that

$$(4.13) \quad M_{s,\lambda}(q) = \sup_x \int |q(y)| G_{2s,\lambda}(x-y) dy$$

is finite.

B. Let $W_{s,\lambda}$ denote the set of tempered distributions $q(x)$ such that

$$(4.14) \quad W_{s,\lambda}(q) = \sup_\xi \int |\hat{q}(\eta)| (\lambda + |\xi - \eta|^2)^{-s} d\eta$$

is finite.

C. Let $B_{s,\lambda}$ denote the set of those measurable functions $q(x)$ such that

$$(4.15) \quad B_{s,\lambda}(q) = \inf_{\Psi \geq 0} \sup_x \frac{1}{\Psi(x)} \int |q(y)| G_{2s,\lambda}(x-y) \Psi(y) dy$$

is finite.

When $n = 3$ we have $M_{\mathbb{E}}(V) = M_{1,\mathbb{E}}(V)$, etc. in the notation of Section 2. Note that

$$B_{s,\lambda}(q) \leq M_{s,\lambda}(q).$$

We now come to one of our basic theorems. We let $H^t + (B_{s,\lambda})_{\epsilon}$ denote the set of those tempered distributions q such that for each $\epsilon > 0$ there is a $q_1 \in H^t$ for which

$$B_{s,\lambda}(q - q_1) < \epsilon.$$

We have

Theorem 4.1. Assume that $q(x)$ is a real valued locally integrable function such that $q = q_1 + q_2 + q_3 + q_4$, where $q_1 \geq 0$, $q_2 \in H^{n/2-2s} + (B_{s,\lambda})_{\epsilon}$ and $W_{s,\lambda}(q_3) + B_{s,\lambda}(q_4) < 1$. Then $(\lambda - \Delta)^s + q$ has a self-adjoint forms extension H_{λ} such that

$$(4.16) \quad \sigma_{\epsilon}(H_{\lambda}) \subset [0, \infty).$$

If $s = 1$, then $-\Delta + q$ has such an extension H satisfying

$$(4.17) \quad \sigma_{\epsilon}(H) \subset [-\lambda, \infty).$$

Moreover, H_{λ} has at most finite number of eigenvalues in $(-\infty, 0)$ and H has at most a finite number of eigenvalues in $(-\infty, -\lambda)$.

This theorem will be proved in Section 5. Meanwhile we discuss the spaces introduced here and in Section 2. Standard methods in integration theory give

Theorem 4.2. $M_{s,\lambda}$ and $W_{s,\lambda}$ are Banach spaces for each $s > 0$ and $\lambda \geq 0$.

We also have

Theorem 4.3. $R \subset H^{-1/2}$. Hence every function in R is a tempered distribution.

Proof. Suppose $V \in R$. Set $V_n(x) = V(x)$ for $|x| < n$ and zero otherwise. Then $V_n \in L^1$. By Fubini's theorem and Parseval's identity

$$(4.18) \quad \|V_n\|_G^2 = 2\pi^2(V_n, G_{1,0}V_n) = 2\pi^2 \int |\hat{V}_n(p)|^2 \frac{dp}{|p|}.$$

Thus

$$(4.19) \quad 2\pi^2 \int |\hat{V}_n(p)|^2 \frac{dp}{|p|} \leq \|V_n\|_R^2 \leq \|V\|_R^2.$$

Let u be a function in C_0^∞ . Then

$$(4.20) \quad |(V_n, u)| = |(\hat{V}_n, \hat{u})| \leq \|\hat{V}_n p^{-1/2}\| \|\hat{u} p^{1/2}\| \leq \text{constant} \|V\|_R \|u\|_{1/2}.$$

Now

$$(4.21) \quad (V_n, u) = (V, u)$$

for n sufficiently large. Hence

$$|(V, u)| \leq \text{constant } \|V\|_R \|u\|_{1/2}, \quad u \in C_0^\infty.$$

This implies that $V \in H^{-1/2}$.

Proof of Lemma 2.1. Let V be a function in R . Define $V_n(x)$ as above. Then $\|V - V_n\|_G \leq \|V - V_n\|_R \rightarrow 0$. By (4.18) this gives

$$2\pi^2 \|\hat{V}_n p^{-1/2}\|^2 \rightarrow \|V\|_G^2.$$

If $\|V\|_G = 0$, then for any $u \in C_0^\infty$ we have by (4.20)

$$(V_n, u) = (\hat{V}_n p^{-1/2}, p^{1/2}u) \rightarrow 0.$$

Since (4.21) holds for n sufficiently large, we have $(V, u) = 0$ for all $u \in C_0^\infty$. This implies $V \equiv 0$.

Proof of Lemmas 2.2, 2.3 and Theorems 2.4, 2.5. Note first that the space T is just $H^{-1/2}(\mathbf{R}^3)$. Next note that

$$(4.22) \quad \|V\|_S = \|G_{1/2}V\| = (2\pi)^{n/2} \|\hat{G}_{1/2}\hat{V}\| = \|V\|_{1/2}.$$

This shows that S is the same space and that its norm equals that of T . Now suppose $V \in R$. Then by Fubini's theorem

$$\|V\|_U^2 = (V, G_1V) = \|G_{1/2}V\|^2 = \|V\|_{-1/2}^2.$$

This shows that $\|V\|_U$ is a norm on R (proving Lemma 2.2) and that it coincides with the norm of $H^{-1/2}$ on R . Since $C_0^\infty \subset R$ and C_0^∞ is dense in $H^{-1/2}$, we see that $U = S = T = H^{-1/2}$. This completes the proof of Theorem 2.4 and part of Lemma 2.3. Since G is the completion of a pre-Hilbert space, the conclusion concerning it is trivial. Since $M_E = M_{E,1}$ and $W_E = W_{E,1}$, the remainder follows from Theorem 4.2. To prove Theorem 2.5 note that

$$(4.23) \quad \frac{1}{2\pi^2} \|u\|_S \leq \|u\|_G \leq \|u\|_R, \quad u \in C_0^\infty.$$

We have shown that $R \subset S$ (Theorem 4.3) and it is immediate from the definition that $R \subset G$. Now suppose $V \in G$. Then there is a sequence $\{V_k\} \subset R$ such that $\|V_k - V\|_G \rightarrow 0$. By (4.23) there is a $V' \in S$ such that $\|V_k - V'\|_S \rightarrow 0$. Moreover, any other sequence converging to V in G converges to V' in S . Note also that $V' = 0$ implies $V = 0$. For if $u \in C_0^\infty$, then

$$|(V_k, u)| \leq \|V_k\|_{-1/2} \|u\|_{1/2} \rightarrow 0.$$

But $(V_k, u) = (\hat{V}_k/p^{1/2}, p^{1/2}u) \rightarrow (\hat{V}, u)$. Thus $V = 0$.

5. Integral operators. We now prove certain theorems for integral operators on $L^2(\mathbf{R}^n)$. These results are used in the proofs of the theorems of Sections 2–4. However, some of them are of interest in their own right. We follow the development of Aronszajn–Mulla–Szeptycki [1] and Gagliardo [2].

Theorem 5.1. Let $K(x, y) \geq 0$ be a measurable function on $\mathbf{R}^{2n}$ and define

$$(5.1) \quad Tu(x) = \int K(x, y)u(y) dy.$$

Then T is a bounded operator on $L^2(\mathbf{R}^n)$ if and only if

$$(5.2) \quad C_0 = \inf_{\varphi \geq 0} \sup_y \frac{1}{\varphi(y)} \iint K(x, y)K(x, z)\varphi(z) dx dz$$

is finite. The value of C_0 is not affected if we restrict the infimum to those $\varphi \geq 0$ which are in L^2 . Moreover $\|T\|^2 = C_0$.

Proof. Suppose $C_0 < \infty$, and let $\epsilon > 0$ be given. Then there is a $\varphi \geq 0$ such that

$$\iint K(x, y)K(x, z)\varphi(z) dx dz \leq (C_0 + \epsilon)\varphi(y).$$

Set

$$\Psi(x) = \int K(x, z)\varphi(z) dz.$$

Then $\Psi(x) \geq 0$ and it is finite a.e. Also,

$$\int K(x, y)\Psi(x) dx \leq (C_0 + \epsilon)\varphi(y).$$

For $u, v \in C_0^\infty$ we have

$$\begin{aligned} |(Tu, v)| &\leq \iint K(x, y) |u(y)v(x)| dx dy \\ &\leq \left(\iint K(x, y)\Psi(x)\varphi(y)^{-1} |u(y)|^2 dx dy \right)^{1/2} \\ &\quad \times \left(\iint K(x, y)\varphi(y)\Psi(x)^{-1} |v(x)|^2 dx dy \right)^{1/2} \\ &\leq (C_0 + \epsilon)^{1/2} \|u\| \|v\|. \end{aligned}$$

Thus T is bounded on L^2 and $\|T\|^2 \leq C_0 + \epsilon$. Since $\epsilon > 0$ was arbitrary, we have $\|T\|^2 \leq C_0$.

Conversely, suppose T is bounded. Then so is T^*T . Let C be any number greater than $\|T^*T\|$, and let $h(x)$ be any positive function in L^2 . Set $\varphi_0 = 0$ and define

$$\varphi_k = h + C^{-1}T^*T\varphi_{k-1}, \quad k = 1, 2, \dots$$

Then $\varphi_k \geq 0$ by induction, and

$$\|\varphi_{k+1} - \varphi_k\| \leq \|T^*T\| \|\varphi_k - \varphi_{k-1}\|/C \leq \|T^*T\|^k \|\varphi_1\|/C^k.$$

Thus $\{\varphi_k\}$ converges in L^2 to a non-negative function φ satisfying

$$\varphi = h + C^{-1}T^*T\varphi.$$

Thus $\varphi \geq h > 0$ and $T^*T\varphi \leq C\varphi$. Thus C_0 is finite and $C_0 \leq \|T^*T\| \leq \|T\|^2$. This completes the proof.

Theorem 5.2. *The inequality*

$$(5.3) \quad (|q| u, u) \leq C([\lambda - \Delta]^s u, u), \quad u \in C_0^\infty,$$

holds if and only if

$$(5.4) \quad B_{s,\lambda}(q) \leq C.$$

Proof. Assume (5.4) holds. Set

$$K(x, y) = G_{s,\lambda}(x - y) |q(y)|^{1/2},$$

and define the operator T by (5.1). For each $\epsilon > 0$ there is a function $\Psi \geq 0$ such that

$$\int G_{2s,\lambda}(y - z) |q(z)| \Psi(z) dz \leq (B_{s,\lambda}(q) + \epsilon)\Psi(y).$$

Set $\varphi(z) = |q(z)|^{1/2} \Psi(z)$. Then $\varphi(z) \geq 0$ and

$$\begin{aligned} \iint K(x, y)K(x, z)\varphi(z) dx dz \\ &= \int \left[\int G_{s,\lambda}(x - y)G_{s,\lambda}(x - z) dx \right] |q(y)|^{1/2} |q(z)|^{1/2} \varphi(z) dz \\ &= \int G_{2s,\lambda}(y - z) |q(z)| \Psi(z) dz |q(y)|^{1/2} \leq [B_{s,\lambda}(q) + \epsilon]\varphi(y). \end{aligned}$$

Thus by Theorem 5.1, T is bounded and $\|T\| \leq B_{s,\lambda}(q) + \epsilon$. Since this is true for every $\epsilon > 0$, we have $\|T\| \leq B_{s,\lambda}(q)$.

Let u be any function in C_0^∞ and set $v = (\lambda - \Delta)^{s/2}u$. Then $u = G_{s,\lambda}v$. Thus

$$\begin{aligned} (|q| u, u) &= (|q| G_{s,\lambda}v, G_{s,\lambda}v) = \|T^*v\|^2 \\ &\leq B_{s,\lambda}(q) \|v\|^2 = B_{s,\lambda}(q)([\lambda - \Delta]^s u, u). \end{aligned}$$

This proves (5.3). Conversely, if (5.3) holds, then T^* is bounded and $\|T\| = \|T^*\| \leq C$. Hence for each $\epsilon > 0$ there is a $\varphi \geq 0$ such that

$$\iint K(x, y)K(x, z)\varphi(z) dz \leq (C + \epsilon)\varphi(y)$$

(Theorem 5.1). This is equivalent to

$$\int G_{2s,\lambda}(y - z) |q(y)|^{1/2} |q(z)|^{1/2} \varphi(z) dz \leq (C + \epsilon)\varphi(y).$$

Set $\Psi(z) = \varphi(z)/|q(z)|^{1/2}$ when $q(z) \neq 0$ and $\Psi(z) = 0$ otherwise. Then $\Psi(z) \geq 0$ and

$$\int G_{2s,\lambda}(y-z) |q(z)| \Psi(z) dz \leq (C + \epsilon)\Psi(y).$$

This shows that $B_{s,\lambda}(q) \leq C + \epsilon$. Since this is true for every ϵ , we have (5.4). The proof is complete.

Theorem 5.3. Let q be an element of $H^{n/2-s-t}(\mathbf{R}^n)$, where $s > 0$ and $t > 0$. Then the operator $G_{t,\lambda}qG_{s,\lambda}$ is a Hilbert-Schmidt operator on $L^2(\mathbf{R}^n)$ for each $\lambda > 0$.

Proof. The operator is defined by (5.1), where

$$(5.5) \quad K(x, y) = \int G_{t,\lambda}(x-z)q(z)G_{s,\lambda}(z-y) dz.$$

Taking Fourier transforms, we have

$$\hat{K}(\xi, \eta) = (\lambda + |\xi|^2)^{-t/2} \hat{q}(\xi - \eta) (\lambda + |\eta|^2)^{-s/2}.$$

Hence

$$\begin{aligned} \iint |\hat{K}(\xi, \eta)|^2 d\xi d\eta &= \iint (\lambda + |\xi|^2)^{-t} |\hat{q}(\xi - \eta)|^2 (\lambda + |\eta|^2)^{-s} d\xi d\eta \\ &= \iint |\hat{q}(\zeta)|^2 (\lambda + |\zeta + \eta|^2)^{-t} (\lambda + |\eta|^2)^{-s} d\eta d\zeta \\ &\leq C \int |\hat{q}(\zeta)|^2 (1 + |\zeta|^2)^{n/2-s-t} d\zeta = C |q|_{n/2-s-t}^2, \end{aligned}$$

where the constant depends only on λ , s and t . Thus

$$(5.6) \quad \iint |K(x, y)|^2 dx dy < \infty.$$

This shows that T is a Hilbert-Schmidt operator.

Theorem 5.4.

$$(5.7) \quad |(qu, v)| \leq W_{s,\lambda}(q) \|(\lambda - \Delta)^{s/2}u\| \|(\lambda - \Delta)^{s/2}v\|, \quad u, v \in H^s.$$

Proof. Set $f = (\lambda - \Delta)^{s/2}u$, $g = (\lambda - \Delta)^{s/2}v$. Then

$$\begin{aligned} |(qu, v)| &= |(qG_{s,\lambda}f, G_{s,\lambda}g)| \\ &= \left| \iint \hat{q}(\xi - \eta) (\lambda + |\eta|^2)^{-s/2} \hat{f}(\eta) (\lambda + |\xi|^2)^{-s/2} \overline{\hat{g}(\xi)} d\xi d\eta \right| \\ &\leq \left(\iint |\hat{q}(\xi - \eta)| (\lambda + |\eta|^2)^{-s} |\hat{g}(\xi)|^2 d\xi d\eta \right)^{1/2} \end{aligned}$$

$$\times \left(\iint |\hat{q}(\xi - \eta)| (\lambda + |\xi|^2)^{-s} |\hat{f}(\eta)|^2 d\xi d\eta \right)^{1/2} \\ \leq W_{s,\lambda}(q) \|f\| \|g\|.$$

This completes the proof.

Theorem 5.5. *The inequality*

$$(5.8) \quad |(\hat{q}u, u)| \leq C([\lambda - \Delta]^s u, u), \quad u \in C_0^\infty,$$

holds if

$$(5.9) \quad \inf_{\varphi \geq 0} \sup_{\eta} \frac{1}{\varphi(\eta)} \int \gamma_{s,\lambda}(\eta - \xi) |\hat{q}(\xi)| \varphi(\xi) d\xi \leq C,$$

where

$$(5.10) \quad \gamma_{s,\lambda}(\xi) = \int (\lambda + |\xi - \eta|^2)^{-s/2} (\lambda + |\eta|^2)^{-s/2} d\eta.$$

Proof. We merely follow the proof of Theorem 5.2. Set

$$K(\xi, \eta) = (\lambda + |\xi - \eta|^2)^{-s/2} |\hat{q}(\eta)|^{1/2},$$

and define the operator T by

$$Tu(\xi) = \int K(\xi, \eta) \hat{u}(\eta) d\eta.$$

Then we show that (5.8) is implied by $\|T\| \leq C$. We then apply Theorem 5.1.

Corollary 5.6. *If*

$$\int \gamma_{s,\lambda}(\eta - \xi) |\hat{q}(\xi)| d\xi \leq C, \quad \eta \in \mathbf{R}^n,$$

where $\gamma_{s,\lambda}(\xi)$ is given by (5.10), then (5.8) holds.

Remark 5.7. From the proof of Theorem 5.5 we see that (5.9) is a necessary and sufficient condition for the inequality

$$(|\hat{q}| * \hat{u}, \hat{u}) \leq C([\lambda - \Delta]^s u, u), \quad u \in C_0^\infty.$$

Theorem 5.8. *Suppose $q \in B_{s,\lambda}$. Then $T = G_{s,\lambda} q G_{s,\lambda}$ is bounded on L^2 with bound $\leq B_{s,\lambda}(q)$.*

Proof. T is an integral operator with kernel

$$K(x, z) = \int G_{s,\lambda}(x - y) q(y) G_{s,\lambda}(y - z) dy.$$

Set

$$K_1(x, z) = \int G_{s,\lambda}(x - y) |q(y)| G_{s,\lambda}(y - z) dy,$$

and let T_1 be the corresponding operator. Since $|K(x, z)| \leq K_1(x, z)$, we have $\|T\| \leq \|T_1\|$. Now Theorem 5.2 states that T_1 is bounded and its bound is $\leq B_{s,\lambda}(q)$. This completes the proof.

Theorem 5.9. Suppose $q \in H^{n/2-2s} + (B_{s,\lambda})_\epsilon$ for some $s > 0$ and $\lambda > 0$. Then the operator $T = G_{s,\lambda}qG_{s,\lambda}$ is compact on L^2 .

Proof. For each $\epsilon > 0$ there is a function $f \in B_{s,\lambda}$ such that $B_{s,\lambda}(f) < \epsilon$ and $q - f$ is in $H^{n/2-2s}$. By Theorem 5.8 the operator $G_{s,\lambda}fG_{s,\lambda}$ is bounded with bound $< \epsilon$. By Theorem 5.3, the operator $G_{s,\lambda}(q - f)G_{s,\lambda}$ is compact. Thus $G_{s,\lambda}qG_{s,\lambda}$ is the limit in norm of compact operators. Consequently it is also compact.

We can now give the

Proof of Theorem 4.1. Set $A = (\lambda - \Delta)^{s/2}$, $b(u, v) = (qu, v)$ and $K = G_{s,\lambda}q_2G_{s,\lambda}$ in Theorem 3.8. We take $D(A) = D(b) = C_0^\infty$ (note that this can be done because q is locally in L^1 even though qu need not be in L^2 for each $u \in C_0^\infty$). By Theorem 5.9, the operator K is compact. By Theorems 5.2 and 5.4 there is a constant $\theta < 1$ such that

$$|(q_3u, u)| + |(q_4u, u)| \leq \theta \|Au\|^2, \quad u \in C_0^\infty.$$

Since

$$(q_2u, u) = (G_{s,\lambda}q_2G_{s,\lambda}(\lambda - \Delta)^{s/2}u, (\lambda - \Delta)^{s/2}u),$$

we have

$$b(u) \geq (KAu, Au) - \theta \|Au\|^2, \quad u \in C_0^\infty.$$

The conclusion now follows from Theorem 3.8.

Theorem 5.10. Let s, t be positive numbers such that $\max(s, t) < n/2 < s + t$. Let $q(x)$ be a function such that $|\xi|^{n/2-s-t} \hat{q}$ is in $L^2(\mathbb{R}^n)$. Then the operator $T = G_{s,0}qG_{t,0}$ is a Hilbert-Schmidt operator in $L^2(\mathbb{R}^n)$ with Hilbert-Schmidt norm satisfying

$$(5.11) \quad \|T\|_{HS}^2 \leq \frac{\Gamma\left(\frac{n}{2} - s\right)\Gamma\left(\frac{n}{2} - t\right)\Gamma\left(t + s - \frac{n}{2}\right)}{2^n \pi^{n/2} \Gamma(s)\Gamma(t)\Gamma(n - s - t)} \| |\xi|^{n/2-s-t} \hat{q} \|^2.$$

Proof. We follow the proof of Theorem 5.3. T is an integral operator with kernel given by

$$K = G_{s,0} * (qG_{t,0}).$$

Consequently

$$\hat{K} = (2\pi)^{n/2} \hat{G}_{s,0}(\hat{q} * \hat{G}_{t,0}).$$

Hence

$$(5.12) \quad \|T\|_{HS}^2 \leq (2\pi)^{-n} \int |\hat{q}(\xi)|^2 \int |\eta|^{-2s} |\xi - \eta|^{-2t} d\eta d\xi.$$

This gives (5.11).

Proof of Theorem 2.6. Note that $S = H^{-1/2}$. Apply Theorem 4.1.

Proof of Theorem 2.7. Set $A = (-\Delta)^{1/2}$, $b(u, v) = (qu, v)$, and $K = G_{1,0}V_sG_{1,0}$ in Theorem 3.8. By Theorem 5.10, K is a Hilbert-Schmidt operator with

$$(5.13) \quad \|K\|_{HS}^2 \leq \|p^{-1/2}\hat{V}_2\|^2/8 = \|V_2\|_G^2/(4\pi)^2.$$

By Theorem 3.8, $-\Delta + V$ has a forms extension H satisfying (2.9). Moreover, H has no more negative eigenvalues than K has in the interval $[-\|K\|, \theta - 1]$, where

$$\theta = B_0(V_3) + W_0(V_4).$$

If K has m eigenvalues in this interval, then $m(1 - \theta)^2$ must be $\leq \|K\|_{HS}^2$. This combined with (5.13) gives (2.10). The proof is complete.

Proof of Theorem 2.8. Set

$$\alpha = \|V_{(+)}\|_G^{-2} \max [(V, V_{(+)})_G, 0],$$

and put $V_1 = U - V + \alpha V_{(+)}$, $V_2 = V - \alpha V_{(+)}$, $V_3 = V_4 = 0$ in Theorem 2.7. Thus

$$\|V_2\|_G^2 = \|V\|_G^2 - 2\alpha(V, V_{(+)})_G + \alpha^2 \|V_{(+)}\|_G^2 = (4\pi)^2 N(V).$$

This completes the proof.

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