Boundary Behavior of Non-parametric Minimal Surfaces

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1. It is well known that the construction of barriers is an essential step in the proof of the existence and regularity of the solutions to the Dirichlet problem for nonlinear, non-uniformly elliptic equations.

In particular, Jenkins and Serrin ([3]) have considered the equation of non-parametric minimal surfaces in an open set Ω in $\mathbf{R}^n$, and using the barrier technique have proved that there exists a unique solution to the minimal surface equation in Ω , which takes prescribed values $\varphi(x)$ on the boundary of Ω , provided $\partial\Omega$ is a C^3 -hypersurface with non-negative mean curvature, and the function φ is twice-continuously differentiable. This solution is twice-continuously differentiable in all of $\bar{\Omega}$. Also, if $\partial\Omega$ and the boundary data are more regular, say $C^{k+\alpha}$ with $k \geq 2$, then the solution is $C^{k+\alpha}$ in all of $\bar{\Omega}$. On the other hand, by means of an approximation technique (see [4]), one can show the existence and the uniqueness of a continuous non-parametric minimal surface, provided the function $\varphi(x)$ is continuous and the condition on the mean curvature of $\partial\Omega$ is satisfied.

The *a priori* gradient estimate of Bombieri, De Giorgi and M. Miranda ([1]) implies that this solution is analytic in Ω , and hence is a solution to the Dirichlet problem for the equation of minimal surfaces with continuous boundary data.

The aim of this paper is to study the global regularity (*i.e.*, regularity up to the boundary) of minimal surfaces in the intermediate case in which the function $\varphi(x)$ is more than merely continuous, but is not twice differentiable. We get the following results:

- a) If φ belongs to $C^{1+\alpha}$, $0 < \alpha \leq 1$, then the solution $u(x)$ is Lipschitz-continuous up to the boundary.
- b) If φ is Lipschitz-continuous, then $u(x)$ is Hölder-continuous in $\bar{\Omega}$, with some positive exponent σ .

The statement (b) can be improved if we make an additional assumption on $\partial\Omega$. In fact, we have:

- c) If the mean curvature of $\partial\Omega$ is strictly positive, and if φ belongs to $C^{0+\alpha}$, $0 < \alpha \leq 1$, then $u(x)$ is Hölder-continuous in $\bar{\Omega}$, with exponent $\alpha/2$.

The same results hold for the solutions to the Plateau problem with obstacles; the precise statement is given in Theorem 4.

The core of the proof consists of the construction of barriers in the various cases. This is done in Sections 2–5. In Section 6, we derive the announced results.

2. Let Ω be a bounded open set in $\mathbf{R}^n$, with boundary $\partial\Omega$ of class C^3 , and let $d(x)$ be the distance from the boundary:

$$(2.1) \quad d(x) = \text{dist}(x, \partial\Omega).$$

For $t > 0$, let

$$\begin{aligned} \Omega_t &= \{x \in \Omega : d(x) < t\}, \\ \Gamma_t &= \{x \in \Omega : d(x) = t\}. \end{aligned}$$

It is well known (see [5], Lemma 2) that there exists a positive number ϵ_0 such that $d(x)$ is twice-continuously differentiable in Ω_{ϵ_0} , and that, for every point $x \in \Omega_{\epsilon_0}$, there exists a unique point $y \in \partial\Omega$ with the property $d(x) = |x - y|$. In addition, we have $|d_x| \equiv 1$ in Ω_{ϵ_0} so that, if $d(x_0) = t$, the sum of the principal curvatures of the surface Γ_t at x_0 is given by

$$(2.2) \quad \varepsilon_0(d) \equiv -\frac{\Delta d}{|d_x|} + \frac{d_{x_i} d_{x_i}}{|d_x|^3} d_{x_i x_i} = -\Delta d(x_0)$$

(here and in the following, summation on repeated indices is understood).

If y_0 is the point of $\partial\Omega$ corresponding to x_0 , and if $k_i(y_0)$ are the principal curvatures of $\partial\Omega$ at y_0 , we have

$$(2.3) \quad -\Delta d(x_0) = \sum_{i=1}^{n-1} \frac{k_i(y_0)}{1 - k_i(y_0) d(x_0)} \geq \sum_{i=1}^{n-1} k_i(y_0).$$

Let $\varphi(x)$ be a continuous function defined on $\partial\Omega$, and let $K = \max_{\partial\Omega} |\varphi|$.

A lower barrier (relative to φ) is a function $\lambda(x)$, defined in some $\bar{\Omega}_\epsilon$, $\epsilon > 0$, such that

$$(\lambda_1) \quad \lambda \in C^2(\Omega_\epsilon) \cap C^0(\bar{\Omega}_\epsilon),$$

$$(\lambda_2) \quad \lambda = \varphi \quad \text{in} \quad \partial\Omega; \quad \lambda \leq -K - 1 \quad \text{in} \quad \Gamma_\epsilon,$$

$$(\lambda_3) \quad \varepsilon(\lambda) \equiv (1 + |\lambda_x|^2) \Delta \lambda - \lambda_{x_i} \lambda_{x_i} \lambda_{x_i x_i} \geq 0 \quad \text{in} \quad \Omega_\epsilon.$$

Since we want also to consider problems with obstacles, let us introduce a continuous function $\psi(x)$, defined in $\bar{\Omega}$, and such that $\psi(x) \leq \varphi(x)$ in $\partial\Omega$. Let

$$K_1 = \max \{K, \max_{\bar{\Omega}} \psi\}.$$

An upper barrier (relative to φ and ψ) is a function $\mu(x) : \bar{\Omega}_\epsilon \rightarrow \mathbf{R}$ such that

$$\begin{aligned}
(\mu_1) \quad & \mu \in C^2(\Omega_\epsilon) \cap C^0(\bar{\Omega}_\epsilon), \\
(\mu_2) \quad & \mu = \varphi \text{ in } \partial\Omega; \quad \mu \geq K_1 + 1 \text{ in } \Gamma_\epsilon; \quad \mu \geq \psi \text{ in } \Omega_\epsilon, \\
(\mu_3) \quad & \varepsilon(\mu) \leq 0 \text{ in } \Omega_\epsilon.
\end{aligned}$$

In the following, we will show how to construct an upper barrier; the construction of a lower barrier can be made in the same way.

We will suppose that the function $\varphi(x)$ is of class $C^{k+\alpha}$, $k = 0, 1$ and $0 < \alpha \leq 1$. It is clearly possible to extend φ to a new function defined in all of $\mathbf{R}^n$ and with compact support, having the same regularity of φ (i.e., of class $C^{k+\alpha}(\mathbf{R}^n)$). Such a function will be denoted again by $\varphi(x)$. Let $\eta(x)$ be an infinitely differentiable function with support in the unit ball, and such that $\int \eta(x) dx = 1$. For $t > 0$, let

$$(2.4) \quad w(x, t) = t^{-n} \int \eta\left(\frac{y-x}{t}\right) \varphi(y) dy = \int \eta(z) \varphi(x + tz) dz,$$

and let $w(x, 0) = \varphi(x)$.

Lemma 1. *If $\varphi(x)$ belongs to $C^{0+\alpha}(\mathbf{R}^n)$, we have:*

$$(2.5) \quad \begin{cases} |w(x, t)| \leq c_1, \\ |w_i|, |w_0| \leq c_1 t^{\alpha-1}, \\ |w_{ij}|, |w_{i0}|, |w_{00}| \leq c_1 t^{\alpha-2}; \end{cases}$$

and if $\varphi \in C^{1+\alpha}(\mathbf{R}^n)$:

$$(2.6) \quad \begin{cases} |w|, |w_i|, |w_0| \leq c_1, \\ |w_{ij}|, |w_{i0}|, |w_{00}| \leq c_1 t^{\alpha-1}, \end{cases}$$

where $w_i = \partial w / \partial x_i$, $w_0 = \partial w / \partial t$, etc., and c_1 is a constant depending on φ and η .

Proof. It is sufficient to show how to get one of these inequalities, the method being always the same. Let $\varphi \in C^{0+\alpha}$. We have

$$w_i = -t^{-1-n} \int \eta_i\left(\frac{y-x}{t}\right) \varphi(y) dy = -t^{-1} \int \eta_i(z) \varphi(x + tz) dz.$$

On the other hand,

$$\int \eta_i(z) dz = 0,$$

and therefore

$$w_i = t^{\alpha-1} \int \eta_i(z) \frac{\varphi(x) - \varphi(x + tz)}{t^\alpha} dz.$$

Remembering that η has support in the unit ball, we easily get $|w_i| \leq c_1 t^{\alpha-1}$.
Q.E.D.

We will consider barriers of the type

$$(2.7) \quad \mu(x) = w(x, f(d)) + L(f(d)),$$

where $d = d(x)$ and $f(s)$, $L(s)$ are defined for positive s and verify $f(0) = L(0) = 0$.

We have:

$$\begin{aligned} \varepsilon(\mu) &= [(w_{00} + L')f'^2 + (w_0 + L')f'']\{1 + |w_x|^2 - (w_i d_i)^2\} \\ &\quad + (1 + |w_x|^2) \Delta w - w_i w_i w_{i,i} + 2f'\{(1 + |w_x|^2)w_{i0} d_i - w_{i0} w_i w_{i,i}\} \\ &\quad + 2f''(w_0 + L')\{w_{i0} d_i w_{i,i} - w_{i0} w_i\} \\ &\quad + f'(w_0 + L')\{2 \Delta w w_{i,i} d_i - 2w_{i,i} w_i d_i + (1 + |w_x|^2) \Delta d - d_{i,i} w_i w_{i,i}\} \\ &\quad + f'^2(w_0 + L')^2 \{\Delta w - w_{i,i} d_i d_i + 2 \Delta d w_{i,i} d_i\} \\ &\quad + f'^3(w_0 + L')^3 \Delta d. \end{aligned}$$

3. In this section we will consider the case $\varphi \in C^{1+\alpha}$. We have the following theorem.

Theorem 1. *Let Ω be an open bounded set in $\mathbb{R}^n$, and let $\partial\Omega$ be a surface of class C^3 with non-negative mean curvature. Let $\varphi(x)$ be a function of class $C^{1+\alpha}$, and let ψ be a Lipschitz-continuous function in $\bar{\Omega}$, such that $\psi \leq \varphi$ in $\partial\Omega$. Then there exist $\epsilon > 0$ and an upper barrier $\mu(x)$ (relative to φ and ψ) which is Lipschitz-continuous in $\bar{\Omega}_\epsilon$.*

Proof. Let us observe first that, because of (2.3) and the hypothesis on the mean curvature of $\partial\Omega$, we have $\Delta d \leq 0$ in Ω_ϵ . Let ϵ , $0 < \epsilon < \epsilon_0$, be a positive number which will be fixed later. We will choose f and L , verifying the following conditions:

$$(f_1) \quad 0 \leq f(s) \leq 1 \quad \text{for } 0 \leq s \leq \epsilon; \quad f(0) = 0; \quad f(\epsilon) = 1,$$

$$(f_2) \quad f'(s) \geq 1 \quad \text{and} \quad f''(s) \leq 0 \quad \text{for } 0 < s < \epsilon,$$

$$(L_1) \quad L(0) = 0, \quad L(1) \geq K_1 + c_1 + 1,$$

$$(L_2) \quad L'(s) \geq c_1 + |\psi|_1 + |\varphi|_1 + 1, \quad L''(s) \leq 0 \quad \text{for } 0 < s < 1.$$

($|\psi|_1$ and $|\varphi|_1$ are the Lipschitz-constants of ψ and φ .) With these conditions, one easily verifies (μ_1) and (μ_2) . Let us show for instance that $\mu(x) \geq \psi(x)$ in Ω_ϵ . Let $y \in \partial\Omega$ be such that $d(x) = |x - y|$. We have

$$\begin{aligned} \mu(x) - \mu(y) &= w(x, f(d)) + L(f(d)) - w(y, 0) \\ &\geq L(f(d)) - |w(x, f(d)) - w(x, 0)| - |w(x, 0) - w(y, 0)| \\ &\geq (c_1 + |\psi|_1 + |\varphi|_1)f(d) - c_1 f(d) - |\varphi|_1 |x - y| \\ &\geq |\psi|_1 |x - y|. \end{aligned}$$

On the other hand,

$$|\psi(x) - \psi(y)| \leq |\psi|_1 |x - y|,$$

and therefore, since $\mu(y) = \varphi(y) \geq \psi(y)$, we get $\mu(x) \geq \psi(x)$.

In addition, taking into account the estimates (2.6), one easily gets from (2.8):

$$(3.1) \quad \varepsilon(\mu) \leq (L' - c_2)f'' + (L'' + c_2L'^2f^{\alpha-1})f'^2.$$

Let us choose

$$(3.2) \quad L(s) = 2Ms \left(1 - \frac{s^{\alpha/2}}{\alpha + 2}\right)$$

with

$$M = K_1 + 1 + 2c_1 + |\varphi|_1 + |\psi|_1.$$

In this way, the conditions (L_1) and (L_2) are satisfied, and we have

$$(3.3) \quad \varepsilon(\mu) \leq \frac{M}{2} \{f'' + (8Mc_2f^{\alpha/2} - \alpha)f^{\alpha/2-1}f'^2\}.$$

It is obvious that

$$\varepsilon(\mu) \leq 0 \quad \text{if} \quad f \leq a = \left(\frac{\alpha}{8Mc_2}\right)^{2/\alpha},$$

so that we must consider only the set where $f > a$. In this case, we have

$$\varepsilon(\mu) \leq \frac{M}{2} \{f'' + 8Mc_2a^{\alpha-1}f'^2\}.$$

If we put

$$f(s) = A \log (Bs + 1)$$

with

$$A = (8c_2Ma^{\alpha-1})^{-1},$$

we have

$$\varepsilon(\mu) \leq 0.$$

The conditions (f_1) and (f_2) will be satisfied if one chooses B in such a way that

$$(3.4) \quad \epsilon \equiv \frac{1}{B} (e^{1/A} - 1) \leq \epsilon_0$$

and

$$(3.5) \quad BAe^{-1/A} \geq 1.$$

This concludes the proof of Theorem 1.

Q.E.D.

4. We will consider now the case $\varphi \in C^{0+\alpha}$. This case can be handled if we assume a more stringent condition on $\partial\Omega$; namely, if we assume that the mean curvature of $\partial\Omega$ is strictly positive. Remembering the inequality (2.3), we have

$$(4.1) \quad \Delta d \leq -h_0 < 0 \quad \text{in } \Omega_\epsilon.$$

We shall consider obstacles ψ which are Hölder-continuous in $\bar{\Omega}$ with exponent $\alpha/2$. If $v(x)$ is a Hölder-continuous function in $\bar{\Omega}$, with exponent σ , we will put

$$|v|_\sigma = \sup_{\substack{x, y \in \bar{\Omega} \\ x \neq y}} \frac{|v(x) - v(y)|}{|x - y|^\sigma}.$$

Let us now choose the function $L(s)$:

$$(4.2) \quad L(s) = \frac{2}{\alpha} M s^\alpha$$

with

$$(4.3) \quad M = 1 + c_1 + K_1 + |\varphi|_\alpha + |\psi|_{\alpha/2},$$

and let us suppose that the function $f(s)$ verifies conditions (f_1) and (f_2) . Then (μ_1) and (μ_2) are satisfied, with the possible exception of the inequality $\mu(x) \geq \psi(x)$, which will be discussed later. In addition, taking in account the estimates (2.5) and the inequality (4.1), we get

$$(4.4) \quad \varepsilon(\mu) \geq M f'' f^{\alpha-1} + M^2 f'^2 f^{3\alpha-4} (c_3 - M h_0 f' f).$$

Therefore, if we put

$$(4.5) \quad f(s) = A s^{1/2},$$

we have $\varepsilon(\mu) \leq 0$, provided

$$(4.6) \quad A \geq (2c_3/Mh_0)^{1/2}.$$

It remains to verify the conditions (f_1) , (f_2) , and the inequality $\mu(x) \geq \psi(x)$. For (f_1) , it is sufficient to take $\epsilon = A^{-2}$, which is possible if

$$(4.7) \quad A \geq \epsilon_0^{-1/2}.$$

It is also clear that (f_2) is satisfied if $A \geq 2$.

Finally, we have

$$\mu(x) = w(x, A d^{1/2}) + \frac{2M}{\alpha} A^\alpha d^{\alpha/2}.$$

Let $x \in \Omega_\epsilon$, and let $y \in \partial\Omega$ be such that $d(x) = |x - y|$. We have

$$\begin{aligned} \mu(x) - \mu(y) &\geq M A^\alpha d^{\alpha/2} - c_1 A^\alpha d^{\alpha/2} - |\varphi|_\alpha |x - y|^\alpha \\ &\geq [(M - c_1) A^\alpha - |\varphi|_\alpha] |x - y|^{\alpha/2} \geq (M - c_1 - |\varphi|_\alpha) |x - y|^{\alpha/2}. \end{aligned}$$

Recalling that $\mu(y) \geq \psi(y)$, and that

$$|\psi(x) - \psi(y)| \leq |\psi|_{\alpha/2} |x - y|^{\alpha/2},$$

we get $\mu(x) \geq \dot{\psi}(x)$ from (4.3).

In this way, we have proved the following theorem.

Theorem 2. *Let Ω be an open bounded set in $\mathbf{R}^n$, and let $\partial\Omega$ be a C^3 -surface with strictly positive mean curvature. Let $\varphi(x)$ be a function of class $C^{0+\alpha}$, and let $\psi(x)$ be in $C^{0+\alpha/2}(\bar{\Omega})$, $\psi(x) \leq \varphi(x)$ in $\partial\Omega$. Then there exist $\epsilon > 0$ and an upper barrier $\mu(x)$ (relative to φ and ψ) which is Hölder-continuous, with exponent $\alpha/2$, in $\bar{\Omega}_\epsilon$.*

5. The hypothesis that the mean curvature of $\partial\Omega$ is strictly positive enters in an essential way in the proof of Theorem 2, and it does not seem easy to replace this assumption with the weaker hypothesis of non-negative mean curvature. Nevertheless, some result can also be obtained in this case, at least if the boundary datum φ is Lipschitz-continuous. More precisely, we have the following theorem.

Theorem 3. *Let Ω be a bounded open set in $\mathbf{R}^n$ such that $\partial\Omega$ is a C^3 -surface of non-negative mean curvature. Let φ be a Lipschitz-continuous function, and let ψ be Hölder-continuous with exponent β . Then there exist $\epsilon > 0$ and an upper barrier (relative to φ and ψ), which is Hölder-continuous in $\bar{\Omega}_\epsilon$, with some exponent $\sigma \leq \beta$.*

Proof. In (2.7) we choose

$$f(s) = s^\sigma; \quad L(s) = 2Ms$$

with $\sigma \leq \beta$, and

$$(5.1) \quad M \geq \frac{1}{2}(1 + c_1 + K_1 + |\varphi|_1 + |\psi|_\beta)\epsilon^{-\sigma}.$$

In that way, (μ_1) and (μ_2) are satisfied, and we have

$$(5.2) \quad \mathcal{E}(\mu) \leq -\sigma M d^{\sigma-2}(1 - c_4 \sigma M) + c_4 d^{-\sigma}.$$

If we put $M = T\epsilon^{-\sigma}$, we easily get

$$(5.3) \quad \mathcal{E}(\mu) \leq c_4^{-1} d^{\sigma-2} \{-c_4 T \sigma \epsilon^{-\sigma}(1 - c_4 T \sigma \epsilon^{-\sigma}) + c_4^2 \epsilon^{2-2\sigma}\}.$$

Let us suppose $\sigma \leq 1/2$, and let

$$(5.4) \quad \epsilon = \min \left\{ \epsilon_0, \frac{1}{8c_4^2} \right\}.$$

Let

$$q_0 = \sup \{c_4 \tau \epsilon^{-\tau}; \quad 0 < \tau \leq \min [1/2, \beta]\}.$$

We can choose $T \geq \frac{1}{2}(1 + c_1 + K_1 + |\varphi|_1 + |\psi|_\beta)$ in such a way that $q_0 \geq 1/2T$. It follows that there exists a σ , $0 < \sigma < \min (1/2, \beta)$ such that $c_4 T \sigma \epsilon^{-\sigma} = 1/2$. With this choice of σ , one easily gets $\mathcal{E}(\mu) \leq 0$. Q.E.D.

6. The barriers constructed in the preceding sections will now be used in order to prove the global regularity of the solution to the non-parametric Plateau problem; more precisely, to estimate the modulus of continuity of solutions to the Dirichlet problem for the equation of minimal surfaces. This is done in two steps: first, one reduces the interior estimates to boundary estimates, and then uses the barriers in order to get the required bounds near the boundary. The latter technique is standard, and has been widely used (see [5] and [6]). The first also is not new, although perhaps less known; the idea is essentially the same as in [4].

Let Ω be, as usual, a bounded open set whose boundary $\partial\Omega$ is a C^3 -surface with non-negative mean curvature, and let $\varphi: \partial\Omega \rightarrow \mathbf{R}$ and $\psi: \bar{\Omega} \rightarrow \mathbf{R}$ be two continuous functions such that $\psi \leq \varphi$ in $\partial\Omega$.

The Plateau problem with obstacle ψ and boundary data φ has a unique continuous solution. (This is proved in [2] under stricter assumptions on $\partial\Omega$. It is not difficult, however, to see that the result holds also in the present case.) We want to estimate the modulus of continuity of this solution. If $f(x)$ is a continuous function in $\bar{\Omega}$, for $r > 0$ let us define:

$$(6.1) \quad F(r) = \sup \{|f(x) - f(y)|; \ x, y \in \bar{\Omega}, \ |x - y| < r\}$$

and

$$(6.2) \quad F^*(r) = \sup \{|f(x) - f(y)|; \ x \in \bar{\Omega}, \ y \in \partial\Omega, \ |x - y| < r\}.$$

We have the following

Lemma 2. *Let Ω , φ , and ψ be as described, and let $u(x)$ be the solution to the Plateau problem relative to φ and ψ . Then, for every $r > 0$, we have*

$$(6.3) \quad U(r) \leq \max \{\Psi(r), U^*(r)\}.$$

Proof. Suppose first that $\varphi \in C^2(\partial\Omega)$, and ψ is Lipschitz-continuous, so that $u(x)$ is Lipschitz-continuous. For $x, x' \in \Omega$, let $\tau = x' - x$ and

$$\Omega_\tau = \{z \in \mathbf{R}^n : z - \tau \in \Omega\}.$$

If $f(x)$ is a function defined in Ω , let $f_\tau(x)$ be the function defined by $f_\tau(z) = f(z - \tau)$.

As in [2], Prop. 1.2, one can prove that there exist $y \in \partial(\Omega \cap \Omega_\tau)$ and $y' \in \Omega \cap \Omega_\tau$ such that

$$(6.4) \quad |u(x') - u_\tau(x')| \leq \max \{|u(y) - u_\tau(y)|, |\psi(y') - \psi_\tau(y')|\},$$

and, since either y or $y - \tau$ belongs to $\partial\Omega$, (6.3) follows at once from (6.4).

Now let φ and ψ be continuous, and let $\varphi_i(x)$ and $\psi_i(x)$ be two sequences of C^2 -functions, converging uniformly to φ and ψ , respectively, and verifying $\psi_i(x) \leq \varphi_i(x)$ in $\partial\Omega$ ($j = 1, 2, \dots$).

The Plateau problem relative to φ_i and ψ_i has a unique Lipschitz-continuous solution u_i , which satisfies the inequality (6.3).

Passing to the limit as $j \rightarrow \infty$ and recalling that $u_j(x)$ converges uniformly to $u(x)$, we get the conclusion of the lemma. Q.E.D.

Let us come now to the boundary estimate. A supersolution (for the area functional) is a function $m(x) \in C^{0+1}(\Omega) \cap C^0(\bar{\Omega})$, such that, for every non-negative Lipschitz-continuous function $\eta(x)$ with $S = \text{supp } \eta \subset \Omega$, we have

$$(6.5) \quad \int_S (1 + |D(m + \eta)|^2)^{1/2} dx \geq \int_S (1 + |Dm|^2)^{1/2} dx.$$

In a similar way, one can define subsolutions. It follows at once from (6.5) that, if $u(x)$ is a Lipschitz-continuous function in $\bar{\Omega}$, solution of the Plateau problem with obstacle $\psi(x)$, and if we have

$$\psi(x) \leq m(x) \quad \text{in } \bar{\Omega}, \quad u(x) \leq m(x) \quad \text{in } \partial\bar{\Omega}, \quad \text{then } u(x) \leq m(x) \quad \text{in } \bar{\Omega}.$$

With an approximation procedure similar to that used in Lemma 2, one gets the same conclusion for continuous solutions.

With an identical argument, one concludes that, if $\ell(x)$ is a subsolution and $u(x) \geq \ell(x)$ in $\partial\Omega$, then $u(x) \geq \ell(x)$ in $\bar{\Omega}$.

We have thus proved the following result:

Lemma 3. *Let Ω , φ , and ψ be as before, and let $m(x)$ and $\ell(x)$ be a super- and subsolution such that*

$$(6.6) \quad \ell(x) = m(x) = \varphi(x) \quad \text{in } \partial\Omega,$$

$$(6.7) \quad m(x) \geq \psi(x) \quad \text{in } \bar{\Omega}.$$

Let $u(x)$ be the solution of the Plateau problem relative to φ and ψ . Then, for every $r > 0$

$$(6.8) \quad U^*(r) \leq \max \{M^*(r), L^*(r)\}.$$

The connection between super- and subsolutions and barriers is clear if one observes that, if $\mu(x)$ is an upper barrier, then the function $m(x) = \min \{\mu(x), K_1\}$ is a supersolution such that $m(x) \geq \psi(x)$. Similarly,

$$\ell(x) = \max \{\lambda(x), -K\}$$

is a subsolution.

Combining the results of this section with the barriers constructed in the previous paragraphs, one gets the following theorem.

Theorem 4. *Let Ω be an open bounded set in $\mathbf{R}^n$ whose boundary $\partial\Omega$ is a C^3 -surface with non-negative mean curvature. Let $\varphi: \partial\Omega \rightarrow \mathbf{R}$ and $\psi: \bar{\Omega} \rightarrow \mathbf{R}$ be continuous functions such that $\psi \leq \varphi$ in $\partial\Omega$, and let $u(x)$ be the solution to the Plateau problem relative to φ and ψ . Then:*

(a₁) *if φ is of class $C^{1+\alpha}$ ($\alpha > 0$), and ψ is Lipschitz-continuous, $u(x)$ is Lipschitz-continuous in $\bar{\Omega}$;*

- (a₂) if the mean curvature of $\partial\Omega$ is strictly positive, φ is in $C^{0+\alpha}$ and ψ is in $C^{0+\alpha/2}$, $u(x)$ is $C^{0+\alpha/2}(\bar{\Omega})$;
 (a₃) if φ is Lipschitz-continuous and ψ is in $C^{0+\beta}$, $u(x)$ belongs to $C^{0+\sigma}(\bar{\Omega})$ for some $\sigma \leq \beta$.

7. The following example, suggested by H. Weinberger, shows that the result (a₂) is the best possible. Let Ω be the unit circle in $\mathbb{R}^2$, and let

$$\varphi(\theta) = \log((5 + 4 \cos \theta)^{1/2} + 2(1 + \cos \theta)^{1/2}), \quad 0 \leq \theta < 2\pi.$$

(Note that φ is Lipschitz-continuous in $\partial\Omega$, but is not differentiable at $\theta = \pi$)

The function

$$u(x, y) = \log((x^2 + y^2 + 4x + 4)^{1/2} + (x^2 + y^2 + 4x + 3)^{1/2})$$

is a solution of the minimal surfaces equation in Ω , equal to φ in $\partial\Omega$, and is Hölder-continuous with exponent $1/2$.

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