Existence Theorems for Parametric Surfaces of Prescribed Mean Curvature

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1. Introduction. Let D be a domain in E^3 and let $H(x)$ be a given real valued function defined in D . In this work we treat the following problem: Given an oriented closed Jordan curve Γ contractible in $\bar{D}$, when can we find a surface of prescribed mean curvature $H(x)$ contained in $\bar{D}$ which spans Γ in an orientation preserving fashion? That is, we seek a mapping $x: \bar{B} \rightarrow \bar{D}$, B denoting the two dimensional unit disc, satisfying the following conditions:

- (i) $x \in C^2(B) \cap C^0(\bar{B})$;
- (ii) x maps ∂B homeomorphically onto Γ with degree 1;
- (iii) x satisfies in B the equations

$$(1.1) \quad x_{uu} + x_{vv} = 2H(x)x_u \wedge x_v,$$

$$(1.2) \quad x_u^2 - x_v^2 = x_u \cdot x_v = 0.$$

Equations (1.2) state that x is a conformal mapping of B onto its image; under that condition equations (1.1) become the equations for mean curvature $H(x)$.

The case where D is *convex* and $H \equiv 0$ corresponds to the classical Plateau problem which was solved by Radó [21] and Douglas [5]. Heinz [9] considered the case of constant mean curvature H and showed that if D is a *ball* of radius $(17^{1/2} - 1)/(8|H|)$ then the problem is solvable. The radius was improved by Werner [29] to $1/(2|H|)$ and finally sharpened to the best possible value $1/|H|$ by Hildebrandt [13]. Hildebrandt has generalized this result to allow variable $H(x)$ [12] and also has considered the case where D is an *ellipsoid* [14]. In another direction, the authors [7] showed that if D is a *cylinder* of radius $1/(2|H|)$, then the problem is solvable; they also allowed variable $H(x)$.

These special results, although quite interesting in themselves, point to the need for a general theory to explain the phenomenon involved. This is the motivation for the problem we have considered. It is the object of this paper to provide such a general theory.

We shall show that it is the inward mean curvature of ∂D which is the crucial factor for the solvability of the problem. For example, in the case of constant

H , our results say that if D is compact, ∂D is C^2 and has inward mean curvature $\geq |H|$, then the problem is solvable. For noncompact D we require a certain uniformity on the behavior of ∂D at infinity.

Our method is to obtain the desired mapping as the minimum among mappings “approximately” satisfying conditions (i) and (ii) above for the functional

$$E[x] = D[x] + W[x]$$

where $D[x]$ is the Dirichlet integral of x , and $W[x]$ is an oriented volume integral of the form $W[x] = \int \int_B Q(x) \cdot x_u \wedge x_v \, du \, dv$.

The condition that equations (1.1) be the Euler equations of $E[x]$ is precisely $\operatorname{div} Q = 4H(x)$ (Lemma 2.1). In order that $E[x]$ have a minimum, we must further have $\sup |Q(x)| \leq 2\alpha < 2$ for some $\alpha < 1$. In the cases mentioned above of known results in the literature, the vector field $Q(x)$ is always written down explicitly. However for a general domain D , the existence of $Q(x)$ satisfying the required conditions is a non-trivial problem. It turns out that the existence of $Q(x)$ is closely connected with Serrin’s existence theory for *non-parametric* surfaces of prescribed mean curvature [23]. Since this theory applies only to bounded domains, we have found it necessary to extend Serrin’s theory to a class of unbounded domains. This extension is carried out in the appendix.

In the first part of our paper §2–4, we start out by assuming the existence of $Q(x)$ in a domain D and proceed to find sufficient conditions for the solution of our parametric existence problem. In §2 we set up a variational problem and show some regularity properties of a solution. Much of the material of this section is well-known [13], [15]. In §3 we prove a strong geometric maximum principle which allows us to show that solutions of our variational problem lie strictly interior to D . This maximum principle lies at the heart of our method and is a generalization of the corresponding results in [13] and [7]. In §4 we apply the results of §2 and §3 and prove an existence theorem for the Dirichlet problem for equations (1.1), as well as existence theorems giving sufficient conditions for the “Plateau-type” problem stated at the beginning, to be solvable. §5 contains our main existence results. These follow by combining the existence theorem for $Q(x)$ proved in the appendix and the explicit construction of $Q(x)$ in a number of interesting cases, with the results of §4. We illustrate our results with a number of simple geometric examples, and discuss the relationship of our work with previous results in the literature. In §6 we show that for constant H , our results are best possible for a large class of domains D . The results of this section are closely related to those of the authors’ paper [8].

2. A variational problem with fixed boundary values.

Definition. Let D be a domain in E^3 (possibly unbounded) such that ∂D is of class C^2 with uniformly bounded principal curvatures. We assume further that for ϵ sufficiently small, ∂D has globally defined inward and outward parallel surfaces formed by moving a distance ϵ along the inward and outward normal

to ∂D , respectively. For a given C^1 function $H(x): D \rightarrow \mathbf{R}$, we say that such a D is *admissible* if there exists a C^1 vector field $Q(x) = Q^H(x)$ such that

$$(2.1) \quad \operatorname{div} Q = 4H(x),$$

$$(2.2) \quad \sup |Q(x)| \leq 2\alpha < 2 \quad \text{for some constant } \alpha(H) < 1.$$

Given an admissible D with respect to a C^1 function $H(x)$ in D , we define the functional

$$E[x] = \iint_B (x_u^2 + x_v^2 + Q(x) \cdot x_u \wedge x_v) \, du \, dv.$$

Then the following lemma holds [7, Lemma 8]:

Lemma 2.1. *If $Q(x): E^3 \rightarrow E^3$ is a C^1 vector field, then the Euler equations associated with $E[x]$ are*

$$2(x_{uu} + x_{vv}) = (\operatorname{div} Q(x))x_u \wedge x_v.$$

Thus if (2.1) holds, equations 1.1 are the associated Euler equations of the functional $E[x]$.

Definitions. We define $\mathfrak{D} = \{x \in H_1(B): x(w) \in \bar{D}, \text{ for almost all } w \in B\}$. Here $H_1(G)$ is the Sobolev space of vector functions $x \in L^2(G)$ with distributional derivatives $x_u, x_v \in L^2(G)$, under the norm

$$\|x\|_{1,G} = \left\{ \iint_G (|x|^2 + x_u^2 + x_v^2) \, du \, dv \right\}^{1/2}.$$

We abbreviate $\|x\|_{1,B}$ simply as $\|x\|_1$. We denote weak convergence of a sequence $x_n \in H_1(B)$ to $x \in H_1(B)$ by $x_n \xrightarrow{H_1(B)} x$. We shall also use the norm $|x|_0 = \operatorname{ess\,sup} |x(w)|$.

For $f \in \mathfrak{D}$ we define $\mathfrak{D}(f) = \{x \in H_1(B): x - f \in H_1^0(B)\}$. Here $H_1^0(B)$ is the closure of smooth functions with compact support in B under $\|\cdot\|_1$. We can now define the variational problem

$$\mathfrak{P}(f, D, H): E[x] \rightarrow \min \text{ among } x \in \mathfrak{D}(f)$$

and we denote this minimum by $e(f, D, H)$.

Suppose $x \in \mathfrak{D}$. Since $2|x_u \wedge x_v| \leq x_u^2 + x_v^2$ and $|Q(x)| \leq 2\alpha < 2$ (by assumption (2.2)) there holds

$$(2.3) \quad (1 - \alpha)D[x] \leq E[x] \leq (1 + \alpha)D[x].$$

The same relations hold on subdomains of B .

Lemma 2.2. (Lower semi continuity). *Let $\{x_n\}$ be a sequence from $\mathfrak{D}$ such that $x_n \xrightarrow{H_1(B)} x$. Then $x \in \mathfrak{D}$ and $E_G[x] \leq \liminf E_G[x_n]$ for each measurable $G \subseteq B$.*

Proof. Follows easily from Theorem 1.8.2 of [18], p. 22.

Lemma 2.3. *There exists a solution $x \in \mathfrak{D}(f)$ of the variational problem $\mathfrak{P}(f, D, H)$.*

Proof. Choose a sequence $x_n \in \mathfrak{D}(f)$ such that $\lim E[x_n] = e(f, D, H)$. Since (by (2.3)) $E[x_n] \geq (1 - \alpha)D[x_n]$, we have $D[x_n]$ uniformly bounded. Since $x_n - f \in H_1^0(B)$, we have by Poincaré's inequality

$$\iint_B |x_n - f|^2 du dv \leq C \iint_B |\nabla(x_n - f)|^2 du dv.$$

Now using the inequality $2ab \leq a^2/2 + 2b^2$, we obtain

$$\iint_B x_n^2 du dv \leq 4CD[x_n] + 4CD[f] + 2 \iint_B f^2 du dv.$$

Therefore $\|x_n\|_1$ is uniformly bounded, and after choosing a subsequence, we have $x_n \xrightarrow{H_1(B)} x$. By Lemma 2.2 $x \in \mathfrak{D}(f)$ and

$$e(f, D, H) \leq E[x] \leq \lim E[x_n] = e(f, D, H).$$

Thus $E[x] = e(f, D, H)$.

Q.E.D.

We next discuss some regularity properties of solutions to $\mathfrak{P}(f, D, H)$. Certain steps in the proof of regularity rely on convexity properties of the admissible set for the variational problem. Roughly speaking, a quasiregular set is one which differs only by a uniform amount from being locally convex. The precise definition and basic properties were given by Hildebrandt and Kaul [15; cf. Definition 1]. For our purposes, suppose $\Omega \subset E^3$ is the closure of its interior. Ω is quasiregular if there are positive numbers δ_0 , δ_1 and d such that for all $x \in \Omega$ there is a C^1 diffeomorphism g of $B_d(x) \cap \Omega$ with a convex set in E^3 , such that for every vector η

$$(2.4) \quad \delta_0 |\eta|^2 \leq |Dg(\eta)|^2 \leq \delta_1 |\eta|^2$$

where $B_d(x)$ is a ball of radius d about x . In particular Ω will be quasiregular if it is compact and has C^1 boundary. It is desirable to find geometric conditions which ensure quasiregularity for unbounded domains.

Lemma 2.4. *Assume $\partial\Omega$ is of class C^2 , and that there exist positive numbers ρ , M such that for all $x \in \Omega$ and $\epsilon \leq \rho$, $B_\epsilon(x) \cap \partial\Omega$ is connected; and for all $p \in \partial\Omega$, $|k_1(p)| \leq M$ and $|k_2(p)| \leq M$ where k_1 , k_2 denote the principal curvatures of $\partial\Omega$. Then Ω is quasiregular.*

Proof. Take $d = \min(\rho, 1/(10M))$. Given $x \in \Omega$ if $B_d(x) \cap \partial\Omega$ is empty we take g to be the identity so that (2.4) is satisfied for $\delta_0 \leq 1 \leq \delta_1$. If $B_d(x) \cap \partial\Omega$ is nonempty, let $q \in \partial\Omega$ be a point whose distance from x is the smallest possible. We introduce orthogonal coordinates (x^1, x^2, x^3) centered at q such that the interior normal vector at q is in the x^3 -direction. Let Z denote the cylinder $(x^1)^2 + (x^2)^2 < d^2$. We shall show that the component of $\partial\Omega \cap Z$ containing q and therefore $\partial\Omega \cap B_d(x)$, in particular, may be represented non-parametrically

by $x^3 = \varphi(x^1, x^2)$. To begin with, some neighborhood of q on $\partial\Omega \cap Z$ may be represented this way, for $(x^1, x^2) \in U$, where U is a neighborhood of $(0, 0)$. Thus $\nabla\varphi(0, 0) = 0$.

To estimate the value of $|\nabla\varphi|$ at $(x^1, x^2) \in U$, let $\sigma(t)$ be a line segment in U parameterized by arc length, with $\sigma(0) = (0, 0)$ and $\sigma(r) = (x^1, x^2)$, $r^2 = (x^1)^2 + (x^2)^2$. Let s denote arc length along the curve on the surface $x^3 = \varphi(x^1, x^2)$ whose projection onto the (x^1, x^2) plane is σ . Then $ds/dt \leq W$, where $W^2 = 1 + |\nabla\varphi|^2$. Since W^{-1} is the x^3 -component of the interior unit normal, our curvature assumptions imply $|d/ds(W^{-1})| \leq M$. Together these relations imply $d/dt(W^{-2}) \geq -2M$. On integrating this along σ we find

$$(2.5) \quad |\nabla\varphi|^2 \leq \frac{2rM}{1 - 2rM} \leq \frac{1}{4},$$

since $rM \leq dM \leq 1/10$. It follows that φ may be extended to give a non-parametric representation of the component of $Z \cap \partial\Omega$ containing q .

Define $g: Z \rightarrow Z$ by $g(x^1, x^2, x^3) = (x^1, x^2, x^3 - \varphi(x^1, x^2))$. Then for a vector $\eta = (\eta^1, \eta^2, \eta^3)$

$$Dg(\eta) = (\eta_1 - \varphi_{x^1}\eta_3, \eta_2 - \varphi_{x^2}\eta_3, \eta_3).$$

It may be seen using standard inequalities that

$$(2.6) \quad |(|Dg(\eta)|^2 - |\eta|^2)| \leq (|\nabla\varphi| + |\nabla\varphi|^2) |\eta|^2.$$

The relation (2.4) with $\delta_0 = 1/4$, $\delta_1 = 7/4$, follows from (2.5) and (2.6).

It remains to show $g(B_d(x) \cap \Omega)$ is convex. Observe that $\partial(B_d(x) \cap \Omega)$ consists of portions of $\partial B_d(x)$ with part of the surface $x^3 = \varphi(x^1, x^2)$. It therefore suffices to show $g(B_d)$ is convex.

To show that $g(B_d)$ is convex, we need only show that every tangent line to its boundary lies locally in the closure of the complement of $g(B_d)$. It is easy to see that this holds for lines in the x^3 -direction. Given any other line, let $\gamma: \mathbf{R} \rightarrow \mathbf{R}^2$ be its projection in the (x^1, x^2) plane, parametrized by arc length. In some neighborhood of the point of tangency, the boundary of $g(B_d)$ may be described by

$$x^3 = \varphi_0(x^1, x^2) - \varphi(x^1, x^2)$$

where $x^3 = \varphi_0(x^1, x^2)$ describes a piece of the boundary of B_d . We need to show that $\varphi_0 \circ \gamma - \varphi \circ \gamma$ has a maximum or minimum at the point of tangency, depending on whether φ_0 describes a piece of the upper or lower hemisphere. Thus it suffices to show that $\varphi_0' - \varphi''$ has the same sign as φ_0' , i.e., $|\varphi''| \leq |\varphi_0'|$. Here φ' denotes the derivative of $\varphi \circ \gamma$. It may be shown in a straightforward fashion that $|\varphi_0'| \geq 1/d$.

Let v be the vector tangent to the surface $x^3 = \varphi(x^1, x^2)$ whose projection on the (x^1, x^2) plane is γ' . Then $\varphi'' = WB(v, v)$, where B is the second fundamental form of the surface. Since $|\gamma'| = 1$, $|v| \leq W$. We now have

$$|\varphi''| = W |B(v, v)| \leq WM |v|^2 \leq MW^3.$$

We have assumed $10 dM \leq 1$; with (2.5) this implies

$$|\varphi''| \leq (10d)^{-1}(1 + |\nabla\varphi|^2)^{3/2} < 1/d \leq |\varphi_0''|$$

as required.

Q.E.D.

Remark. The above lemma holds for a set in E^n ; the proof is completely analogous.

Corollary 2.1. *Admissible domains have quasiregular closures.*

We may now apply a regularity result of Hildebrandt and Kaul.

Lemma 2.5. *Let x be a solution to $\mathcal{O}(f, D, H)$ where D is admissible for H . Then x is μ -Hölder continuous in B , where μ depends on α , D , and a bound for $D[x]$. Suppose further that $f \in H_1(B) \cap C^0(\partial B)$. Then $x \in C^0(\bar{B})$ and $x = f$ on ∂B .*

Proof. Follows directly from Corollary 2.1 and Theorem 2 of [15].

Lemma 2.6. *Let x be a solution to $\mathcal{O}(f, D, H)$ where D is admissible for H and suppose $x(w_0) \in D$ for $w_0 \in B$. Then there is a neighborhood N of w_0 such that $x \in C^2(N)$ and satisfies equations 1.1 in N .*

Proof. Since x is continuous by Lemma 2.5, there is a neighborhood N' of w_0 such that $x(N')$ is contained in a relatively compact subdomain of D . Therefore for any $\varphi \in H_1^0(N') \cap L_\infty(N')$, $x + \epsilon\varphi \in \mathcal{D}(f)$ for sufficiently small ϵ . Hence by a well-known argument we can derive the variational equation

$$\iint_B (\nabla x \cdot \nabla \varphi + 2H(x)\varphi \cdot x_u \wedge x_v) du dv = 0.$$

Using the results of Lemma 2.5 the statement of the lemma follows for $N \subset N'$ from regularity results of Morrey [18].

3. Geometric maximum principles. Let Ω be the closure of a domain in E^3 with $\partial\Omega$ of class C^2 . Denote by k_1, k_2 the principal curvatures of $\partial\Omega$ with respect to the inward normal N to $\partial\Omega$. For $P \in \Omega$ denote by $r(P)$ the distance from P to $\partial\Omega$. There is a neighborhood V of $\partial\Omega$ such that for $P \in V$, there is a unique point $\pi(P) \in \partial\Omega$ with $|P - \pi(P)| = r(P)$; π is a C^1 mapping of V onto $\partial\Omega$. Given $R > 0$ we define a truncation $T: \Omega \rightarrow E^3$ by $T(P) = \pi(P) + RN(\pi(P))$ for $P \in V$ with $r(P) \leq R$, and $T(P) = P$ otherwise. Observe that T need not be continuous.

Theorem 3.1. *Suppose $x: B \rightarrow \Omega$, $x \in C^0(B) \cap H_1(B)$. Assume $H: \Omega \rightarrow \mathbf{R}$ is of class C^1 and let a C^1 vector field Q on Ω be given with $\operatorname{div} Q = 4H$. Assume either*

- (1) $k_i(p) > |H(p)|$ for all $p \in \partial\Omega$, $i = 1, 2$, or
- (2) $(k_1(p) + k_2(p))/2 > |H(p)|$ for all $p \in \partial\Omega$ and x satisfies the conformality conditions (1.2) a.e. in B .

Then there is $R_0 > 0$ such that if $0 < R \leq R_0$ and $\inf \{r(x(w)) : w \in B\} < R$, we have $T \circ x \in C^0(B) \cap H_1(B)$ and $E[T \circ x] < E[x]$.

Proof. Let $B_1 = \{w \in B : x(w) \in V\}$. Then $K = \pi \circ x(B_1)$ is a relatively compact set in $\partial\Omega$. Observe that there is $R_2 > 0$ such that for all $q \in K$ and $r \in (0, R_2]$, $q + rN(q) \in V$. This forces $k_i(p)R_2 < 1$ for all $p \in K$, $i = 1, 2$. For future purposes, we shall assume $4k_i(p)R_2 \leq 1$ for all $p \in K$, $i = 1, 2$. Then for $R \leq R_2$, the truncated mapping $y = T \circ x$ is continuous and of class $H_1(B)$. Denote $B_0 = \{w \in B_1 : r(x(w)) < R\}$.

Let D be the oriented 3-chain in E^3 whose boundary is the oriented 2-chain given by $x|_{B_0}$ minus that given by $y|_{B_0}$. Then

$$W[x] - W[y] = \iiint_D \operatorname{div} Q \, dV = \iiint_D 4H \, dV.$$

Likewise, if Q_0 is any C^1 vector field on D , (that is on the support of D), with $\operatorname{div} Q_0 = 4H$, we have

$$W[x] - W[y] = \iiint_D \operatorname{div} Q_0 \, dV = \iint_{B_0} (Q_0(x) \cdot x_u \wedge x_v - Q_0(y) \cdot y_u \wedge y_v) \, du \, dv.$$

Thus if $i(z) = |\nabla z|^2 + Q_0(z) \cdot z_u \wedge z_v$, we have

$$E[x] - E[y] = \iint_{B_0} (i(x) - i(y)) \, du \, dv.$$

Now for any point $p \in V$, introduce local parallel coordinates (x^0, x^1, x^2) centered at $\pi(p)$ such that the coordinate vector fields $E_i = \partial/\partial x^i$ satisfy $E_0 = N(\pi(p))$ and E_i is a curvature direction for the k_i , $i = 1, 2$. Thus $r(x^0, x^1, x^2) = x^0$ and $\pi(x^0, x^1, x^2) = (0, x^1, x^2)$. We define the vector field Q_0 by

$$Q_0(x^0, x^1, x^2) = \frac{1}{\gamma(x^0)} \int_{x^0}^{x^0} 4H(r, x^1, x^2) \gamma(r) \, dr \, E_0$$

where $\gamma(r) = E_0 \cdot E_1 \wedge E_2(r, x^1, x^2)$. It may be shown in a standard way that $\operatorname{div} Q_0 = 4H$ and that Q_0 does not depend on the choice of parallel coordinates.

Extend the functions k_i on $\partial\Omega$ over V by defining $k_i(p) = k_i(\pi(p))$. Then (see [11], p. 36)

$$(3.1) \quad \begin{aligned} E_i(x^0, x^1, x^2) &= (1 - k_i x^0) E_i(0, x^1, x^2), & i = 1, 2, \\ E_0(x^0, x^1, x^2) &= E_0(0, x^1, x^2) \end{aligned}$$

modulo the parallel translation of vectors in E^3 . In particular, $\gamma(r) = (1 - k_1 r)(1 - k_2 r)\gamma(0)$.

We shall concern ourselves with the value of $i(x) - i(y)$ on B_0 , disregarding sets of measure zero. Let the components of x be written (x^0, x^1, x^2) ; then the components of y are (R, x^1, x^2) . Define a real-valued function on the support

of D by

$$h(x^0, x^1, x^2) = \sup \{|H(r, x^1, x^2)| : 0 \leq r \leq R\}.$$

Observe $Q_0(y) = 0$, so that

$$\begin{aligned} & |Q_0(x) \cdot x_u \wedge x_v - Q_0(y) \cdot y_u \wedge y_v| \\ (3.2) \quad &= \left| \frac{1}{\gamma(x^0)} \int_R^{x^0} 4H(r, x^1, x^2) \gamma(r) \, dr \, E_0 \cdot x_u \wedge x_v \right| \\ &\leq \frac{2h |\nabla x|^2}{\gamma(x^0)} \int_{x^0}^R \gamma(r) \, dr = 2h |\nabla x|^2 (R - x^0) f(R, x^0) \end{aligned}$$

where $f(R, x^0)$ is a quotient of polynomials, each having constant term 1 and coefficients involving k_1 and k_2 . Thus given $\epsilon > 0$ there is R_0 , $0 < R_0 < R_2$ such that

$$(3.3) \quad f(R, x^0) \leq 1 + \epsilon \quad \text{whenever} \quad x^0 \leq R \leq R_0.$$

Again applying (3.1) we have

$$(3.4) \quad |\nabla y|^2 = \sum |\nabla x^i|^2 \left(\frac{1 - k_i R}{1 - k_i x^0} \right)^2$$

and

$$(3.5) \quad |\nabla x|^2 - |\nabla y|^2 = 2(R - x^0) \sum k_i |\nabla x^i|^2 f_i(R, x^0) + |\nabla x^0|^2$$

where the summations are over $i = 1, 2$ and $f_i(R, x^0)$ is a quotient of polynomials with constant term 1 and coefficients involving k_i . Thus for given $\epsilon > 0$ we may assume the R_0 of (3.3) is chosen so that

$$(3.6) \quad |f_i(R, x^0) - 1| \leq \epsilon \quad \text{whenever} \quad x^0 \leq R \leq R_0.$$

First assume the hypotheses of part (1), that is $k_i > |H|$ on $\partial\Omega$, $i = 1, 2$. Then there exists $R_3 > 0$ such that if $R \leq R_3$, $k_i > h$ on the relatively compact set $K \subset \partial\Omega$. Further, $k_i \geq h + \delta$ on K for some $\delta > 0$; thus $k_i \geq h + \delta$ on D (recall that h and k_i are defined to be constant on normal lines to $\partial\Omega$). Likewise, h , $|k_1|$ and $|k_2|$ have a common upper bound M on D . Thus we may choose $\epsilon > 0$ small enough that

$$(3.7) \quad \delta_0 = \delta(1 - \epsilon) - 2M\epsilon > 0.$$

Now choose $R_0 \leq \min(R_2, R_3)$ and small enough to satisfy (3.3) and (3.6) with this ϵ . From (3.5) and (3.6) we now have that

$$\begin{aligned} |\nabla x|^2 - |\nabla y|^2 &\geq 2(R - x^0)(h + \delta)(|\nabla x^1|^2 + |\nabla x^2|^2)(1 - \epsilon) + |\nabla x^0|^2 \\ &\geq 2(R - x^0) |\nabla x|^2 \delta_0. \end{aligned}$$

The last inequality follows by observing

$$2(h + \delta)(R - x^0) \leq 2k_1 R_2 < 1.$$

Together with (3.2), (3.3) and (3.7) this yields the estimate

$$(3.8) \quad \begin{aligned} i(x) - i(y) &\geq |\nabla x|^2 - |\nabla y|^2 - |Q_0(x) \cdot x_u \wedge x_v - Q_0(y) \cdot y_u \wedge y_v| \\ &\geq 2(R - x^0) |\nabla x|^2 \delta_0 \end{aligned}$$

on B_0 (almost everywhere). We shall return to this inequality after considering the other case.

So assume the hypotheses of part (2), that is, that $(k_1 + k_2)/2 > |H|$ on $\partial\Omega$ and that

$$(3.9) \quad |x_u| = |x_v|, \quad x_u \cdot x_v = 0$$

holds almost everywhere in B . Then by a similar argument to that used above, there is $R_3 > 0$ such that $R \leq R_3$ implies $(k_1 + k_2)/2 \geq h + \delta$ on D , for some $\delta > 0$. It follows from (3.9) that for $i = 1, 2$ $|\nabla x^i|^2 \leq \frac{1}{2} |\nabla x|^2$. Using this with (3.4) and (3.6), we obtain

$$\begin{aligned} |\nabla x|^2 - |\nabla y|^2 &\geq |\nabla x|^2 \left\{ 1 - \frac{1}{2} \sum \left(\frac{1 - k_i R}{1 - k_i x^0} \right)^2 \right\} \\ &= (R - x^0) |\nabla x|^2 \sum k_i f_i(R, x^0) \\ &\geq 2(R - x^0) |\nabla x|^2 (h + \delta - M\epsilon). \end{aligned}$$

Together with (3.2), (3.3) and (3.7), and the observation that $\delta - 2M\epsilon > \delta_0$, this implies as before

$$i(x) - i(y) \geq 2(R - x^0) |\nabla x|^2 \delta_0.$$

In either case, since $R_4 = \inf \{x^0(w) : w \in B\} < R$, the set $B_2 = \{w \in B : R_4 \leq x^0(w) \leq (R_4 + R)/2\}$ has nonempty interior, and x is not locally constant on B_2 . Thus $D_{B_*}[x] > 0$. Therefore

$$\begin{aligned} E[x] - E[y] &= \iint_{B_0 \setminus B_*} (i(x) - i(y)) \, du \, dv + \iint_{B_*} (i(x) - i(y)) \, du \, dv \\ &\geq (R - R_4) \delta_0 D_{B_*}[x] > 0. \end{aligned} \quad \text{Q.E.D.}$$

It is of interest in the case $H \equiv 0$ to have a sharper version of Theorem 3.1.

Theorem 3.2. Suppose $x: B \rightarrow \Omega$, $x \in C^0(B) \cap H_1(B)$. Assume either

(1) $k_i(p) \geq 0$ for all $p \in \partial\Omega$, $i = 1, 2$, or

(2) $(k_1(p) + k_2(p))/2 \geq 0$ for all $p \in \partial\Omega$ and x satisfies the conformality condition (1.2) a.e. in B .

Then there is an $R_0 > 0$ and $z: B \rightarrow \Omega$ such that if $0 < R \leq R_0$, $\inf \{r(x(w)) : w \in B\} < R$, $\sup \{r(x(w)) : w \in B\} > R$ we have $z \in C^0(B) \cap H_1(B)$ and $D[z] < D[x]$, where z is a reparametrization of $T \circ x$.

Proof. Set $y = T \circ x$ and let R_0 be chosen as in Theorem 3.1. First assume the hypotheses of part (1) are satisfied. Then since $k_i \geq 0$ we have

$$(1 - k_i R) \leq (1 - k_i x^0), \quad i = 1, 2.$$

Using (3.4) this gives $|\nabla y|^2 \leq |\nabla x^1|^2 + |\nabla x^2|^2$ or

$$|\nabla y|^2 \leq |\nabla x|^2 - |\nabla x^0|^2.$$

By assumption, x^0 is not constant on B_0 . Hence

$$D[y] - D[x] \leq - \iint_{B_0} |\nabla x^0|^2 du dv < 0$$

and we may take $z = y$.

Now assume the hypotheses of part (2), that is $(k_1 + k_2) \geq 0$ and (3.9) holds. We denote the area functional $A[x] = \iint_B |x_u \wedge x_v| du dv$. Using (3.1) we have

$$|y_u \wedge y_v| \leq \frac{(1 - k_1 R)(1 - k_2 R)}{(1 - k_1 x^0)(1 - k_2 x^0)} |x_u \wedge x_v|$$

so that

$$\begin{aligned} |y_u \wedge y_v| - |x_u \wedge x_v| &\leq \left\{ \frac{(1 - k_1 R)(1 - k_2 R)}{(1 - k_1 x^0)(1 - k_2 x^0)} - 1 \right\} |x_u \wedge x_v| \\ &= (R - x^0) \frac{(-(k_1 + k_2) + k_1 k_2 (R + x^0))}{(1 - k_1 x^0)(1 - k_2 x^0)} |x_u \wedge x_v|. \end{aligned}$$

Since $2k_1 k_2 \leq (k_1 + k_2)^2$, equality holding only if k_1 and k_2 both vanish, this gives

$$\begin{aligned} |y_u \wedge y_v| - |x_u \wedge x_v| &\leq \frac{(R - x^0)(k_1 + k_2)(-1 + (k_1 + k_2)(R + x^0)/2)}{(1 - k_1 x^0)(1 - k_2 x^0)} |x_u \wedge x_v| \\ &\leq -\frac{1}{2}(R - x^0)(k_1 + k_2) |x_u \wedge x_v|. \end{aligned}$$

This last inequality follows since $x^0 \leq R \leq R_0 \leq 1/(4k_i)$, $i = 1, 2$. Observe that equality in the combined relation is possible only if $R = x^0$ or $k_1 = k_2 = 0$, or $x_u \wedge x_v = 0$. We may assume that k_1 and k_2 do not vanish identically on $\pi \circ x(B)$ since this case may be treated as in the proof of part (1). Hence $B_3 = \{w \in B_0 : k_1^2 + k_2^2 \neq 0 \text{ at } \pi(x(w))\}$ is nonempty and open.

Let $B_4 = \{w \in B_0 : r(x(w)) > 0\}$. On B_4 , $x_u \wedge x_v = 0$ only at isolated points since by Lemma 2.6 x satisfies equations (1.1) (cf. E. Heinz, Math. Z., **113**(1970), 99–105). Therefore unless $r \circ x \equiv 0$ on B_3 , $B_3 \cap B_4$ must contain an open set on which $A[x] > A[y]$. Assuming $r \circ x \equiv 0$ on B_3 , then unless $x_u \wedge x_v = 0$ a.e. on B_3 , we have $A_{B_3}[x] > A_{B_3}[y]$.

We will consider first the case $r \circ x \equiv 0$ on B_3 and $x_u \wedge x_v = 0$ a.e. on B_3 . It may be readily seen from the conformality relation (1.2) that $x_u \wedge x_v = 0$ only when $x_u = x_v = 0$. Hence from (3.4) we have

$$\begin{aligned} |\nabla y|^2 &= |\nabla x|^2 = 0 \quad \text{on } B_3, \\ |\nabla y|^2 &= |\nabla x|^2 - |\nabla x^0|^2 \quad \text{on } B_0 \setminus B_3. \end{aligned}$$

Since $x^0 = 0$ on B_3 and x^0 is not constant on B_0 we have

$$D[y] - D[x] \leq - \iint_{B_0 \setminus B_3} |\nabla x^0|^2 du dv < 0$$

and we may take $z = y$.

In the alternative case, $A[y] < A[x]$. Since x satisfies (3.9) we have $A[x] = \frac{1}{2} D[x]$ so that $A[y] < \frac{1}{2} D[x]$. Set $\epsilon = \frac{1}{2} D[x] - A[y] > 0$. According to Morrey's ϵ -conformal mapping theorem [19, Theorem 2.1], there is a reparametrization z of y such that

$$\begin{aligned} D[z] &\leq 2A[z] + \epsilon = 2A[y] + \epsilon \\ &= A[y] + \frac{1}{2} D[x] < D[x]. \end{aligned} \quad \text{Q.E.D.}$$

4. Existence theorems in admissible domains.

Definition. Let Γ be an oriented closed Jordan curve contractible in $\bar{D}$. Denote by $\mathfrak{D}(\Gamma)$ the set of mappings $x \in \mathfrak{D} \cap C^0(\partial B)$ such that $x|_{\partial B}$ is equal a.e. to a continuous monotone mapping of degree 1 (over the integers) onto Γ . Let D be admissible with respect to $H(x): D \rightarrow \mathbb{R}$. We define the variational problem

$$\mathcal{O}(\Gamma, D, H): E[x] \rightarrow \min \quad \text{among } x \in \mathfrak{D}(\Gamma)$$

and we denote this minimum by $e(\Gamma, D, H)$.

Proposition 4.1. Let $H: \bar{D} \rightarrow \mathbb{R}$ be of class C^1 where D is an admissible domain in E^3 with respect to H . Suppose the inward principal curvatures k_1, k_2 of ∂D satisfy $k_i(p) > |H(p)|$ for all $p \in \partial D, i = 1, 2$. Then given $f \in \mathfrak{D} \cap C^0(\partial B)$ with $f(\partial B)$ strictly interior to D , there exists a solution $x \in C^2(B) \cap C^0(\bar{B})$ to $\mathcal{O}(f, D, H)$ which satisfies equations (1.1) in B , and $x = f$ on ∂B .

Proof. Let x be the solution to $\mathcal{O}(f, D, H)$ given by Lemma 2.3. According to Lemma 2.5 $x \in C^0(\bar{B})$ so that the maximum principle Theorem 3.1 applies. Therefore $x(B)$ is strictly interior to D and Lemma 2.6 now shows $x \in C^2(B)$ and satisfies the Euler equations (1.1). Q.E.D.

Lemma 4.1. Suppose $\{H_n\}, n \geq 0$ is a sequence of C^1 functions in D with respect to which D is admissible, such that $|H_n - H_0|_0 \rightarrow 0$ and the corresponding vector fields Q_n satisfy $\sup |Q_n| \leq 2\alpha < 2$ for some constant $\alpha < 1$ independent of n . Let $f \in \mathfrak{D} \cap C^0(\partial B)$ and let Γ be an oriented closed Jordan curve contractible in $\bar{D}$ such that $\mathfrak{D}(\Gamma) \neq \emptyset$. Suppose x_n solves $\mathcal{O}(f, D, H_n)$, respectively $\mathcal{O}(\Gamma, D, H_n)$. Then $\{x_n\}$ is a minimizing sequence for $\mathcal{O}(f, D, H_0)$, respectively $\mathcal{O}(\Gamma, D, H_0)$.

Proof. We will give a proof for $\mathcal{O}(\Gamma, D, H_0)$ which is also valid for $\mathcal{O}(f, D, H_0)$. Let

$$E_n[x] = D[x] + \iint_B Q_n(x) \cdot x_u \wedge x_v du dv.$$

Define

$$G_n[x] = E_n[x] - E_0[x] = \iint_B (Q_n - Q_0) \cdot x_u \wedge x_v \, du \, dv.$$

Let z be a solution of $\mathcal{O}(\Gamma, D, H_0)$ and let $R(z, x_n)$ be the oriented 3-chain with oriented boundary $z(B)$ minus that given by $x_n(B)$. Then

$$G_n[z] - G_n[x_n] = \iiint_{R(z, x_n)} \operatorname{div} (Q_n - Q_0) \, dV = \iiint_{R(z, x_n)} 4(H_n - H_0) \, dV.$$

Let $f \in \mathcal{D}(\Gamma)$. Then $D[x_n] \leq ((1 + \alpha)/(1 - \alpha)) D[f] = C$. Then since $z, x_n \in C^0(\bar{B})$ (Lemma 2.5) we can apply the isoperimetric inequality [1] and conclude

$$|G_n[z] - G_n[x_n]| \leq C |H_n - H_0|_0 \quad \text{for some new constant } C.$$

Hence

$$|G_n[z] - G_n[x_n]| \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

But we may write

$$G_n[z] - G_n[x_n] = (E_n[z] - E_n[x_n]) + (E_0[x_n] - E_0[z])$$

and each term on the right hand side is non-negative by the minimizing properties of x_n and z . In particular

$$E_0[x_n] \rightarrow E_0[z] = e(\Gamma, D, H_0). \quad \text{Q.E.D.}$$

In the following discussion, $\|x\|_{2, \mathcal{G}}$ is defined by

$$\|x\|_{2, \mathcal{G}}^2 = \iint \sum_{|\alpha| \leq 2} |D^\alpha x|^2 \, du \, dv.$$

Lemma 4.2. *Let $H \in C^0(\bar{D})$ and suppose $x \in \mathcal{D} \cap C^0(\bar{B}) \cap C^2(B)$ is a solution of equations (1.1). Then x satisfies the a priori estimate*

$$\|x\|_{2, B'} \leq C(D, D[x], w_x, h, B') \quad \text{for any } B' \subset \subset B$$

where $w_x(r) = \sup \{|x(w) - x(w')| : |w' - w| \leq r, w, w' \in \bar{B}\}$.

Proof. [13, Lemma 7].

Theorem 4.1. (Dirichlet problem). *Let D be an admissible domain in E^3 with respect to $H : \bar{D} \rightarrow \mathbb{R}$ of class C^1 . Given $f \in \mathcal{D} \cap C^0(\partial B)$ suppose the inward principal curvatures k_1, k_2 of ∂D satisfy either*

- (1) $k_i(p) \geq |H(p)| + \delta$ for all $p \in \partial D$, $i = 1, 2$, some $\delta > 0$;
- (2) $k_i(p) \geq |H|_0 > 0$ for all $p \in \partial D$, $i = 1, 2$; or
- (3) $k_i(p) > |H(p)|$ for all $p \in \partial D$, $i = 1, 2$, and $f(\partial B)$ strictly interior to D .

Then there exists a solution $x \in C^2(B) \cap C^0(\bar{B})$ to $\mathcal{O}(f, D, H)$ satisfying equations (1.1) in B with $x = f$ on ∂B .

Proof. Extend H to $C^1(E^3)$, and extend $Q = Q^H$ to $C^1(E^3)$ arbitrarily. By assumption there is an ϵ_0 such that for $\epsilon \leq \epsilon_0$, ∂D has globally defined outward parallel surfaces at distance ϵ . Denote by D_ϵ the domain formed by expanding D by moving a distance ϵ along the exterior normal to ∂D . For any point $p \in D_\epsilon \setminus D$ introduce local parallel coordinates (x^0, x^1, x^2) centered at $\pi(p)$ (the projection of p onto ∂D) exactly as in Section 3. We define the vector field Q_0 in $D_\epsilon \setminus D$ by

$$Q_0(x^0, x^1, x^2) = \frac{1}{\gamma(x^0)} \int_0^{x^0} (4H(r, x^1, x^2) - \operatorname{div} Q(r, x^1, x^2)) \gamma(r) dr E_0.$$

Then Q_0 does not depend on the choice of parallel coordinates and $\operatorname{div} Q_0 = 4H - \operatorname{div} Q$. We define a Lipschitz vector field Q_1 in D_ϵ by

$$Q_1 = \begin{cases} Q & \text{in } \bar{D}, \\ Q_0 + Q & \text{in } D_\epsilon \setminus \bar{D}. \end{cases}$$

Observe $\operatorname{div} Q_1 = 4H$.

Now suppose the hypothesis of part (1) holds, that is $k_i(p) \geq |H(p)| + \delta$ for all $p \in \partial D$, $i = 1, 2$, and some $\delta > 0$. Denote by k_i^ϵ the inward principal curvatures of ∂D_ϵ . Then

$$k_i^\epsilon(p) = \frac{k_i(\pi(p))}{1 + \epsilon k_i(\pi(p))} \quad i = 1, 2, p \in \partial D_\epsilon.$$

It follows that for ϵ sufficiently small (depending on a bound for k_1, k_2 and a Lipschitz constant for H) $k_i^\epsilon(p) > |H(p)|$ for all $p \in \partial D_\epsilon$, $i = 1, 2$, and $\sup_{D_\epsilon} |Q_1| \leq 2\beta < 2$ for some constant $\beta < 1$. Set

$$E_1[x] = D[x] + \iint_B Q_1(x) \cdot x_u \wedge x_v du dv.$$

Let x be the solution to $\mathcal{O}(f, D_\epsilon, H)$ given by Lemma 2.3. According to Lemma 2.5 $x \in C^0(\bar{B})$ and $x = f$ on ∂B . We may now apply Theorem 3.1 to show $x(B) \subset \bar{D}$; then $Q_1(x) = Q^H(x)$. Since x solves $\mathcal{O}(f, D_\epsilon, H)$, a simple modification of Lemma 2.6 shows $x \in C^2(B)$ and satisfies equations (1.1). That $x(B)$ is actually interior to D follows in a straightforward way from a "local version" of Theorem 3.1.

Next suppose the hypothesis of part (2) is satisfied, that is $k_i(p) \geq |H|_0 > 0$ for all $p \in \partial D$, $i = 1, 2$. We can find sequences $\epsilon_n \downarrow 0$ and $\alpha_n \uparrow 1$ such that

$$k_i^{\epsilon_n}(p) > \alpha_n \sup_{D_\epsilon} |H| \quad \text{for all } p \in \partial D_\epsilon, \quad i = 1, 2, \quad \text{where } s = \epsilon_n.$$

Set $H_n = \alpha_n H$, $Q_n = \alpha_n Q_1$ ($n > 1$) and define

$$E_n[x] = D[x] + \iint_B Q_n(x) \cdot x_u \wedge x_v du dv.$$

Repeating the reasoning above, we can find surfaces $x_n \in C^2(B) \cap C^0(\bar{B}) \cap \mathfrak{D}(f)$ and $x_n = f$ on ∂B solving $\mathcal{O}(f, D, H_n)$ and satisfying equations (1.1) in B . As in the proof of Lemma 2.3 $x_n \xrightarrow{H_1(B)} x$ and $x \in \mathfrak{D}(f)$. Using Lemmas 2.2 and 4.1

we have $E[x] \leq \liminf E[x_n] = e(f, D, H)$ so that x solves $\mathcal{O}(f, D, H)$ and $x \in C^0(\bar{B})$, $x = f$ on ∂B (Lemma 2.5). Since $D[x_n]$ is uniformly bounded Lemma 2.5 provides a uniform Hölder estimate for x_n on compact subdomains of B . Thus from Lemma 4.2 $\|x_n\|_{2, B'} \leq C$ for $B' \subset \subset B$. By a diagonal argument and Rellich's lemma we may assume $\|x_n - x\|_{1, B'} \rightarrow 0$ and $x_n(w) \rightarrow x(w)$ for almost all $w \in B$. It follows easily that

$$\iint_B (\nabla x \cdot \nabla \varphi + 2H(x)\varphi \cdot x_u \wedge x_v) du dv = 0$$

for all test functions $\varphi \in H_1^0(B) \cap L_\infty(B)$ with support contained in B' . The regularity results of Heinz and Tomi [cf. 26] show that $x \in C^2(B)$ and x satisfies equations (1.1).

Part (3) is precisely Proposition 4.1.

Q.E.D.

Lemma 4.3. *Suppose $x \in H_1(B)$ and $D[x] \leq D[x \circ \varphi]$ for every diffeomorphism $\varphi: \bar{B} \rightarrow \bar{B}$ with positive jacobian. Then x satisfies the conformality relations (1.2) almost everywhere in B .*

Proof. Follows easily from [2; pp. 107–112.]

Proposition 4.2. *Let $H: \bar{D} \rightarrow \mathbb{R}$ be of class C^1 where D is admissible with respect to H . Let Γ be an oriented closed Jordan curve strictly interior to D , such that $\mathfrak{D}(\Gamma) \neq \emptyset$. Suppose that the inward mean curvature of $\partial D > |H(p)|$ for all $p \in \partial D$. Then there exists a solution $x \in C^2(B) \cap C^0(\bar{B})$ of the variational problem $\mathcal{O}(\Gamma, D, H)$ mapping ∂B homeomorphically into Γ , in an orientation preserving fashion, and satisfying equations (1.1) and (1.2) in B .*

Proof. Let f_n be a minimizing sequence of mappings, $f_n \in \mathfrak{D}(\Gamma)$. Let x_n be the solution of $\mathcal{O}(f_n, D, H)$ given by Lemma 2.3. It follows in a standard way [13] that a modified sequence $x_n \xrightarrow{H_1(B)} x$ and that $x \in \mathfrak{D}(\Gamma)$ is a solution of $\mathcal{O}(\Gamma, D, H)$. In the process of the proof, the original sequence $\{x_n\}$ is modified to satisfy a three point condition. This can be done without changing $E[x_n]$ since E is conformally invariant. We require the choice of the three points to be such that every monotone map: $\partial B \rightarrow \Gamma$ satisfying the three point condition will be of degree 1. Lemma 4.3 shows that x satisfies the conformality conditions (1.2) almost everywhere in B so that the maximum principle Theorem 3.1 applies. Hence $x(B)$ is strictly interior to D and Lemma 2.6 implies $x \in C^2(B)$ and satisfies (1.1). That $x|_{\partial B}$ is a homeomorphism follows as in [12, pp. 211–212].

Q.E.D.

Theorem 4.2. (Plateau problem). *Let $H: \bar{D} \rightarrow \mathbb{R}$ be of class C^1 where D is admissible with respect to H . Let Γ be an oriented closed Jordan curve contractible in $\bar{D}$, such that $\mathfrak{D}(\Gamma) \neq \emptyset$. Suppose either*

- (1) *The inward mean curvature of $\partial D \geq |H(p)| + \delta$, for all $p \in \partial D$, some $\delta > 0$;*
- (2) *The inward mean curvature of $\partial D \geq |H|_0 > 0$; or*

- (3) *The inward mean curvature of $\partial D > |H(p)|$ for all $p \in \partial D$ and Γ strictly interior to D .*

Then there exists a solution $x \in C^2(B) \cap C^0(\bar{B})$ to the variational problem $\mathcal{P}(\Gamma, D, H)$ mapping ∂D homeomorphically onto Γ in an orientation preserving fashion, and satisfying equations (1.1) and (1.2) in B .

Proof. Follows in a straightforward way by applying the proof of Theorem 4.1 in the context of the proof of Proposition 4.2. That $x(B)$ is interior to D in case (1) follows from the proof of Theorem 2.5 of [8] instead of the argument given in Theorem 4.1. Q.E.D.

Theorem 4.3. *Let D be an admissible domain in E^3 with respect to $H(x) \equiv 0$. Let Γ be a closed Jordan curve contractible in $\bar{D}$. Then if the inward mean curvature of $\partial D \geq 0$ there is a minimal surface in D with a conformal representation $x \in C^2(B) \cap C^0(\bar{B})$, mapping ∂B homeomorphically onto Γ .*

Proof. Suppose to begin with that Γ is rectifiable and strictly interior to D . Since Γ is rectifiable $\mathfrak{D}(\Gamma) \neq \emptyset$. Then using Theorem 3.2 in place of Theorem 3.1, we can conclude as in the proof of Proposition 4.2 that there is a minimal surface in D “spanning Γ ”. To take care of the more general case we approximate in the Frechét sense any Γ contained in $\bar{D}$ by rectifiable (contractible) Jordan curves Γ_n strictly interior to D . For each Γ_n there is a solution x_n . Applying the approximation procedure of Douglas ([22], p. 72) it follows that the $\{x_n\}$ converge to a minimal surface in D with a conformal representation $x \in C^2(B) \cap C^0(\bar{B})$ mapping ∂B homeomorphically onto Γ . Q.E.D.

Remark. Since minimal surfaces “never get very far from their boundaries” that is they satisfy the convex hull property, the behavior of ∂D in Theorem 4.3 at infinity is superfluous. Therefore it is natural to assume only that ∂D is of class C^2 , instead of admissibility. This will be done in the next section.

5. Concrete results and examples. In the previous section we have given geometrical conditions that are sufficient to solve the Dirichlet problem for equation (1.1) and the “Plateau type” problem of the introduction in an admissible domain D . Thus we have reduced our problem to the question of the existence of a C^1 vector field $Q(x)$ in D satisfying conditions (2.1) and (2.2).

In the classical minimal surface case $H(x) \equiv 0$, we may take $Q \equiv 0$ so that the functional $E[x]$ reduces to $D[x]$. We then have the following result.

Theorem 5.1. *Let D be a domain in E^3 with ∂D of class C^2 . Let Γ be a simple closed Jordan curve contractible in $\bar{D}$, such that $\mathfrak{D}(\Gamma)$ is nonempty. Then if the inward mean curvature of $\partial D \geq 0$ and > 0 outside some compact set there is a minimal surface in D with a conformal representation $x \in C^2(B) \cap C^0(\bar{B})$ mapping ∂B homeomorphically onto Γ .*

Proof. Consider a family of concentric spheres of increasing radius containing Γ . We can find a sphere C in this family which intersects ∂D trans-

versally and only at points where the inward mean curvature of $\partial D > 0$. Let D_1 be the component of $D \cap C$ containing Γ . Then ∂D_1 is piecewise C^2 with positive inward mean curvature where defined. We may then smooth the edges of ∂D_1 to obtain a bounded domain D_2 with ∂D_2 of class C^2 and having positive inward mean curvature. Theorem 4.3 now applies to the domain D_2 . Q.E.D.

Our method for showing the existence of the vector field Q for non-zero $H(x)$ is to look for Q in the form

$$Q(x) = 2D\varphi/(1 + |D\varphi|^2)^{1/2}$$

where $\varphi(x)$ is a C^2 function in $\bar{D}$. According to (2.1) φ must satisfy the partial differential equation

$$(5.1) \quad \operatorname{div} D\varphi/(1 + |D\varphi|^2)^{1/2} = 2H(x).$$

If we set $\Lambda(x) = \frac{2}{3}H(x)$ we may rewrite (5.1) as

$$(5.2) \quad \operatorname{div} D\varphi/(1 + |D\varphi|^2)^{1/2} = 3\Lambda(x).$$

Equation (5.2) means that $z = \varphi(x)$ is a three dimensional *non-parametric* surface of prescribed mean curvature $\Lambda(x)$ defined over D .

For bounded domains D very general existence theorems for this class of equations have been proven by Serrin [23]. In the appendix we show that Serrin's results extend naturally to unbounded D . In order to state the resulting existence theorems, we make the following

Definitions. Let $D \subset E^3$ be a domain with C^2 boundary and let $k_1 \geq k_2$ be the inward principal curvatures of ∂D . As usual, let $r(x)$ denote the distance from $x \in D$ to ∂D . Let D_1 be the open subset of points $x \in D$ such that

- (a) the closed ball of radius $r(x)$ centered at x touches ∂D at a single point $y = y(x)$ and
- (b) $r(x) < 1/k_1(y)$.

For $x \in \bar{D}$ define

$$\mathcal{H}(x) = \begin{cases} \frac{1}{2}[k_1(y)/(1 - k_1(y)r(x)) + k_2(y)/(1 - k_2(y)r(x))], & x \in D_1 \cup \partial D, \\ + \infty & \text{otherwise.} \end{cases}$$

Observe that for $x \in \partial D$ $\mathcal{H}(x)$ is the inward mean curvature of ∂D at x . Set $r_1 = \sup_D r(x)$.

For the remainder of the section we will assume unless otherwise specified that ∂D is of class C^2 with uniformly bounded principle curvatures, and that for sufficiently small $\epsilon > 0$, ∂D has globally defined inward and outward parallel surfaces at distance ϵ .

Theorem 5.2. (Dirichlet problem). *Let $H: \bar{D} \rightarrow \mathbb{R}$ be of class C^1 . Given $f \in \mathcal{D} \cap C^0(\partial B)$ suppose that there exists a constant a , $0 \leq a \leq 1/r_1$ such that*

$$|H(x)| \leq (1 - ar(x))\mathcal{H}(x) + a/2 \quad \text{in } D$$

and either

- (1) $k_i(p) \geq |H(p)| + \delta$ for all $p \in \partial D$, $i = 1, 2$ some $\delta > 0$;
- (2) $k_i(p) \geq |H|_0 > 0$ for all $p \in \partial D$, $i = 1, 2$; or
- (3) $k_i(p) > |H(p)|$ for all $p \in \partial D$, $i = 1, 2$ and $f(\partial B)$ strictly interior to D .

Then there exists a solution $x \in C^2(B) \cap C^0(\bar{B})$ to $\mathcal{P}(f, D, H)$ satisfying equations (1.1) in B with $x = f$ on ∂B .

Proof. Immediate from Theorem 4.1 and Theorem A.2.

Q.E.D.

Theorem 5.3. (Plateau problem). Let Γ be an oriented simple closed Jordan curve contractible in $\bar{D}$ such that $\mathfrak{D}(\Gamma) \neq \emptyset$. Let $H: \bar{D} \rightarrow \mathbb{R}$ be of class C^1 and suppose there exists a constant a , $0 \leq a \leq 1/r_1$, such that

$$|H(x)| \leq (1 - \ar(x))\mathfrak{C}(x) + a/2 \quad \text{in } D$$

and either

- (1) $\mathfrak{C}(p) \geq |H(p)| + \delta$ for all $p \in \partial D$, $i = 1, 2$ some $\delta > 0$;
- (2) $\mathfrak{C}(p) \geq |H|_0$ for all $p \in \partial D$; or
- (3) $\mathfrak{C}(p) > |H(p)|$ for all $p \in \partial D$ and Γ strictly interior to D .

Then there exists a solution $x \in C^2(B) \cap C^0(\bar{B})$ to the variational problem $\mathcal{P}(\Gamma, D, H)$ mapping ∂D homeomorphically onto Γ in an orientation preserving fashion, and satisfying equations (1.1) and (1.2) in B .

Proof. Immediate from Theorem 4.2 and Theorem A.2.

Q.E.D.

If we observe that $\mathfrak{C}(x)$ assume its minimum on ∂D we obtain

Corollary 5.1. Let Γ be an oriented simple closed Jordan curve contractible in $\bar{D}$ such that $\mathfrak{D}(\Gamma) \neq \emptyset$. Let $H: \bar{D} \rightarrow \mathbb{R}$ be of class C^1 and suppose that the inward mean curvature of $\partial D \geq |H|_0$. Then there exists a solution $x \in C^2(B) \cap C^0(\bar{B})$ to the variational problem $\mathcal{P}(\Gamma, D, H)$ mapping ∂B homeomorphically onto Γ in an orientation preserving fashion, and satisfying equations (1.1) and (1.2) in B .

Before we illustrate our results by some simple examples, we will show how to construct the vector field Q explicitly in a number of interesting cases. This approach, although not generally applicable has two advantages. The first is that we may relax the regularity assumption on $H(x)$ to uniform Hölder continuity. The second is that our sufficient conditions for existence will then involve only assumptions about ∂D . However, assumptions concerning $|H|_0$ are built into the construction.

Let C_R be a closed ball of radius R with center at the origin. Let H be a C^1 function in C_R . We define

$$Q(x) = \left(\int_0^1 4t^2 H(tx) dt \right) x.$$

A straightforward integration by parts shows that $\operatorname{div} Q = 4H$. Set $h = |H|_0$.

Then

$$|Q(x)| \leq \frac{4}{3} hR.$$

In particular (2.2) is satisfied if $R < 3/2h$.

Next consider a slab S_R of width R . Let us choose coordinates (x^1, x^2, x^3) so that S_R is described by $|x^3| \leq R/2$. We define

$$Q(x) = \left(\int_0^1 4H(x^1, x^2, tx^3) dt \right) x^3 e_3.$$

Again $\operatorname{div} Q = 4H$ and

$$|Q(x)| \leq 2hR.$$

In particular (2.2) is satisfied if $R < 1/h$.

These two constructions are actually special cases of the construction of the vector field Q_0 in parallel coordinates given in Section 3.

Corollary 5.2. *Let Γ be an oriented simple closed Jordan curve contractible in $\bar{D}$ such that $\mathfrak{D}(\Gamma) \neq \emptyset$. Let $H : \bar{D} \rightarrow \mathbf{R}$ be uniformly Hölder continuous and suppose that D is contained in C_R , $R < 3/2h$ or S_R , $R < 1/h$. Assume either*

(1) $\mathfrak{H}(p) \geq |H(p)| + \delta$ for all $p \in \partial D$, $i = 1, 2$ some $\delta > 0$;

(2) $\mathfrak{H}(p) \geq |H|_0$ for all $p \in \partial D$; or

(3) $\mathfrak{H}(p) > |H(p)|$ for all $p \in \partial D$ and Γ strictly interior to D .

Then there exists a solution $x \in C^2(B) \cap C^0(\bar{B})$ to the variational problem $\mathfrak{P}(\Gamma, D, H)$ mapping ∂B homeomorphically onto Γ in an orientation preserving fashion, and satisfying equations (1.1) and (1.2) in B .

Proof. Follows from Theorem 4.2 and an approximation argument of [7].
Q.E.D.

Example 1. Let T be the torus generated by revolving the disk $(x - a)^2 + y^2 \leq b^2$ about the y axis. Then T has non-negative inward mean curvature for $a \geq 2b$. The hypotheses of Theorem 5.1 are satisfied so that for Γ a Jordan curve contractible in T there is a simply connected minimal surface in T spanning Γ . That Γ must be contractible in order for a solution to exist follows by considering the non-contractible fundamental cycle γ of T ; the unique minimal surface of the type of the disk spanning γ is a circular disk.

Example 2. Let C_R be a ball of radius R . Then for $H(x)$ uniformly Hölder continuous in C_R with $|H|_0 \leq 1/R$ the hypotheses of part (2) of Corollary 5.2 are satisfied. Then for Γ contained in C_R with $\mathfrak{D}(\Gamma) \neq \emptyset$, the “Plateau problem of the introduction is solvable.” This gives a new proof of the results of Heinz, Werner and Hildebrandt mentioned in the introduction. Moreover, we have proved the following extension: if $|H|_0 < 3/2R$ and $|H| < 1/R$ on ∂C_R , then the Plateau problem is solvable. This also follows from a sharpening of the method of Hildebrandt in [12].

Example 3. Let Z_R be a circular cylinder of radius R . Then for $H(x)$ uniformly Hölder continuous in Z_R with $|H|_0 \leq 1/2R$ the hypotheses of Corollary 5.2 are again satisfied. This gives another proof of the result of the authors mentioned in the introduction. Again it suffices to require $|H|_0 < 1/R$, $|H| < 1/2R$ on ∂Z_R .

Example 4. Consider the cylinder Z_R and the ball C_{2R} . Both of these domains are part of a one-parameter family of domains whose boundaries are surfaces of revolution of constant mean curvature $1/R$. These surfaces of revolution were treated by Delaunay [4]. Delaunay found the generating curve of these surfaces of revolution is obtained by tracing out the path of the focus of an ellipse as the ellipse rolls without slipping along the axis of revolution. This ellipse will have a major semi-axis of length $R/2$ and a minor semi-axis of length $(AR)^{1/2}$ where $0 < A < R/4$. Let $D(A)$ be the simply-connected domain bounded by the Delaunay surface corresponding to parameter A , $0 < A < R/4$. Each $D(A)$ is non-convex; as $A \rightarrow 0$ $D(A)$ tends to Z_R while as $A \rightarrow R/4$ $D(A)$ tends to a "sequence of C_{2R} stacked one on top of another."

For $H(x)$ uniformly Hölder continuous in $\overline{D(A)}$ with $|H|_0 \leq 1/R$ the hypotheses of Corollary 5.2 are satisfied. Thus for Γ contained in $\overline{D(A)}$ with $\mathfrak{D}(\Gamma) \neq \emptyset$ the Plateau problem of the introduction is solvable.

Example 5. Let D be an ellipsoid with semi-axes of lengths $a \geq b \geq c$. A simple computation shows that the minimal value of the inward mean curvature of ∂D is $(c/2)(1/a^2 + 1/b^2)$. Therefore for $|H|_0 \leq (c/2)(1/a^2 + 1/b^2)$, $H : \bar{D} \rightarrow R$ of class C^1 the hypotheses of Corollary 5.1 are satisfied. Thus for Γ contained in $\bar{D}$ with $\mathfrak{D}(\Gamma) \neq \emptyset$ the Plateau problem is solvable. This is an improvement of a result of Hildebrandt [14] who required $|H|_0 < c/a^2$. In fact, using Theorem 5.2 the condition $|H|_0 \leq c/a^2$ is sufficient to solve the Dirichlet problem for equation (1.1).

The interested reader will have little difficulty in constructing further examples that are of interest to him.

Remarks.

1) In many instances the regularity assumptions of ∂D can be relaxed. For example if ∂D has global *outward* parallel surfaces at distance ϵ , for small ϵ , then a slightly expanded domain will have a boundary having global inward and outward parallel surfaces for small distances.

2) One of the authors [6] has demonstrated that the solution mapping x that we obtain is actually an immersion, that is $x_u^2 + x_v^2 \neq 0$ in B .

3) Our results should be compared with the work of Steffen [25] and Wente [28]. These authors find sufficient conditions for a *particular boundary curve* Γ to be spanned by a surface of *constant mean curvature* H , requiring $|H| (A(\Gamma))^{1/2} < \text{constant}$ where $A(\Gamma)$ is the lower bound for areas of surfaces spanning Γ . As a general rule, if Γ is stubby or not too violently bent, the results of this

paper provide a better estimate with fairly obvious choices for D . It seems clear however that for certain curves having both long and thin elements and acute angles, the results of Wentz and Steffen are more useful.

6. Non-existence theorems. Let H be a constant ≥ 0 . In this section we show that for $H(x) \equiv H$, our existence results of the previous section are best possible in many interesting cases. The discussion that follows uses results of the authors' paper [8].

We begin with the minimal surface case $H = 0$.

Theorem 6.1. *Let Ω be the closure of a region in E^3 with $\partial\Omega$ of class C^2 , and having the following property: for every C^2 Jordan curve $\Gamma \subset \Omega$ of sufficiently small diameter, there is a mapping $y : \bar{B} \rightarrow \Omega$, $y \in C^2(B) \cap C^0(\bar{B})$, $y : \partial B \rightarrow \Gamma$ a homeomorphism, satisfying (1.1) and (1.2) with $H(y) \equiv 0$. Then the inward mean curvature of $\partial\Omega$ at each point is ≥ 0 .*

Proof. Suppose at some point $p \in \partial\Omega$ the inward mean curvature of $\partial\Omega < 0$. Introduce coordinates (z^1, z^2, z^3) with e_3 the interior normal to $\partial\Omega$ at p . There is a neighborhood U of p on $\partial\Omega$ which may be represented non-parametrically by $z^3 = \varphi(z^1, z^2)$ and such that the inward mean curvature of $\partial\Omega < 0$ in U . Let Γ be the intersection of U with a circular cylinder whose radius is chosen small enough so that the diameter of Γ satisfies the hypotheses of the theorem.

By hypothesis there exists a mapping $y : \bar{B} \rightarrow \Omega$ satisfying (1.1) and (1.2) and mapping ∂B onto Γ homeomorphically. According to a theorem of Radó [22] (see also [8]) $y(B)$ is the graph of a C^2 function $\psi(z^1, z^2)$. We introduce the minimal surface operator

$$Lu = (1 + u_{z^1}^2)u_{z^2 z^2} - 2u_{z^1}u_{z^2}u_{z^1 z^2} + (1 + u_{z^2}^2)u_{z^1 z^1}.$$

Then if D is the region in the (z^1, z^2) plane bounded by the projection of Γ , we have

$$\begin{aligned} L\psi &= 0 \quad \text{and} \quad L\varphi < 0 \quad \text{in } D, \\ \psi &= \varphi \quad \quad \quad \text{on } \partial D. \end{aligned}$$

The maximum principle now implies (see [20]) that $\psi < \varphi$ in D . In particular $y(B) \subset \Omega$. Q.E.D.

Theorem 6.2. *Let a constant $H > 0$ be given and let Ω be the closure of a convex region in E^3 with $\partial\Omega$ of class C^2 , and having the following property: for every C^2 oriented Jordan curve $\Gamma \subset \Omega$ of sufficiently small diameter, there is a mapping $y : \bar{B} \rightarrow \Omega$, $y \in C^2(B) \cap C^0(\bar{B})$, $y : \partial B \rightarrow \Gamma$ homeomorphically with degree 1 satisfying (1.1) and (1.2) with $H(y) \equiv H > 0$. Then the inward mean curvature of $\partial\Omega$ at each point is $\geq H$.*

Proof. Suppose at some point $p \in \partial\Omega$ the inward mean curvature of $\partial\Omega < H$. Introduce coordinates (z^1, z^2, z^3) with e_3 the interior normal to $\partial\Omega$ at p . There is a neighborhood U of p on $\partial\Omega$ which may be represented non-parametrically

by $z^3 = \varphi(z^1, z^2)$, and such that the inward mean curvature of $\partial\Omega < H$ in U . Let Γ be the intersection of U with a circular cylinder of small radius $< 1/2H$ (as in the proof of Theorem 6.2), with the orientation inherited from the positive orientation of its projection γ in the (z^1, z^2) plane. Then there is a surface $x: \bar{B} \rightarrow E^3$, $x \in C^2(B) \cap C^1(\bar{B})$ satisfying (1.1) and (1.2) with $H(x) \equiv H > 0$, mapping ∂B diffeomorphically with degree 1 onto Γ , such that $x(B)$ is the graph of a function $\psi(z^1, z^2)$ defined on the interior of γ ; this was shown in [8]. Applying E. Hopf's boundary point lemma [20] $d\varphi/ds > d\psi/ds$ at all points of γ , where s represents distance along a ray into the interior of γ . Denote by V the unit vector field along Γ which is tangent to $\partial\Omega$ and which points toward the interior of Γ (i.e., toward that part of U bounded entirely by Γ). Let V_x be the unit vector field $\partial x/\partial n$ along Γ , where n represents the interior normal direction to ∂B . Then since $d\varphi/ds > d\psi/ds$ we have $e_3 \cdot V > e_3 \cdot V_x$.

Now let $y: \bar{B} \rightarrow E^3$ be any mapping satisfying (1.1) and (1.2) with $H(y) \equiv H > 0$ and mapping ∂B onto Γ with the correct orientation. Denote $V_y = \partial y/\partial n$. Then by a result of Heinz [10] we have

$$\int_{\Gamma} V_y \, ds = H \int_{\Gamma} y \wedge dy = H \int_{\Gamma} x \wedge dx = \int_{\Gamma} V_x \, ds.$$

The middle inequality holds since y and x give the same orientation to Γ . Hence

$$\int_{\Gamma} V_y \, ds > \int_{\Gamma} V_x \cdot e_3 \, ds = \int_{\Gamma} V_y \cdot e_3 \, ds.$$

Therefore $V \cdot e_3 > V_y \cdot e_3$ at some point q of Γ . Since Ω is convex, $V \cdot e_3 \leq 0$ and since V and V_y are normal to Γ , this implies that $V_y(q)$ points toward the exterior of Ω . In particular $y(B) \not\subset \Omega$. Q.E.D.

Theorem 6.3. *Let a constant $H > 0$ be given and let Ω be the closure of a region in E^3 with $\partial\Omega$ of class C^2 and having the following property: for every C^2 oriented Jordan curve $\Gamma \subset \Omega$ of sufficiently small diameter, there is a mapping $y: \bar{B} \rightarrow \Omega$, $y \in C^2(B) \cap C^0(\bar{B})$, $y: \partial B \rightarrow \Gamma$ homeomorphically with degree 1 satisfying (1.1) and (1.2) with $H(y) \equiv H > 0$. Suppose Ω is contained in the closure of a domain D , where ∂D is of class C^2 and has inward mean curvature $\geq H$. Then the inward mean curvature of ∂D at each point is $\geq H$.*

Proof. Suppose at some point $p \in \partial\Omega$ the inward mean curvature of $\partial\Omega < H$. Introduce coordinates (z^1, z^2, z^3) with e_3 the interior normal to $\partial\Omega$ at p . There is a neighborhood U of p on $\partial\Omega$ which may be represented non-parametrically by $z^3 = \varphi(z^1, z^2)$ and such that the inward mean curvature of $\partial\Omega < H$ in U . Let L be the operator

$$\begin{aligned} Lu &= (1 + u_x^2)u_{x^1x^1} - 2u_x u_{x^1x^2} + (1 + u_x^2)u_{x^2x^2} \\ &\quad - 2H(1 + u_x^2 + u_{x^2}^2)^{3/2}. \end{aligned}$$

Observe that p does not lie on ∂D . For if $p \in \partial D$ we can represent a neighborhood

of p on ∂D non-parametrically as $z^3 = f(z^1, z^2)$ where f satisfies $Lf \geq 0$. Then $\varphi - f$ has an interior minimum at p . But $L\varphi < 0$ contradicting the maximum principle.

Let Γ be the intersection of U with a circular cylinder $Z_\epsilon(p)$ of radius ϵ , with its orientation inherited from the positive orientation of its projection γ in the (z^1, z^2) plane. For ϵ small enough, Γ is contained in a closed ball $C_{2\epsilon}(p)$ of radius 2ϵ centered at p , with $C_{2\epsilon}(p)$ strictly interior to D .

By hypothesis there exists a mapping $y : \bar{B} \rightarrow \Omega$ satisfying (1.1) and (1.2) with $H(y) \equiv H$ and mapping ∂B homeomorphically onto Γ with the correct orientation. We claim that $y(B)$ is contained in $C_{2\epsilon}(p)$. To prove this, let $R = \sup \{r : C_r(p) \subset D\}$. Then it may be readily seen that $R \leq 1/H$ by applying a contact argument between $\partial C_R(p)$ and ∂D . If $y(B) \not\subset C_R(p)$ let $y(w_0)$, $w_0 \in B$ be the point on $y(B)$ farthest from p . There is a rotation S of E^3 leaving p fixed such that $S \circ y(w_0) \notin \bar{D}$. Otherwise D would contain a ball of radius $|y(w_0) - p| > R$, contradicting the definition of R . S is an element of a one-parameter group of rotations S_t for $t \leq 0$, leaving p fixed with S_0 the identity. Observe that $S_t \circ y(\partial B)$ is strictly interior to D . There is a smallest value $t_0 > 0$ such that $S_{t_0} \circ y(\bar{B}) \subset D$; hence $S_{t_0} \circ y(B)$ has an interior contact with ∂D . By the proof of Theorem 2.5 of [8] this is impossible. Hence, $y(\bar{B}) \subset C_R(p)$.

Now if $y(B)$ is not contained in $C_{2\epsilon}(p)$, then $R_0 = \sup |y(w) - p| \leq 1/H$ is assumed at an interior point $w_0 \in B$. This contact of $y(B)$ with $C_{R_0}(p)$ violates the maximum principles [8; Theorem 2.5]. (This is essentially the reasoning of Serrin [24, Theorem 1].)

Therefore $y(B)$ is contained in $C_{2\epsilon}(p)$ and in particular is contained in $Z_{2\epsilon}(p)$. Since $y(\partial B) \subset Z_\epsilon(p)$ it follows easily that $y(B) \subset Z_\epsilon(p)$ if $\epsilon < 1/4H$ [8; Lemma 4.1]. This being the case, $y(B)$ is the graph of a C^2 function $\psi(z^1, z^2)$ which represents a non-parametric surface of constant mean curvature H . This was shown by the authors in [8]. Therefore if D is the region in the (z^1, z^2) plane bounded by γ , we have

$$L\psi = 0 \quad \text{and} \quad L\varphi < 0 \quad \text{in} \quad D,$$

$$\psi = \varphi \quad \text{on} \quad \partial D.$$

The maximum principle implies that $\psi < \varphi$ in D . Thus $y(B) \subsetneq \Omega$. Q.E.D.

Appendix. In this section we prove an existence theorem for non-parametric surfaces of prescribed mean curvature in unbounded domains. As a corollary we obtain quite general sufficient conditions for a domain to be admissible. The basic tools for the proof are due to Serrin [23].

Let Ω be a domain in E^n and suppose $\partial\Omega$ is of class C^2 . For points $y \in \partial\Omega$ let $k_1(y) \geq k_2(y) \geq \dots \geq k_{n-1}(y)$ be the principle curvatures of $\partial\Omega$ with respect to the inward normal at y . For points $x \in \Omega$ let $r(x)$ denote the distance from x to $\partial\Omega$. We define Ω_1 to be the open subset of points $x \in \Omega$ such that

- (a) the closed ball of radius $r(x)$ centered at x touches $\partial\Omega$ at a single point $y = y(x)$ and
 (b) $r(x) < 1/k_1(y)$.

For $x \in \bar{\Omega}$ define

$$\mathcal{H}(x) = \begin{cases} \frac{1}{n-1} \sum k_i(y)/(1 - k_i(y)r(x)), & x \in \Omega_1 \cup \partial\Omega, \\ +\infty & \text{otherwise.} \end{cases}$$

Set $r_1 = \sup_{\Omega} r(x)$.

Theorem A.1. Let $\Omega \subset E^n$ be a domain with C^2 boundary, and suppose the principal curvatures of $\partial\Omega$ are uniformly bounded. Let $\Lambda(x)$ be a bounded function of class C^1 in $\bar{\Omega}$ and suppose there is a constant a , $0 \leq a \leq 1/r_1$, such that

$$\frac{n}{n-1} |\Lambda(x)| \leq (1 - ar(x))\mathcal{H}(x) + \frac{a}{n-1} \quad \text{in } \Omega,$$

$$\mathcal{H}(y) \geq \frac{n}{n-1} |\Lambda(y)| \quad \text{for all } y \in \partial\Omega.$$

Then the Dirichlet problem for the equation

$$* \quad (1 + |\nabla u|^2)\Delta u - u_i u_i u_{ii} = n\Lambda(x)(1 + |\nabla u|^2)^{3/2}$$

is solvable for arbitrary continuous (bounded) boundary data.

For Ω bounded and C^2 boundary data, this is precisely the theorem of Serrin [23, p. 481]. The extension to unbounded domains is via the Perron process. According to the results of Ladyzhenskaya and Ural'tseva [17] (see also Trudinger [27]) solutions of $*$ satisfy an interior gradient bound. That is, at any point $x \in \Omega$, the gradient of a solution at x can be bounded in terms of $r(x)$, $\Lambda(x)$ and a bound for the solution in a ball of radius $r(x)$ centered at x . It follows in a well known way that bounded families of solutions of $*$ are compact. Using this fact and Serrin's theorem, the Dirichlet problem for $*$ is solvable in any closed ball C_R of radius $R \leq (n-1)/(n|\Lambda|_0)$ contained in Ω , for arbitrary continuous boundary data. Thus given a continuous function v in Ω and a sufficiently small closed ball $C \subset \Omega$, there exists a unique solution $u(v, C)$ of $*$ in C which coincides with v on ∂C . Let $M_c[v]$ denote the continuous function which is equal to $u(v, C)$ in C and agrees with v in $\Omega \setminus C$.

Definition. v is said to be a subsolution (supersolution) of $*$ if for every sufficiently small ball $C \subset \Omega$, $v \leq M_c[v]$ ($v \geq M_c[v]$).

Lemma A.1. (Serrin [23, p. 481]). For $0 < a < 1/r$, set $v(x) = 1/a(2ar - a^2 r^2)^{1/2}$ and for $a = 0$ set $v(x) = c^{-1}e^{cr}(1 - e^{-cr})$ where $c = n \sup |\Lambda(x)| + 1$. Then $v(x)$ is a supersolution of $*$ and $-v(x)$ is a subsolution of $*$.

Now let f denote the assigned continuous (bounded) boundary data. If v is a subsolution in Ω and $v \leq f$ on $\partial\Omega$ we say v is a subfunction. Similarly if v is

a supersolution in Ω and $v \geq f$ on $\partial\Omega$, v is said to be a *superfunction*. Since adding constants preserves the class of subsolutions and the class of supersolutions, the above lemma shows the set of all subfunctions and the set of all superfunctions is nonempty. Let $\varphi(x)$ be a fixed bounded superfunction. We define $\mathfrak{F}$ as the class of all subfunctions v in Ω satisfying $v \leq \varphi$. Hence the function

$$u(x) = \text{lub } v(x), \quad v \in \mathfrak{F},$$

is well defined in Ω . Using the compactness principle stated earlier and the maximum principle, it follows (see [3]) that $u(x)$ is a solution of $*$.

We still must show that u takes on the boundary values f . This requires the construction of local barrier functions at each point Q of $\partial\Omega$. That is, for each pair of positive numbers M and δ , it is sufficient to find two neighborhoods N^+ and N^- of Q and two continuous functions w^+ and w^- defined in $N^+ \cap \Omega$ and $N^- \cap \Omega$ respectively, such that

- (a) N^+ and N^- are contained in a ball of radius δ about Q .
- (b) w^+ is a positive supersolution and w^- a negative subsolution.
- (c) $w^+ \geq M$ on $\partial N^+ \cap \Omega$ and $w^- \leq -M$ on $\partial N^- \cap \Omega$.
- (d) $w^+ = w^- = 0$ at Q .

The construction we give is essentially that of Jenkins and Serrin [16] for the minimal surface case. Set

$$Lw = (1 + |\nabla w|^2)\Delta w - w_i w_j w_{ij} - n\Lambda(x)(1 + |\nabla w|^2)^{3/2}$$

where we have used the usual summation convention.

In a neighborhood of the point $Q \in \partial\Omega$ we may represent $\partial\Omega$ non-parametrically in the form $x_n = \varphi(x_1, \dots, x_{n-1}) \in C^2$ where the x_n axis is along the inner normal to $\partial\Omega$ at Q . Let us put $r = (x_1^2 + \dots + x_{n-1}^2)^{1/2}$ and suppose $\varphi(x_1, \dots, x_{n-1})$ is defined for $r < a$. We may assume $|\nabla \varphi| < 1$. Now let α, β, c be positive constants such that

$$\alpha < \beta, \quad c < \pi/2\beta, \quad \pi/2\alpha \leq a.$$

Let D be the domain in E^n determined by the inequalities

$$\varphi + c - \pi/2\beta < x_n < \varphi + c - \alpha/\beta r, \quad r < \pi/2\alpha.$$

In D , define the functions

$$\begin{aligned} s &= \sec \alpha r, & t &= \tan \alpha r, \\ S &= \sec \beta(\varphi + c - x_n), & T &= \tan \beta(\varphi + c - x_n). \end{aligned}$$

Let the principal curvatures of $\partial\Omega$ be bounded by K , and denote $\hat{x} = (x_1, \dots, x_{n-1}, \varphi(x_1, \dots, x_{n-1})), \nu = (x_1/r, \dots, x_{n-1}/r)$.

Lemma A.2. *The function $w(x) = (1/\alpha) \log s - (1/\beta) \log S$, $x \in D$, satisfies*

$$\begin{aligned} Lw &\leq -s^2 S^2 \{ \beta - 2(n + (n+2)^{1/2})K - (3n-5)\alpha \} \\ &\quad - (n-1)(1 + |\nabla \varphi|^2)^{3/2} \mathfrak{C}(\hat{x}) T^3 - n\Lambda(x) \{ T^2(1 + |\nabla \varphi|^2) + s^2 - 2tT\nu \cdot \nabla \varphi \}^{3/2}. \end{aligned}$$

The proof is by direct calculation; the details are carried out in [16, p. 174].

Lemma A.3. *Let $h(u) = u \tan(\beta c - \beta u)$ where $0 < u < c$ and $0 < \beta c < \pi/2$. Then $h(u) < 1/2\beta$.*

Proof. $h(0) = h(c) = 0$ so that h has an interior maximum. $h'(u) = \tan(\beta c - \beta u) - \beta u \sec^2(\beta c - \beta u)$ so that at the maximum point $2\beta u = \sin 2(\beta c - \beta u)$. A straightforward calculation shows that $h(u) = (1 - (1 - 4\beta^2 u^2)^{1/2})/2\beta$ at this point. Q.E.D.

Recalling the definition of D , we see that $\Omega \cap D$ is determined by the inequality

$$0 < x_n - \varphi < c - (\alpha/\beta)r.$$

Since Λ is of class C^1 in $\bar{\Omega}$ there is a constant L so that $|\Lambda(x) - \Lambda(\hat{x})| \leq L(x_n - \varphi)$ in $\Omega \cap D$. By hypothesis $\mathcal{H}(\hat{x}) \geq (n/n-1)|\Lambda(\hat{x})|$. It now follows in a straightforward way from Lemmas A.2 and A.3 (with $u = x_n - \varphi$ in Lemma A.2) that

$$Lw \leq -s^2 S^2 \{\beta - c_1 \alpha - c_2 - c_3/\beta\}$$

where c_1, c_2, c_3 are constants depending only on n, L , and K . Therefore choosing $\beta = (1 + c_1)\alpha$ with α large gives $Lw \leq 0$ in $\Omega \cap D$, i.e., w is a supersolution in $\Omega \cap D$. If we replace $\Lambda(x)$ by $-\Lambda(x)$, exactly the same calculations show that $-w$ is a subsolution in $\Omega \cap D$.

Now set

$$\begin{aligned} w^+ &= M + w(x), & N^+ &= \Omega \cap D, \\ w^- &= -w^+, & N^- &= \Omega \cap D. \end{aligned}$$

In order that $w^+(Q) = w^-(Q) = 0$, we must have

$$0 = M - (1/\beta) \log \sec \beta c,$$

that is,

$$c = (1/\beta) \sec^{-1}(e^{\beta M})$$

where the inverse secant is taken in the first quadrant so that $\beta c < \pi/2$. For α sufficiently large (depending on δ) the functions w^+, w^- are the required local barriers.

The proof that our Perron solution of $*$ takes on the boundary values f requires only a slight modification of the usual argument, and we shall omit the details.

It remains to treat the case $a = 1/r_1$. To do so let $\{\lambda_k\}$ be a monotone increasing sequence with $\lambda_k < 1$, $\lim \lambda_k = 1$. Set $\Lambda_k = \lambda_k \Lambda$. Then for Λ_k , the hypotheses of the theorem are satisfied for $a_k = \lambda_k a < 1/r_1$.

Therefore for each k , there is a solution with the given boundary data. Using the compactness principle and the Schauder theory, it follows that the cor-

responding solutions converge to a solution of $*$ for $\Lambda(x)$. The same barrier argument shows that this solution takes on the required boundary values. Q.E.D.

Remark. Serrin's proof of Lemma A.1 assumes that $r(x)$ is of class C^2 in Ω_1 . In our discussion we can only assert $r(x) \in C^1$ in Ω_1 . However since $*$ is equivalent to a divergence structure equation, we can interpret Serrin's argument in the weak sense and the lemma is valid.

For application to the Plateau problem let us take $n = 3$ and $\Lambda(x) = \frac{2}{3}H(x)$. The solvability conditions then become

$$** \quad \begin{cases} \text{There exists a constant } a, 0 \leq a \leq 1/r_1 \text{ such that} \\ |H(x)| \leq (1 - ar(x))\mathcal{H}(x) + a/2 \quad \text{in } \Omega, \\ \mathcal{H}(y) \geq |H(y)| \quad \text{for all } y \in \partial\Omega. \end{cases}$$

Suppose $\Omega \subset E^3$ has a C^2 boundary with bounded principle curvatures and satisfies $**$.

We now solve the Dirichlet problem for $*$ with zero boundary data. That is, there exists a C^2 function $\varphi(x)$ in Ω , satisfying

$$\operatorname{div} \frac{\nabla \varphi}{(1 + |\nabla \varphi|^2)^{1/2}} = 2H(x) \quad \text{in } \Omega$$

with $\varphi = 0$ on $\partial\Omega$. We now set

$$Q(x) = \frac{2\nabla \varphi}{(1 + |\nabla \varphi|^2)^{1/2}}.$$

Then $\operatorname{div} Q = 4H(x)$.

Lemma A.4. Suppose ∂D has globally defined inward parallel surfaces at distance ϵ , for ϵ sufficiently small. Then there exists a constant $\alpha < 1$ depending on the geometry of Ω and a uniform Lipschitz bound for $H(x)$ such that

$$\sup_{\Omega} |Q(x)| \leq 2\alpha < 2.$$

Proof. According to the proof of Theorem A.1, φ is bounded. For a point $P \in \Omega$ we denote by $C_r(P)$, a closed ball of radius $r(P)$ centered at P . The interior gradient estimates of [17] say that there are absolute constants C_1 and C_2 (depending on $H(x)$) such that

$$|\nabla \varphi(P)| \leq C_1 \exp(C_2 \sup_{C_r(P)} |\varphi|/r(P)).$$

It is therefore sufficient to estimate $|\nabla \varphi|$ at points near $\partial\Omega$. As in Section 3, there is a (uniform) neighborhood V of $\partial\Omega$ such that for each $P \in V$, there is a unique point $\pi(P) \in \partial\Omega$ with $|P - \pi(P)| = r(P)$. For $P \in V$ let $\Delta(P) = \{Q \in \Omega : |Q - \pi(P)| \leq 3r(P)\}$. We suppose $r(P)$ is small enough so that D

is contained in $N^+ \cap \Omega = N^- \cap \Omega$, the neighborhoods of $\pi(P)$ in which our barrier functions were constructed. We note that because of our assumptions on the geometry of Ω each point in $\partial\Omega$ has a uniform neighborhood in which our barrier constructions are valid. Let

$$M = \sup_{\Delta(P)} |\varphi| \geq \sup_{C^+(P)} |\varphi|.$$

Then

$$M = \sup_{Q \in \Delta(P)} |\varphi(Q) - \varphi(\pi(P))| \leq 3r(P) \sup_{Q \in \Delta(P)} \frac{|\varphi(Q) - \varphi(\pi(P))|}{|Q - \pi(P)|}.$$

But $w^- \leq \varphi \leq w^+$ in $\Delta(P)$ and

$$w^-(\pi(P)) = \varphi(\pi(P)) = w^+(\pi(P)).$$

Therefore there exists a constant C such that

$$\sup_{Q \in \Delta(P)} \frac{|\varphi(Q) - \varphi(\pi(P))|}{|Q - \pi(P)|} \leq C.$$

Hence $M \leq 3Cr(P)$ which gives

$$|\nabla\varphi(P)| \leq C_1 \exp 3C_2C. \quad \text{Q.E.D.}$$

From the lemma we see that φ is Lipschitz continuous in $\bar{\Omega}$. It now follows from standard theory that $\varphi \in C^2(\bar{\Omega})$. In particular Q is a C^1 vector field in $\bar{\Omega}$.

We summarize this as

Theorem A.2. *Let D be a domain in E^3 such that ∂D is of class C^2 with uniformly bounded principal curvatures. Assume further that for ϵ sufficiently small, ∂D has globally defined inward parallel surfaces at distance ϵ . Given $H(x) : \bar{D} \rightarrow \mathbf{R}$ of class C^1 suppose there exists a constant a , $0 \leq a \leq 1/r_1$ such that*

$$|H(x)| \leq (1 - ar(x))\mathcal{H}(x) + a/2 \quad \text{in } D,$$

$$\mathcal{H}(y) \geq |H(y)| \quad \text{for all } y \in \partial D.$$

Then there exists a C^1 vector field Q in $\bar{D}$ satisfying (2.1) and (2.2).

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