The Construction and Application of Contractive Retractions in 2-Dimensional Normed Linear Spaces

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Communicated by the Editors

1. Introduction and results. In this note we exhibit a special geometric property of 2-dimensional normed linear spaces and its application to the study of nonlinear mappings. This involves the extension of a Hilbert space construction, which cannot be extended to general Banach spaces, to all 2-dimensional spaces.

Recently the study of certain classes of nonlinear mappings on Banach spaces led to considerable work on the following problem. Let X be a Banach space and C a closed convex subset. Find a *retraction* T of X onto C (i.e., a mapping $T : X \rightarrow C$ such that $Tx = x$ for each $x \in C$) with the additional property that T is also *contractive* (i.e., $\|Tx_1 - Tx_2\| \leq \|x_1 - x_2\|$ for each pair $x_1, x_2 \in X$). For a Hilbert space this is always possible. In fact, the solution T can be defined by yet another property; namely, for each $x \in X$, $Tx \in C$ and $\|Tx - x\| = \inf \{\|z - x\| : z \in C\}$. In a general Banach space, any mapping which satisfies the latter property is said to be a *proximity (or nearest point) mapping* of X onto C . For a general Banach space, a proximity mapping may fail to be contractive. For example, it was shown by DeFigueiredo and the author [3] that a real normed space of dimension 3 or higher is an inner product space if and only if a specific proximity mapping of the space onto its unit ball (the so-called radial retraction) is contractive. This result is also contained in Schäffer [6] in a different form. Moreover, in [4], DeFigueiredo and the author showed that in many situations, *every* retraction of X onto C fails to be contractive. These negative results exempted, among other classes, all 2-dimensional spaces.

In this note we show that if X is 2-dimensional then a contractive retraction always exists. Moreover it can be realized as a proximity mapping, *in a new norm* $\|\cdot\|_*$, of X onto C . The norm $\|\cdot\|_*$ can be identified with the norm of the dual space of X . In the general case, all proximity mappings in $\|\cdot\|$ of X onto C may fail to be contractive in $\|\cdot\|$, while there exists a proximity mapping in

$\|\cdot\|^*$ of X onto C which is contractive in $\|\cdot\|$. If X is a Hilbert space $\|\cdot\|^* = \|\cdot\|$.

Some of the arguments used in this note are purely 2-dimensional. By virtue of the negative results of [4], the results (e.g., Theorem 1) are not valid in general, even for spaces of dimension 3. However, some of the methods and constructions can be usefully generalized to certain higher dimensional situations. This will be the subject of another paper.

In many cases the contractive retraction can be given in closed form, and we state some applications to the existence and iterative construction of fixed points of mappings of the plane into itself.

We begin with an existence theorem for the desired contractive retraction. The realization of this as a proximity mapping in $\|\cdot\|^*$ is given in the subsequent two theorems. Examples, corollaries and applications are also given in this section. All proofs are given in the next section.

Theorem 1. *Let X be a 2-dimensional real linear space with norm $\|\cdot\|$. Let C be a closed convex subset of X . Then there exists a mapping $T : X \rightarrow C$, not necessarily unique, with the properties:*

- (1)
$$Tx = x \text{ for each } x \in C,$$
- (2)
$$\|Tx_1 - Tx_2\| \leq \|x_1 - x_2\| \text{ for each pair } x_1, x_2 \in X,$$
- (3) *if $x' = Tx + \lambda(x - Tx)$ with $\lambda > 0$, then $Tx' = Tx$.*

Property (3) is an additional feature of T which is useful in the applications. It expresses the fact that T retracts X onto C along rays. We now proceed to give an explicit construction for a mapping T which satisfies (1), (2) and (3). Let X be as in Theorem 1 and let E be a closed convex subset of X . We denote by ∂E the boundary of E . The line $\{x + \lambda w : \lambda\}$ separates z and E if there exists a linear functional $f \neq 0$ so that $f(w) = 0$ and $f(y) \leq f(x) \leq f(z)$ for all $y \in E$. We say that E is strictly convex if $y_1, y_2, \frac{1}{2}y_1 + \frac{1}{2}y_2 \in \partial E$ imply that $y_1 = y_2$. We say that E is smooth if at each boundary point of E there is a unique support line. If the origin belongs to the interior of E , then we denote by ρ_E the gauge function of E , i.e., $\rho_E(x) = \inf \{\lambda : \lambda > 0, x/\lambda \in E\}$. Following Birkhoff [1] and James [5], we define the (generally unsymmetric) notion of orthogonality, denoted by $\perp^E$ as follows:

$$x \perp^E w \Leftrightarrow \rho_E(x + \lambda w) \geq \rho_E(x) \quad \text{for all } \lambda.$$

Throughout we denote by B the unit ball $\{x : \|x\| \leq 1\}$ of X .

The following lemma, which cannot in general be extended to higher dimensional spaces, is central to our construction of T .

Lemma 1. *Let X be as in Theorem 1. Let B be the unit ball of X . Then there exists a compact convex set B^* in X which is symmetric about the origin, has non-empty interior and satisfies*

$$(4) \quad x \perp^{B^*} w \Leftrightarrow w \perp^B x.$$

Moreover B^* can be identified with the unit ball of the dual space of X . B^* is strictly convex if and only if B is smooth.

Examples 1. If X is a Hilbert space then we can take $B^* = B$ because $\perp^B$ is a symmetric relation. If $X = \ell^p(2)$, $1 < p < \infty$, then $B = \{(\xi, \eta) : |\xi|^p + |\eta|^p \leq 1\}$ and $B^* = \{(\xi, \eta) : |\xi|^q + |\eta|^q \leq 1\}$, with $1/p + 1/q = 1$. The construction for the general case is given in the proof of Lemma 1.

It follows from Lemma 1 that the gauge function of B^* defines a norm on X . The particular choice of B^* is only a matter of a scalar multiplication and is not important in the sequel.

Definition 1. Let X be as in Theorem 1. Let B^* satisfying Lemma 1 be arbitrarily chosen and fixed. We define the *dual norm* of X , denoted by $\|\cdot\|^*$, by $\|x\|^* = \rho_{B^*}(x)$, for all $x \in X$.

Theorem 2. Let X and C be as in Theorem 1. Suppose that C is strictly convex or B is smooth, or both. Then the proximity mapping T , in the dual norm $\|\cdot\|^*$, of X onto C is uniquely defined for each x by

$$(5) \quad Tx \in C \text{ and } \|Tx - x\|^* = \inf \{\|z - x\|^* : z \in C\},$$

and it satisfies (1), (2), and (3).

Moreover, for each $x \notin C$, Tx is uniquely defined as the point in ∂C that satisfies

$$(6) \quad \{Tx + \lambda w : \lambda\} \text{ separates } C \text{ and } x, \text{ and } w \perp^B (x - Tx),$$

for some $w \neq 0$.

Geometric interpretations. The interpretation of the first half of the theorem is obvious. The second half can be used to describe the rays along which T retracts X onto C without making use of $\|\cdot\|^*$. For each $y \in \partial C$ define

$$Ay = \{z : z \perp^B y \text{ and } \{y + \lambda w : \lambda\} \text{ separates } C \text{ and } y + z \text{ for some } w \neq 0\}.$$

It can be shown that if C has a nonempty interior, then Ay is a closed convex cone; and, in general, C is either a closed convex cone or a closed convex cone and its symmetric image about the origin. In either case, it follows from Theorem 2 that if C is strictly convex or B is smooth, then

$$T^{-1}y = y + Ay.$$

We note parenthetically that from this and from (2) it follows that A is a (multivalued) *accretive* mapping.

Examples 2. If X is a Hilbert space and $C = B$ it follows from the symmetry of $\perp^B$ that $Ay = \{\lambda y : \lambda \geq 0\}$ for each $\|y\| = 1$. If $X = \ell^p(2)$, $1 < p < \infty$, and $C = B$ we readily calculate that $Ay = \{\lambda z : \lambda \geq 0, z \text{ belongs to the same quadrant as } y \text{ and } |\text{slope } z| = |\text{slope } y|^{(p-1)^2}\}$.

It is easily shown that if $X = \ell^p(2)$, $1 < p < \infty$, $p \neq 2$, then $P : X \rightarrow B$, defined by $Px = x$, if $\|x\| \leq 1$, and $Px = x/\|x\|$, if $\|x\| > 1$, is the unique

proximity mapping in $\|\cdot\|$ of X onto B and P fails to be contractive in $\|\cdot\|$. Together with Theorem 2 this shows that: *in some cases every proximity mapping in $\|\cdot\|$ of X onto C fails to be contractive in $\|\cdot\|$, while there exist proximity mappings in $\|\cdot\|^*$ of X onto C which are contractive in $\|\cdot\|$.*

If C is not strictly convex and B is not smooth, the relation (5) may fail to define T uniquely. Hence we have to select a proximity mapping in $\|\cdot\|^*$ of X onto C which will satisfy (1), (2) and (3). To this end we define:

Definition 2. For each $x \notin C$, let: $d(x, C) = \inf \{\|x - y\|^* : y \in C\}$, $V(x) = \{z : \|x - z\|^* = d(x, C)\}$, and $U(x) = V(x) \cap C$. Let $L(x)$ be the affine hull of $U(x)$. Let $u_1(x)$, $u_2(x)$ be the end points of $L(x) \cap C$ and $y_1(x)$, $y_2(x)$ the end points of $L(x) \cap V(x)$ with $y_1(x) - y_2(x) = \lambda(u_1(x) - u_2(x))$ and $\lambda \geq 1$. Let $y(x) = (y_1(x) + y_2(x))/2 = u_1(x) + \alpha(x)(u_2(x) - u_1(x))$, with $\alpha(x) = 0$ if $u_1(x) = u_2(x)$.

Theorem 3. Let X and C be as in Theorem 1. The proximity mapping T , in the dual norm $\|\cdot\|^*$, of X onto C defined by:

$$(7) \quad Tx = \begin{cases} x, & x \in C, \\ y(x), & x \notin C \text{ and } 0 \leq \alpha(x) \leq 1, \\ u_1(x), & x \notin C \text{ and } \alpha(x) < 0, \\ u_2(x), & x \notin C \text{ and } \alpha(x) > 1, \end{cases}$$

satisfies (1), (2) and (3). Moreover for each $x \notin C$ there exists $w \neq 0$ so that Tx and w satisfy (6).

It was noted in Examples 2 that if X is a Hilbert space and $C = B$ then the mapping $T : X \rightarrow B$ of Theorem 2 is a radial retraction, i.e., $Tx = x/\|x\|$ for $x \notin B$. The converse is false in general. However the following provides a proper generalization of the Hilbert space case.

Corollary 1. Let X be as in Theorem 1. Let $C = B^*$, as given by Lemma 1. If B^* is strictly convex, then T of Theorem 2 satisfies

$$Tx = x/\|x\|^* \text{ for } x \notin B^*.$$

We now give two applications. The first, an immediate corollary of Theorem 1, is concerned with the extension of contractive mappings; an issue which motivated much of the aforementioned research on contractive retractions.

Theorem 4. Let X , C and T be as in Theorem 1. Let $U : C \rightarrow X$ be contractive, i.e., $\|Ux - Uy\| \leq \|x - y\|$ for each pair $x, y \in C$. Then $U' = UT : X \rightarrow X$ is a contractive extension of U , i.e., $U = U'$ on C , and U' is also contractive.

The second application is concerned with the existence and iterative construction of fixed points. It is patterned, in part, after an infinite dimensional

result of Browder and Petryshyn [2, Theorem 9]. However the proof contains one purely 2-dimensional argument.

Theorem 5. *Let X , C and T be as in Theorem 1. Suppose that C has a non-empty interior and that the mapping $U : C \rightarrow X$ is contractive. Suppose also that*

$$(8) \quad x \in \partial C, \quad TUx = x \text{ imply } Ux = x.$$

Then U has a fixed point in C , i.e., there exists $x \in C$ such that $Ux = x$. Moreover, for each $\lambda \in (0, 1)$ and each $x_0 \in C$ the sequence $\{x_n\}$ defined by

$$(9) \quad x_{n+1} = \lambda x_n + (1 - \lambda)TUx_n,$$

converges to a fixed point of U in C .

If X is an inner product space and if $C = B$, then from the discussion in Example 2 it follows that (8) reduces to the Leray–Schauder condition:

$$(10) \quad Ux - \lambda x \neq 0 \quad \text{for each } x \in \partial B \quad \text{and each } \lambda > 1.$$

This observation has been made by Browder and Petryshyn [2, Theorem 10]. Since Theorem 6 is not restricted to Hilbert spaces, we can derive, using Theorems 2 and 3, a variety of specialized expressions for (8) which are similar in nature to the Leray–Schauder condition (10). We give one example.

Typical application of Theorem 6. Let $X = \ell^4(2)$ and $C = B$. Suppose that the mapping $U : B \rightarrow X$ is contractive. By Example 2, we have an explicit description of the contractive retraction T of X onto B given by Theorem 2. Using this we can express the hypothesis (8) as follows.

If $x \in \partial B$ and $(Ux - x)$ lies in the same quadrant as x , then

$$\text{slope}(Ux - x) \neq (\text{slope } x)^9.$$

By Theorem 5, if this is satisfied, we can iteratively construct a fixed point of U using the scheme (9).

2. Proofs.

Proof of Lemma 1. Let S be a nontrivial skew bilinear form on X . (This is determined up to a scalar factor.) Define $\|x\|^* = \sup \{S(x, y) : y \in B\}$, and $B^* = \{x : \|x\|^* \leq 1\}$. Then B^* is clearly compact, convex, symmetric about the origin, and has a nonempty interior. Also $\rho_{B^*} = \|\cdot\|^*$. We note that S defines a linear mapping T of X onto its dual space X^* ; namely, $\langle Tx, \cdot \rangle = S(x, \cdot)$. Therefore $\|x\|^* = \|Tx\|$; and, since T is onto, $\|y\| = \sup \{S(x, y) : \|x\|^* \leq 1\}$, (and hence $B^{**} = B$).

Let $x, w \in X$ be linearly independent. We find

$$\begin{aligned} \|x\|^* &= \sup_{\mu} |S(x, w + \mu x)| / \|w + \mu x\| = |S(x, w)| / \inf_{\mu} \|w + \mu x\|, \\ \|w\| &= \sup_{\lambda} |S(x + \lambda w, w)| / \|x + \lambda w\|^* = |S(x, w)| / \inf_{\lambda} \|x + \lambda w\|^*. \end{aligned}$$

Therefore,

$$\inf_{\lambda} \|x + \lambda w\|^* / \|x\|^* = \inf_{\lambda} \|w + \mu x\| / \|w\|,$$

and if one quotient is equal to 1, then so is the other, *i.e.*, $x \perp^{B^*} w$ if and only if $w \perp^B x$. This proves (4).

By construction, B^* is the image, under the linear mapping T^{-1} , of the unit ball of X^* . Consequently B^* is strictly convex if and only if B is smooth. The latter fact also follows directly from (4). This finishes the proof of Lemma 1.

Proof of Theorem 2. By hypothesis, either C is strictly convex or B is smooth. By Lemma 1, this means that either C or B^* is strictly convex. Hence for each x the uniqueness of Tx as defined by (5) (*i.e.*, the uniqueness of the nearest point to x in C in the norm $\|\cdot\|^*$) follows by a standard argument of approximation theory. It also follows by a standard argument of approximation theory that if $x \notin C$, $\inf \{\|x - z\|^* : z \in C\} = d(x, C)$, $x' = Tx + \lambda(x - Tx)$, with $\lambda > 0$, then $\inf \{\|x' - z\|^* : z \in C\} = \lambda d(x, C)$. Therefore, since $\|Tx - x'\| = \lambda d(x, C)$ and since nearest points are unique, it follows that $Tx' = Tx$. Thus T satisfies (3). By definition, T maps X onto C and satisfies (1).

Before showing that T satisfies (2), we prove (6). Let $x \notin C$ be given and let $d(x, C) = \inf \{\|x - z\|^* : z \in C\}$. Let $A = \{z : \|x - z\|^* \leq d(x, C)\}$. It is easily shown that $C \cap \text{interior } A$ is empty. By definition, $Tx \in C \cap A$. Hence, by the separation theorem, there exists $w \neq 0$ so that the line $\{Tx + \lambda w : \lambda\}$ separates C and A . It follows that w supports C at Tx and $(Tx - x) \perp^{B^*} w$. By Lemma 1, the latter is equivalent to $w \perp^B (Tx - x)$. This proves that for $x \notin C$, Tx satisfies (6) for some $w \neq 0$. Conversely, suppose $x \notin C$, and $y \in \partial C$ and $w \neq 0$ satisfy (6) with Tx replaced by y . By virtue of Lemma 1, $(y - x) \perp^{B^*} w$. From this and the separation of C and x we readily deduce that the line $\{y + \lambda w : \lambda\}$ separates C and $\{z : \|z - x\|^* \leq \|y - x\|^*\}$. From this it follows that $\|y - x\|^* = \inf \{\|z - x\|^* : z \in C\}$, *i.e.*, $y = Tx$. This proves (6).

It remains to show that T satisfies (2). If $x_1, x_2 \in C$, then (2) follows immediately from (1). Hence we consider the cases: $x_1 \in C, x_2 \notin C$ and $x_1, x_2 \notin C$. In either case, we choose $w_2 \neq 0$ to satisfy (6). It follows, in particular, that w_2 and $(Tx_2 - x_2)$ are linearly independent. Hence we can choose α_2 and β_2 so that

$$(11) \quad Tx_1 + \alpha_2 w_2 = Tx_2 + \beta_2 (x_2 - Tx_2).$$

We consider the convex function $F(\lambda) = \|Tx_2 + \lambda(x_2 - Tx_2) - Tx_1\|$ and assert that F has a minimum at $\lambda = \beta_2$. For if $\alpha_2 = 0$ this follows directly from (11) and the fact that F is nonnegative. If $\alpha_2 \neq 0$, this follows from (11), the convexity of F , and from $w_2 \perp^B (x_2 - Tx_2)$. We next observe that $\beta_2 \leq 0$. To see this we use the fact, observed above to follow from (6), that the line $\{Tx_2 + \lambda w_2 : \lambda\}$ separates C and $\{z : \|x_2 - z\|^* \leq \|x_2 - Tx_2\|^*\}$. By virtue of this we can choose a linear functional f on X so that $f(Tx_2) \geq f(z)$ for all $z \in C$, $f(w_2) = 0$, and $f(x_2) > f(Tx_2)$. Applying f to (11) we find $\beta_2 \leq 0$. Hence, by the

convexity of F and its minimum at $\beta_2 \leq 0$, we have: $F(\beta_2) \leq F(0) \leq F(1)$. Hence $F(0) = \|Tx_2 - Tx_1\| \leq \|x_2 - Tx_1\| = F(1)$. Thus if $x_1 \in C$, then $Tx_1 = x_1$ and we have shown that (2) holds.

If $x_1, x_2 \notin C$, then by a completely analogous argument we find w_1 satisfying (6), and α_1 and $\beta_1 \leq 0$ so that

$$(12) \quad Tx_2 + \alpha_1 w_1 = Tx_1 + \beta_1(x_1 - Tx_1).$$

Moreover the convex function $G(\lambda) = \|Tx_1 + \lambda(x_1 - Tx_1) - Tx_2\|$ has a minimum at $\lambda = \beta_1$. If $\beta_1, \beta_2 < 0$ we choose $v \neq 0$ so that $(Tx_1 - Tx_2) \perp^B v$ and a linear functional g so that $\|g\| = 1$, $g(v) = 0$, and $g(Tx_1 - Tx_2) = \|Tx_1 - Tx_2\|$. Hence from $G(\beta_1) \leq G(0)$ we deduce that: $\|Tx_1 - Tx_2\| + \beta_1 g(x_1 - Tx_1) \leq \|Tx_1 - Tx_2 + \beta_1(x_1 - Tx_1)\| \leq \|Tx_1 - Tx_2\|$. By $\beta_1 < 0$, it follows that $g(x_1 - Tx_1) \geq 0$. Similarly, using F , we find that $g(x_2 - Tx_2) \leq 0$. Hence $\|x_1 - x_2\| = \|(Tx_1 - Tx_2) + (x_1 - Tx_1) - (x_2 - Tx_2)\| \geq \|Tx_1 - Tx_2\| + g(x_1 - Tx_1) - g(x_2 - Tx_2) \geq \|Tx_1 - Tx_2\|$, i.e., (2) is satisfied. If $\beta_1 = 0 > \beta_2$, we conclude by (11), (12) and (6) that $(Tx_1 - Tx_2) \perp^B (x_1 - Tx_1)$. Hence in the preceding argument we replace v by $(x_1 - Tx_1)$. The argument remains the same except that $g(x_1 - Tx_1) \geq 0$ is replaced by $g(x_1 - Tx_1) = 0$, and we again conclude that $\|x_1 - x_2\| \geq \|Tx_1 - Tx_2\|$. If $\beta_2 = 0 > \beta_1$, we proceed analogously. If $\beta_1 = \beta_2 = 0$, we conclude from (11) and (12) that either $Tx_1 = Tx_2$ or $\alpha_1, \alpha_2 \neq 0$. In the former case, we are finished. In the latter case, by (6), (11) and (12) we find $(Tx_1 - Tx_2) \perp^B (x_i - Tx_i)$, $i = 1, 2$. We choose linear functionals g_1, g_2 such that $\|g_i\| = 1$, $g_i(Tx_1 - Tx_2) = \|Tx_1 - Tx_2\|$, $i = 1, 2$, and $g_1(x_1 - Tx_1) = 0 = g_2(x_2 - Tx_2)$. If $g_1(x_2 - Tx_2) > 0$ and $g_2(x_1 - Tx_1) < 0$ it follows readily that the equation

$$Tx_1 + \sigma(x_1 - Tx_1) = Tx_2 + \tau(x_2 - Tx_2)$$

can be solved with $\sigma, \tau > 0$. By (3), this implies that $Tx_1 = Tx_2$, which contradicts our assumption. Hence either $g_1(x_2 - Tx_2) \leq 0$ or $g_2(x_1 - Tx_1) \geq 0$. Suppose $g_1(x_2 - Tx_2) \leq 0$. Choose $v = (x_2 - Tx_2)$ and $g = g_1$ in the argument above and conclude that $\|x_1 - x_2\| \geq \|Tx_1 - Tx_2\|$. The case $g_2(x_1 - Tx_1) \geq 0$ is analogous. This finishes the proof.

Proof of Theorem 3. For $x \notin C$, using Definition 2, it follows that the intersection of the two line segments with end points $y_1(x), y_2(x)$ and $u_1(x), u_2(x)$, respectively, is equal to $U(x) = V(x) \cap C$. By Definition 2 and by (7), Tx belongs to this intersection; and it follows that T is a proximity mapping in $\|\cdot\|^*$ of X onto C . For each $x \notin C$, the existence of $w \neq 0$ satisfying (6) is then proved in precisely the same manner as in Theorem 2. To prove that T satisfies (3) it is sufficient to consider $x \notin C$ and $x' = Tx + \lambda(x - Tx)$, with $\lambda \geq 1$. Hence $d(x', C) = \lambda d(x, C)$, $U(x) \subseteq U(x')$, and $L(x) = L(x')$. Consequently $u_i(x') = u_i(x)$, $i = 1, 2$; and $y_1(x') = y(x) + \lambda(y_1(x) - y_2(x))/2$, $y_2(x') = y(x) - \lambda(y_2(x) - y_1(x))/2$. Hence $y(x') = y(x)$ and $\alpha(x') = \alpha(x)$; and, by (7), $Tx' = Tx$. Using (6) and the fact that T satisfies (3), we prove that T satisfies (2) in

precisely the same manner as in Theorem 2. By (7), T also satisfies (1). This finishes the proof of Theorem 3.

Proof of Corollary 1. If B^* is strictly convex and $x \notin B^*$, it follows from the triangle inequality that $x/\|x\|^*$ is the unique nearest point to x in B^* in the norm $\|\cdot\|^*$. Hence by Theorem 2, $Tx = x/\|x\|^*$.

Proof of Theorem 4. This is an immediate consequence of Theorem 1.

Proof of Theorem 5. Clearly $S = TU$ is a continuous (in fact, contractive) mapping of C into itself. Therefore, by the classical Brouwer theorem, S has at least one fixed point in C . Moreover, each fixed point x of S is also a fixed point of U . If x belongs to the interior of C , this follows from the definitions of S and T ; and if $x \in \partial C$, it follows from hypothesis (8). We finish the proof by showing that, for given $\lambda \in (0, 1)$ and $x_0 \in C$, the sequence defined by (9) converges to a fixed point of S . Let u be an arbitrary fixed point of S . By (9) and the contractiveness of S ,

$$(13) \quad \begin{aligned} \|u - x_{n+1}\| &\leq \lambda \|u - x_n\| + (1 - \lambda) \|u - Sx_n\| \\ &= \lambda \|u - x_n\| + (1 - \lambda) \|Su - Sx_n\| \leq \|u - x_n\|. \end{aligned}$$

Thus $\{\|u - x_n\|\}$ is a decreasing sequence with limit $d \geq 0$. If $d = 0$, then $x_n \rightarrow u$ and we are finished. If $d > 0$, we next note that from $\|u - Sx_n\| = \|Su - Sx_n\| \leq \|u - x_n\|$ and from (13) it follows readily that the sequence $\{\|u - Sx_n\|\}$ also converges to d . Let E be the set of accumulation points of $\{x_n\}$. We note that for each $w \in E$, we can choose $z = z(w) \in E$ so that

$$(14) \quad w = \lambda z + (1 - \lambda)Sz, \quad z, w, Sz \in \partial B(u, d) = \{x : \|x - u\| = d\}.$$

For if $x_n \rightarrow w$ then, after possible extraction of a further subsequence, $x_{n'} \rightarrow z$ and $Sx_{n'} \rightarrow Sz$, for some z ; whence, by (9) and $\|u - Sx_n\| \rightarrow d$, (14) follows.

If for some $w \in E$, $w = z(w)$, then, by (14), $Sw = w$. By $w \in E$ and by (13), with u replaced by w , it follows that $\|w - x_n\| \rightarrow 0$, and we are finished. We now show that the assumption that $w \neq z(w)$ for each w and $z(w)$ in E leads to a contradiction. Choose w and $z(w)$ in E . By (14), we can define a linear functional f so that $f(x - u) \leq d$, whenever $\|x - u\| \leq d$ and $f(w - u) = f(z(w) - u) = d$. Let $F = E \cap \{x : f(x - u) = d\}$. Since F is nonempty, compact and a subset of a line, we can choose $w_1, w_2 \in F$ so that

$$(15) \quad F \subset \{\xi w_1 + (1 - \xi)w_2 : \xi \in [0, 1]\}.$$

By (14), we can choose $z_1, z_2 \in E$ so that

$$(16) \quad w_i = \lambda z_i + (1 - \lambda)Sz_i \quad \text{and} \quad w_i, z_i, Sz_i \in \partial B(u, d), \quad i = 1, 2.$$

From (14) and (16) it follows that $f(z_i - u) = f(Sz_i - u) = d$, $i = 1, 2$; hence, $z_1, z_2 \in F$. By assumption, $w_i \neq z_i$, $i = 1, 2$. Consequently, by (15) and (16), $\|Sz_1 - Sz_2\| > \|w_1 - w_2\| > \|z_1 - z_2\|$, which contradicts the contractiveness of S . This finishes the proof of Theorem 5.

Proof of Theorem 1. The existence of T satisfying (1), (2) and (3) follows immediately from Theorems 2 and 3. Examples showing the nonuniqueness are easily constructed.

The original version of this paper gives a constructive proof of Theorem 1 without making use of the dual norm.

The author wishes to thank the referee for his comments on the present version, including a new proof of Lemma 1, which have been incorporated.

REFERENCES

1. G. BIRKHOFF, *Orthogonality in linear metric spaces*, Duke Math. J. **1** (1935), 169–172.
2. F. E. BROWDER & W. V. PETRYSHYN, *Construction of fixed points of non-linear mappings in Hilbert spaces*, J. Math. Anal. Appl. **20** (1967), 197–228.
3. D. G. DEFIGUEIREDO & L. A. KARLOVITZ, *On the radial projection in normed spaces*, Bull. Amer. Math. Soc. **73** (1967), 364–368.
4. D. G. DEFIGUEIREDO & L. A. KARLOVITZ, *On the extension of contractions in normed spaces*, Proc. Sympos. Pure Math., vol. XVIII, part 1, Amer. Math. Soc., Providence, R. I., 1970, 95–104.
5. R. C. JAMES, *Orthogonality and linear functionals in normed linear spaces*, Trans. Amer. Math. Soc. **61** (1947), 265–292.
6. J. J. SCHÄFFER, *Another characterization of Hilbert spaces*, Studia Math. **25** (1965), 271–276.

This work was partially supported by N.S.F. Grant GP 20555.

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Date communicated: MAY 19, 1972