A Local Compactness Theorem for Wave Propagation Problems of Classical Physics

JOHN R. SCHULENBERGER

Communicated by RALPH PHILLIPS

1. Introduction. In many of the wave propagation phenomena of classical physics the functions describing the properties of the medium are not smooth but rather only piecewise continuous. When light passes through a lens, for example, the dielectric tensor in the system of Maxwell's equations describing this event has a jump discontinuity at the boundary of the lens. Because of this lack of smoothness, such basic physical phenomena do not fit into the scattering theory for symmetric hyperbolic systems developed earlier by C. H. Wilcox and the author [16, 17, 21]. The result needed here is a local compactness theorem of the type proved in §2 below.

For the purposes of scattering theory, the relevant local compactness theorems have usually been obtained via coerciveness inequalities [6, 7, 8, 10, 19]. However, the appropriate coerciveness results seem to require ellipticity or smooth coefficients or both (*cf.* [18]) and, in addition, special boundary conditions for problems in an exterior domain [7, 10, 12, 19]. Recently it has been possible in some cases to establish local compactness without coerciveness [1, 3, 11]. In particular, M. Š. Birman [1] has proved a local compactness theorem which enables him to include symmetric hyperbolic systems of constant deficit (*cf.* below) on the whole space in his abstract stationary theory, provided that the coefficients of the system are continuous. It is the purpose of this note to prove a local compactness theorem for systems of constant deficit on the whole space with coefficients which may be discontinuous.

Since this work was completed, V. G. Deĭč [3] has published results which include Theorem 3.1 below. This work is based on a rather general local compactness theorem, but this theorem requires a hypothesis in addition to those needed for Theorem 2.1 (*cf.* the remark below following this theorem), and his proof is quite different.

The paper is organized as follows. In §2 the local compactness theorem is formulated and proved. In §3 some applications to scattering theory are mentioned.

2. Systems of constant deficit. Local compactness. Let $E, A_1, \dots, A_n$ be $m \times m$ hermitian matrices; the A_i are constant, while $E = E(x)$ is a measurable function on $\mathbf{R}^n$ which for some positive constants c_0, c_1 satisfies

$$(2.1) \quad c_0 |p|^2 \leq E(x)p \cdot p \leq c_1 |p|^2 < \infty$$

for all $p \in \mathbf{R}^m$. Let $D_i = -i\partial/\partial x_i$, $A(D) = \sum_{i=1}^n A_i D_i$, and $\Lambda(D) = E^{-1}A(D)$. $A(D)$ generates a self-adjoint operator A in $L_2(\mathbf{R}^n)^m$ with domain

$$\mathfrak{D} = \{u \in L_2(\mathbf{R}^n)^m \mid A(D)u \in L_2(\mathbf{R}^n)^m\}.$$

Defining $\mathfrak{D}(\Lambda) = \mathfrak{D}$ and $\Lambda u = E^{-1}Au$ for $u \in \mathfrak{D}$, $\Lambda(D)$ generates a self-adjoint operator Λ on the Hilbert space $\mathfrak{H}$ consisting of functions $f, g \in L_2(\mathbf{R}^n)^m$ with inner product $\langle f, g \rangle = (f, Eg)$ [18, 21]. Here $\langle \cdot, \cdot \rangle$, $|\cdot|$ and $(\cdot, \cdot)$, $\|\cdot\|$ denote the inner products and norms in $\mathfrak{H}$ and $L_2(\mathbf{R}^n)^m$ respectively. If $E = E_0$ is a constant matrix the notation $\Lambda_0, \mathfrak{H}_0, \langle \cdot, \cdot \rangle_0, |\cdot|_0$ is used.

Definition [14, 18]. Λ is of *constant deficit* k , $0 \leq k < m$, if for all $p \in \mathbf{R}^n - \{0\}$ the rank of the matrix $A(p)$ is $m - k$. The system of partial differential equations

$$(2.2) \quad i\partial_t u = \Lambda(D)u \quad (u = u(x, t), \quad t \in \mathbf{R}^1)$$

is of *constant deficit* or *strongly propagative* if Λ is of constant deficit.

Remark 1. Most of the wave equations of classical physics (Maxwell's equations, the equation of elasticity, acoustics, etc.) can be written as systems of constant deficit [14, 21].

If Λ_0 is of constant deficit, let P_0 be the self-adjoint orthogonal projection onto the null space $\mathfrak{N}(\Lambda_0)$ of Λ_0 in $\mathfrak{H}_0$. P_0 is a singular integral operator with smooth symbol which is independent of $x \in \mathbf{R}^n$ [14, 18]. Therefore, for any $\alpha \in C_0^\infty(\mathbf{R}^n)$ $S_\alpha = \alpha P_0 - P_0 \alpha$ is a compact operator in $\mathfrak{H}_0$ and also $\mathfrak{H}$ (cf. [9]). Further, it is obvious from the definition of Λ that as sets

$$(2.3) \quad \mathfrak{D}(\Lambda) = \mathfrak{D}(\Lambda_0) \quad \text{and} \quad \mathfrak{N}(\Lambda_0) = \mathfrak{N}(\Lambda).$$

Finally, let $\mathfrak{H}^{100}$ denote the Fréchet space of measurable functions f such that $\alpha f \in \mathfrak{H}$ for any $\alpha \in C_0^\infty(\mathbf{R}^n)$ (cf. [4]).

Theorem 2.1. Let E be a measurable $m \times m$ hermitian matrix satisfying (2.1), and let Λ be the corresponding self-adjoint operator of constant deficit in $\mathfrak{H}$ described above. If a set of functions $\{u\} \subset \mathfrak{D}(\Lambda) \cap \mathfrak{N}(\Lambda)^\perp$ is bounded in graph norm, i.e.,

$$(2.4) \quad |u| + |\Lambda u| \leq \text{const.},$$

then $\{u\}$ is precompact in $\mathfrak{H}^{100}$.

Remark 2. The terms *locally precompact* and *precompact in local energy norm* are sometimes used in similar contexts and would here mean that the set of functions $\{u\}$ is precompact in $L_2(G)^m$ for every bounded set $G \subset \mathbf{R}^n$. This is clearly equivalent to the precompactness of $\{u\}$ in $\mathfrak{H}^{100}$: a set $S \subset \mathfrak{H}^{100}$ is precompact if and only if αS is precompact in $\mathfrak{H}$ for every $\alpha \in C_0^\infty(\mathbf{R}^n)$ [4, 20].

Remark 3. Theorem 2.1 follows from the local compactness theorem of Deĭč [3] mentioned in the introduction if, in addition, $\sup_{|x| \geq R} |E(x) - E_0| \rightarrow 0$ for $R \rightarrow \infty$.

Proof. With P_0 as above, let $v = P_0 u$ and $w = u - v$. Then (2.4) implies that $\{v\}$ is bounded in $\mathcal{H}$. Further, $P_0 w = 0$ and $\Lambda_0 w = \Lambda_0 u$ by (2.3), so it follows from (2.4) by Fourier transform and Rellich's theorem in the usual way (cf. [18]) that $\{w\}$ is precompact in $\mathcal{H}_0^{1\circ\circ}$ and hence in $\mathcal{H}^{1\circ\circ}$. Now $u \in \mathfrak{N}(\Lambda)^\perp$ if and only if $E_0^{-1} E u \in \mathfrak{N}(\Lambda_0)^\perp$ if and only if $0 = P_0 E_0^{-1} E u = P_0 E_0^{-1} E v + P_0 E_0^{-1} E w$, i.e., if and only if

$$(2.5) \quad v + P_0(E_0^{-1} E - I)v = -P_0 E_0^{-1} E w.$$

Let $\alpha \in C_0^\infty(\mathbb{R}^n)$ and $S_\alpha = \alpha P_0 - P_0 \alpha$. Multiplying (2.5) by α gives

$$(2.6) \quad \alpha v + P_0(E_0^{-1} E - I)\alpha v = f_\alpha \equiv -P_0 \alpha E_0^{-1} E w - S_\alpha(E_0^{-1} E - I)v - S_\alpha E_0^{-1} E w.$$

Since $\{\alpha w\}$ is precompact in $\mathcal{H}$, $\{w\}$ and $\{v\}$ are bounded, and S_α is a compact operator in $\mathcal{H}$, as remarked above, it follows that $\{f_\alpha\}$ is precompact in $\mathcal{H}$. Applying $I - P_0$ to (2.6) gives $(I - P_0)\alpha v = (I - P_0)f_\alpha$, so that $\{(I - P_0)\alpha v\}$ is precompact in $\mathcal{H}$. Now rewrite (2.6) as $P_0 \alpha v + P_0(E_0^{-1} E - I)P_0 \alpha v = h_\alpha \equiv f_\alpha - (I - P_0)\alpha v - P_0(E_0^{-1} E - I)(I - P_0)\alpha v$ or

$$(2.7) \quad [I + P_0(E_0^{-1} E - I)]P_0 \alpha v = h_\alpha.$$

The set $\{h_\alpha\}$ is precompact in $\mathcal{H}$, $\mathcal{H}_0$, and moreover, $\{h_\alpha\} \subset P_0 \mathcal{H}_0$. Now consider the operator $I + P_0(E_0^{-1} E - I)$ occurring in (2.7) as a mapping from $P_0 \mathcal{H}_0$ to $P_0 \mathcal{H}_0$; denote this operator by B . Then $B = P_0 E_0^{-1} E$, and for $f, g \in P_0 \mathcal{H}_0$: $\langle Bf, g \rangle_0 = \langle E_0^{-1} E f, g \rangle_0 = \langle f, E_0^{-1} E g \rangle_0 = \langle f, P_0 E_0^{-1} E g \rangle_0 = \langle f, Bg \rangle_0$. Thus, B is self-adjoint. The range of B is dense in $P_0 \mathcal{H}_0$, since if $g \in P_0 \mathcal{H}_0$ is orthogonal to the range of B , then for all $f \in P_0 \mathcal{H}_0$ $0 = \langle Bf, g \rangle_0 = \langle f, E g \rangle = \langle f, g \rangle$, and taking $f = g$ this implies that $g = 0$. Furthermore, the range of B is closed in $P_0 \mathcal{H}_0$, since $\|Bg\|_0 \cdot \|g\|_0 \geq \langle Bg, g \rangle_0 = \langle g, E g \rangle \geq c_0 \|g\|^2 \geq c_0 c_1^{-1} \|g\|_0^2$. Thus, B is a continuous injection of the closed subspace $P_0 \mathcal{H}_0$ of $\mathcal{H}_0$ onto itself, and therefore by a theorem of Banach it has a continuous inverse B^{-1} in $P_0 \mathcal{H}_0$.

Returning to (2.7), since $\{h_\alpha\}$ is precompact in $\mathcal{H}_0$ and P_0 is continuous, $\{h_\alpha\}$ is also precompact in $P_0 \mathcal{H}_0$. Thus, $\{P_0 \alpha v\} = \{B^{-1} h_\alpha\}$ is precompact in $P_0 \mathcal{H}_0$, hence in $\mathcal{H}_0$ and therefore in $\mathcal{H}$.

Finally, $\{\alpha v\} \subset \{P_0 \alpha v\} + \{(I - P_0)\alpha v\}$ is precompact in $\mathcal{H}$. Therefore, $\{u\} \subset \{v\} + \{w\}$ is precompact in $\mathcal{H}^{1\circ\circ}$, and this completes the proof of the theorem.

3. Applications to scattering theory. Let $U(t)$, $U_0(t)$ denote the groups of unitary operators generated in $\mathcal{H}$, $\mathcal{H}_0$ by $-i\Lambda$, $-i\Lambda_0$ respectively. Define $J : \mathcal{H}_0 \rightarrow \mathcal{H}$ by $Jf = f$ for all $f \in \mathcal{H}_0$. Define the wave operators

$$(3.1) \quad W_\pm(\Lambda, \Lambda_0) = s - \lim_{t \rightarrow \pm\infty} U(-t) J U_0(t) P_0^{ac},$$

where P_0^{ac} is the projection onto the absolutely continuous subspace $\mathcal{H}_0^{ac}$ of Λ_0 in $\mathcal{H}_0$. The wave operators are said to be complete if their range is $\mathcal{H}^{ac}$, the absolutely continuous subspace of Λ in $\mathcal{H}$.

In [21] Wilcox proved the existence of the wave operators (3.1) under very mild conditions on $E(x)$. Their completeness is proved in [16, 17] under the stronger conditions that $E(x)$ has bounded first derivatives and satisfies (2.1) and (3.2), (3.3) below. In [1] Birman proves completeness assuming only (2.1), (3.2), (3.3), and the continuity of $E(x)$. On the basis of Theorem 2.1, precisely the same theorem now holds if $E(x)$ is only measurable and satisfies (2.1), (3.2), (3.3); the proof is the same as in [1]. This theorem has also been obtained by Deİć [3].

Theorem 3.1. *Let E, Λ be as in Theorem 2.1. Suppose that there is a constant hermitian matrix E_0 satisfying (2.1) such that*

$$(3.2) \quad \lim_{|x| \rightarrow \infty} |E(x) - E_0| = 0 \text{ uniformly in } x/|x|;$$

$$(3.3) \quad \int (1 + |x|^2)^s |E(x) - E_0|^2 dx < \infty \text{ for some } s > n/2.$$

Let $\Lambda_0 = E_0^{-1} \Lambda$ be the corresponding self-adjoint operator in $\mathcal{H}_0$. Then the wave operators (3.1) exist and are complete.

Remark 1. For the case in which $E(x)$ has bounded, continuous derivatives Theorem 3.1 was proved in [16, 17] on the basis of a coerciveness inequality proved by the author and Wilcox [18]. Condition (3.2) was used in the proof of the coerciveness theorem. L. Sarason has recently given an improved version of this inequality which does not require condition (3.2). The proof of the completeness theorem given in [16, 17] therefore now also goes through without condition (3.2). Since this condition is not used in Theorem 2.1, it may now be possible to find a proof of Theorem 3.1 which does not require (3.2).

Remark 2. If Λ_0 is uniformly propagative (the roots of $\det [\lambda I - \Lambda_0(p)]$ have constant multiplicities for all nonzero $p \in \mathbf{R}^n$) Deİć [2] has obtained Theorem 3.1 with condition (3.3) replaced by the weaker condition $|E(x) - E_0| = O(|x|^{-1-\epsilon})$ for $|x| \rightarrow \infty$. This work is based on the stationary scattering theory of Kato and Kuroda [5] rather than on Birman's theory. This weaker condition on $E - E_0$ is the same as that under which Wilcox [21] earlier proved the existence (but not completeness) of the wave operators.

Finally, Theorem 2.1 makes it possible to fit systems (2.2) of constant deficit with discontinuous coefficients into the abstract formulation of scattering theory due to Lax and Phillips [7], at least if n is odd and E is constant outside a bounded set. This has now been carried out in [6]. With Theorem 2.1 and the Lax-Phillips radiation condition [7, 8] it should now be possible to treat many wave propagation problems of real physical interest which have so far been neglected.

In conclusion, the author would like to thank W. M. Greenlee and C. H. Wilcox for their attention to this work and to acknowledge his debt to M. Š. Birman whose paper [1] provided the plan of attack for the proof of Theorem 2.1.

REFERENCES

1. M. Š. BIRMAN, *Scattering problems for differential operators with perturbation of the space*, Izv. Akad. Nauk SSSR **35** (1971), 440–455.
2. V. G. DEJČ, *The local stationary method in the theory of scattering with two spaces*, Dokl. Akad. Nauk SSSR **197** (1971), 1247–1250.
3. V. G. DEJČ, *Application of the method of nuclear perturbations in two-space scattering theory*, Izv. Vysš. Učebn. Zaved. (1971), 33–42.
4. L. HÖRMANDER, *Linear partial differential operators*, Springer-Verlag, New York, 1969.
5. T. KATO & S. KURODA, *Theory of simple scattering and eigenfunction expansions*, in *Functional analysis and related fields*, (F. Browder, Ed.) Springer-Verlag, Berlin, 1970.
6. J. LA VITA, J. SCHULENBERGER, & C. WILCOX, *Scattering theory for wave propagation problems of classical physics by the method of Lax and Phillips*, *Applicable Anal.*, (to appear).
7. P. LAX & R. PHILLIPS, *Scattering theory*, Academic Press, New York 1967.
8. P. LAX & R. PHILLIPS, *Scattering theory*, *Rocky Mtn. J. Math.* **1** (1971), 173–224.
9. S. MIKHLIN, *Multidimensional singular integrals and integral equations*, Pergamon Press, London 1965.
10. K. MOCHIZUKI, *Spectral and scattering theory for symmetric hyperbolic systems in an exterior domain*, *Publ. RIMS Kyoto Univ.* **5** (1969), 219–258.
11. R. PHILLIPS & L. SARASON, *Energy decay for the neutrino equation in the exterior of a torus*, *Arch. Rat. Mech. Anal.* **41** (1971), 1–17.
12. J. RALSTON, *Energy Decay Problems in Scattering Theory*, Thesis, Stanford University, 1968.
13. L. SARASON, *A remark on an inequality of Schulenberg and Wilcox*, *Ann. Mat. Pura Appl.*, (to appear).
14. J. SCHULENBERGER & C. WILCOX, *The singularities of the Green's matrix in anisotropic wave motion*, *Indiana Univ. Math. J.* **20** (1971), 1092–1117.
15. J. SCHULENBERGER & C. WILCOX, *Eigenfunction expansions and scattering theory for wave propagation problems of classical physics*, *ONR Technical Summary Report*, No. 14, August (1971).
16. J. SCHULENBERGER & C. WILCOX, *Completeness of the wave operators for perturbations of uniformly propagative systems*, *J. Funct. Anal.* **7** (1971), 447–474.
17. J. SCHULENBERGER & C. WILCOX, *Completeness of the wave operators for scattering problems of classical physics*, *Bull. Amer. Math. Soc.* **77** (1971), 777–782.
18. J. SCHULENBERGER & C. WILCOX, *Coerciveness inequalities for nonelliptic systems of partial differential equations*, *Ann. Mat. Pura Appl.* **88** (1971), 229–306.
19. G. SCHMIDT, *Spectral and scattering theory for Maxwell's equations in an exterior domain*, *Arch. Rat. Mech. Anal.* **28** (1967), 284–322.
20. F. TRÈVES, *Topological vector spaces, distribution, and kernels*, Academic Press, New York 1967.
21. C. WILCOX, *Wave operators and asymptotic solutions of wave propagation problems of classical physics*, *Arch. Rat. Mech. Anal.* **22** (1966), 37–78.

This research was supported by the Office of Naval Research Grant no. N00014-67-A-0325-0006 Task no. NR-041-370

University of Utah

Date communicated: MAY 1, 1972