Properties of the Matrix Range of an Operator

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All Hilbert spaces considered in this paper will be complex, and all operators will be bounded and linear. The algebra of all (bounded linear) operators on a Hilbert space $\mathcal{H}$ will be denoted by $\mathcal{L}(\mathcal{H})$. For a positive integer n , $\mathcal{H}_n$ will designate n -dimensional Hilbert space. If B is a set of vectors in a linear topological space then $[B]$ and $\text{co } (B)$ will denote respectively the closed linear span of the vectors in B and the convex hull of the vectors in B . The spectrum and numerical range of $T \in \mathcal{L}(\mathcal{H})$ will be denoted respectively by $\sigma(T)$ and $W(T)$. The complex numbers will be denoted by $\mathbb{C}$.

1. Introduction. In [2], W. B. Arveson defined, for any positive integer n , the n -dimensional matrix range of $T \in \mathcal{L}(\mathcal{H})$, denoted $W_n(T)$. Specifically, form $C^*(T)$ the C^* -algebra generated by T and the identity and denote by $CP(C^*(T), \mathcal{H}_n ; I)$ the set of all completely positive maps of $C^*(T)$ into $\mathcal{L}(\mathcal{H}_n)$ which preserve the identity (completely positive maps are discussed in [1] and [15]). We then define

$$W_n(T) = \{\varphi(T) : \varphi \in CP(C^*(T), \mathcal{H}_n ; I)\}.$$

If $n = 1$, then $W_1(T) = \overline{W(T)}$ and thus $W_n(T)$ is a generalization of the numerical range (refer to [4] or [1]). In [3] Arveson showed that if T_1 and T_2 are compact irreducible operators, then T_1 is unitarily equivalent to T_2 if and only if $W_n(T_1) = W_n(T_2)$, $n = 1, 2, 3, \dots$.

The purpose of this paper is to uncover information about $W_n(T)$.

2. General results. The next lemma will be used repeatedly.

Lemma 2.1. *Let T and V be operators and assume T is not a multiple of the identity. Define a map by $\varphi : \alpha T + \beta I \rightarrow \alpha V + \beta I$ for all $\alpha, \beta \in \mathbb{C}$. If V is normal and φ is contractive ($\|\varphi\| \leq 1$), then φ has a completely positive linear extension to any C^* -algebra containing T .*

Proof. Since T is not a multiple of the identity φ is well defined and linear

and thus by Proposition 1.2.11 of [1] φ is completely contractive and hence by Theorem 1.2.9 of [1] the conclusion follows.

Now let $T \in \mathfrak{L}(\mathfrak{H})$ have norm one and denote by B_n the solid unit ball of $\mathfrak{L}(\mathfrak{H}_n)$. If $\varphi \in CP(C^*(T), \mathfrak{H}_n; I)$, then $\|\varphi\| = \|\varphi(I)\| = \|I\| = 1$ by [1]. Thus $W_n(T) \subset B_n$ for all n . The next theorem tells us when equality holds.

Theorem 2.1. *If $T \in \mathfrak{L}(\mathfrak{H})$ and $\|T\| = 1$, then the following conditions are equivalent:*

- (i) $\|\alpha T + \beta I\| = |\alpha| + |\beta|$ for all $\alpha, \beta \in \mathbb{C}$,
- (ii) $W_n(T) = B_n$ for all n ,
- (iii) $W_{n_0}(T) = B_{n_0}$ for some n_0 .

Proof. (i) $\Rightarrow$ (ii). To avoid trivialities, we assume T is not a multiple of the identity. Let U be an arbitrary unitary operator in B_n . The map φ defined by $\varphi : \alpha T + \beta I \rightarrow \alpha U + \beta I$ is contractive since by (i) we have $\|\varphi(\alpha T + \beta I)\| = \|\alpha U + \beta I\| \leq |\alpha| + |\beta| = \|\alpha T + \beta I\|$. By Lemma 2.1 we conclude that there exists $\varphi_1 \in CP(C^*(T), \mathfrak{H}_n; I)$ such that $\varphi_1(T) = U$. Since U was arbitrary $W_n(T)$ contains all the unitaries in $\mathfrak{L}(\mathfrak{H}_n)$. Now $W_n(T)$ is always compact and convex and since the unitary operators are precisely the extreme points of B_n we must have $W_n(T) = B_n$. (ii) $\Rightarrow$ (iii) is clear. (iii) $\Rightarrow$ (i). Let $\{e_1, e_2, \dots, e_{n_0}\}$ be an orthonormal basis for $\mathfrak{H}_{n_0}$. Given $\gamma \in \mathbb{C}$, denote by S the operator defined by $Se_1 = (\gamma/|\gamma|)e_1$, $Se_i = 0$, $i \geq 2$. Now $S \in B_{n_0}$ and thus there exists

$$\varphi \in CP(C^*(T), \mathfrak{H}_{n_0}; I)$$

such that $\varphi(T) = S$. A simple calculation shows that $\|S + \gamma I\| = 1 + |\gamma|$ and thus $1 + |\gamma| = \|S + \gamma I\| = \|\varphi(T + \gamma I)\| \leq \|T + \gamma I\| \leq 1 + |\gamma|$ from which the conclusion easily follows.

The above theorem and the fact that $W_1(T) = \overline{W(T)}$ enable us to calculate the matrix range of the unilateral shift (see [10] for a definition and properties).

Corollary 2.1.1. *If U denotes the unilateral shift, then $W_n(U) = B_n$ for all n .*

Proof. Since $\|U\| = 1$ by Theorem 2.1 it suffices to show that $\overline{W(U)} = \{z : |z| \leq 1\}$. However, $\sigma(U) = \{z : |z| \leq 1\}$ and $\sigma(U) \subset \overline{W(U)}$ and so the conclusion follows.

Recall that the numerical radius of an operator, $|W(T)|$, is defined via $|W(T)| = \sup \{|\lambda| : \lambda \in W(T)\}$, and satisfies $\frac{1}{2}\|T\| \leq |W(T)| \leq \|T\|$ [9]. If we analogously define $|W_n(T)| = \sup \{|\lambda| : \lambda \in W_n(T)\}$ for $n \geq 2$, then we have

Theorem 2.2. $|W_n(T)| = \|T\|$ for $n \geq 2$.

Proof. Let π be a faithful representation of $C^*(T)$ on an infinite dimensional Hilbert space $\mathfrak{H}$ [6]. Given $\epsilon > 0$, choose a unit vector x_1 such that

$$(2.1) \quad \|\pi(T)x_1\|^2 > \|\pi(T)\|^2 - \epsilon.$$

If x_1 is not an eigenvector of $\pi(T)$, then by the Gram-Schmidt process we

can find a vector x_2 such that $\{x_1, x_2\}$ is an orthonormal set and $[x_1, x_2] = [x_1, \pi(T)x_1]$. Since $\mathcal{H}$ is infinite dimensional we can choose vectors $x_3, x_4, \dots, x_n$ such that $\{x_1, x_2, x_3, \dots, x_n\}$ is an orthonormal set. Let $\{e_i\}$ be an orthonormal basis for $\mathcal{H}_n$ and define an embedding $V: \mathcal{H}_n \rightarrow \mathcal{H}$ by $Ve_i = x_i$. It is easy to see that $V^*V = I$ (on $\mathcal{H}_n$) and VV^* is the projection onto $[x_1, x_2, \dots, x_n]$. For A in $C^*(T)$ let $\varphi(A) = V^*\pi(A)V$. By Stinespring's Theorem [15] and our construction $\varphi \in CP(C^*(T), \mathcal{H}_n; I)$ and hence $\varphi(T) \in W_n(T)$. Now $\|\varphi(T)\|^2 \geq \|\varphi(T)e_i\|^2 = (VV^*\pi(T)x_1, \pi(T)x_1) = \|\pi(T)x_1\|^2 > \|\pi(T)\|^2 - \epsilon = \|T\|^2 - \epsilon$. The last equality results from (2.1) and the fact that a faithful representation of a C^* -algebra is isometric [6]. Since ϵ was arbitrary we are done.

If x_1 is an eigenvector for $\pi(T)$ we mimic the above proof, except that we do not need the Gram-Schmidt process.

3. Normality and the matrix range. Recall that for a normal operator $\overline{\text{co}}(\sigma(T)) = \overline{W(T)}$. The next theorem, proved in [3], shows that for $n \geq 2$ $W_n(T)$ is actually the closed "matrix-valued" convex hull of the spectrum of T .

Theorem 3.1. (Arveson). *Let T be a normal operator and let n be a positive integer. Then*

$$W_n(T) = CL\left\{\sum_{i=1}^r \lambda_i K_i : r \geq 1, \lambda_i \in \sigma(T), K_i \in \mathcal{L}(\mathcal{H}_n), K_i \geq 0, \sum_{i=1}^r K_i = I\right\}$$

where "CL" denotes closure in the norm topology.

If T is self-adjoint, then since completely positive maps preserve adjoints, $W_n(T)$ consists entirely of self-adjoint operators. Is the converse true? When does $W_n(T)$ consist entirely of normal operators? If $W_n(T)$ consists of normal operators must T be normal? The next theorem resolves these questions. We will need the following simple lemma from [14]:

Lemma 3.1. *In $\mathcal{L}(\mathcal{H}_n)$, $n \geq 2$, there exist positive self-adjoint operators K_1, K_2, K_3 with the following properties:*

- (a) $K_1 + K_2 + K_3 = I$,
- (b) K_2K_3 and K_3K_2 are linearly independent.

Theorem 3.2. *If $T \in \mathcal{L}(\mathcal{H})$, then the following conditions are equivalent:*

- (i) $W_n(T)$ consists entirely of normal operators for all n ,
- (ii) $W_{n_0}(T)$ consists entirely of normal operators for some $n_0 \geq 2$,
- (iii) T is normal and $\sigma(T)$ is contained in a line.

Proof. (i) $\Rightarrow$ (ii) is clear. (ii) $\Rightarrow$ (iii). Assume to the contrary that T is not normal. Let π be a faithful representation of $C^*(T)$ on an infinite dimensional Hilbert space $\mathcal{H}$. Since T is not normal and π is faithful $\pi(T)$ is not normal and therefore we can find a unit vector x in $\mathcal{H}$ such that

$$(3.1) \quad \|\pi(T)x\|^2 \neq \|\pi(T)^*x\|^2.$$

Suppose first that $\{x, \pi(T)x, \pi(T)^*x\}$ is a linearly independent set. Let $S_1 = [x, \pi(T)x]$, $S_2 = [x, \pi(T)^*x]$. By the Gram-Schmidt process we can find vectors y_1 and y_2 such that $S_1 = [x, y_1]$, $S_2 = [x, y_2]$ and $\{x, y_1\}$ and $\{x, y_2\}$ are both orthonormal sets. Since $\mathcal{H}$ is infinite dimensional we can find (if necessary) unit vectors $u_3, u_4, \dots, u_{n_0}$ and $v_3, v_4, \dots, v_{n_0}$ such that $\{x, y_1, u_3, \dots, u_{n_0}\}$ and $\{x, y_2, v_3, v_4, \dots, v_{n_0}\}$ are orthonormal sets. Let $\{e_1, e_2, \dots, e_{n_0}\}$ be an orthonormal basis for $\mathcal{H}_{n_0}$ and define embeddings V_1, V_2 by

$$\begin{aligned} V_1 : e_1 &\rightarrow x, & e_2 &\rightarrow y_1, & e_3 &\rightarrow u_3, & \dots, & e_{n_0} &\rightarrow u_{n_0} \\ V_2 : e_1 &\rightarrow x, & e_2 &\rightarrow y_2, & e_3 &\rightarrow v_3, & \dots, & e_{n_0} &\rightarrow v_{n_0}. \end{aligned}$$

If we set $\varphi_i(\cdot) = V_i^* \pi(\cdot) V_i$ ($i = 1, 2$), then by Stinespring's theorem and our construction $\varphi_i \in CP(C^*(T), \mathcal{H}_{n_0}; I)$ and therefore $\varphi_i(T)$ is normal by hypothesis. Thus $\|\varphi_i(T)z\|^2 = \|\varphi_i(T)^*z\|^2$ for all z in $\mathcal{H}_{n_0}$. Taking $z = e_1$ we obtain

$$(3.2) \quad \|V_i^* \pi(T) V_i e_1\|^2 = \|V_i^* \pi(T)^* V_i e_1\|^2, \quad i = 1, 2.$$

If we let $P_i = V_i^* V_i$, then P_1 is the projection on $[x, y_1, u_3, u_4, \dots, u_{n_0}]$ and P_2 is the projection on $[x, y_2, v_3, v_4, \dots, v_{n_0}]$. Evaluating first the left side and then the right side of (3.2) we obtain

$$\|V_1^* \pi(T) V_1 e_1\|^2 = \langle P_1 \pi(T)x, \pi(T)x \rangle = \|\pi(T)x\|^2$$

and

$$\|V_1^* \pi(T)^* V_1 e_1\|^2 = \langle P_1 \pi(T)^* x, \pi(T)^* x \rangle = \|P_1 \pi(T)^* x\|.$$

Thus we have

$$(3.3) \quad \|\pi(T)x\|^2 = \|P_1 \pi(T)^* x\|^2.$$

Letting $i = 2$ in (3.2), by a similar calculation we obtain

$$(3.4) \quad \|\pi(T)^* x\|^2 = \|P_2 \pi(T)x\|^2.$$

From (3.1) we must have either $\|\pi(T)x\|^2 > \|\pi(T)^* x\|^2$ or $\|\pi(T)^* x\|^2 > \|\pi(T)x\|^2$. If the former holds, then from (3.3) we have $\|\pi(T)x\|^2 = \|P_1 \pi(T)^* x\|_s = \|\pi(T)^* x\|^2 - \|(I - P_1) \pi(T)^* x\|^2$ and therefore $\|\pi(T)^* x\|^2 \geq \|\pi(T)x\|^2$ which is a contradiction. If the latter holds, then from (3.4) we obtain $\|\pi(T)^* x\|^2 = \|P_2 \pi(T)x\|^2 = \|\pi(T)x\|^2 - \|(I - P_2) \pi(T)x\|^2$ and thus $\|\pi(T)x\|^2 \geq \|\pi(T)^* x\|^2$ which is also a contradiction. Thus T must be normal. If $\{x, \pi(T)x, \pi(T)^*x\}$ is a linearly dependent set, then the same proof is valid with just minor changes in notation. We now show that $\sigma(T)$ is contained in a line. Let $\lambda \in \sigma(T)$ and set $S = T - \lambda I$. S is normal and $0 \in \sigma(S)$. Let $K_1, K_2, K_3 \in \mathcal{L}(\mathcal{H}_{n_0})$ be as in Lemma 3.1. If λ_2 and λ_3 are arbitrary points of $\sigma(S)$, then by Theorem 3.1 $U = 0K_1 + \lambda_2 K_2 + \lambda_3 K_3 \in W_{n_0}(S)$. Since $C^*(S) = C^*(T)$ we have $W_{n_0}(S) = W_{n_0}(T) - \lambda I$ and thus $W_{n_0}(S)$ consists entirely of normal operators. Upon simplification the equation $UU^* - U^*U = 0$ becomes $(\lambda_2 \bar{\lambda}_3 - \bar{\lambda}_2 \lambda_3) K_2 K_3 + (\lambda_3 \bar{\lambda}_2 - \bar{\lambda}_3 \lambda_2) K_3 K_2 = 0$

and thus by condition (b) of Lemma 3.1 we must have $\lambda_2 \bar{\lambda}_3 = \bar{\lambda}_2 \lambda_3$ which means that $\lambda_2 \bar{\lambda}_3$ is real. Since λ_2 and λ_3 were arbitrary elements of $\sigma(S)$, a little checking shows that $\sigma(S)$ must be contained in a line through the origin and thus $\sigma(T) = \sigma(S) + \lambda$ is contained in a line.

(iii) $\Rightarrow$ (i). Under the hypothesis T must be a linear function of a self-adjoint operator and this is sufficient to draw the conclusion.

Corollary 3.2.1. *If $W_n(T)$ consists entirely of self-adjoint operators, then T is self-adjoint.*

Proof. By Theorem 3.2 T is normal. If $\lambda \in \sigma(T)$, then by Theorem 3.1, $\lambda I \in W_n(T) \Rightarrow (\lambda I)^* = \lambda I \Rightarrow \lambda$ is real $\Rightarrow T$ is self-adjoint.

4. Applications. A theorem of S. Hildebrandt [12] states that for $T \in \mathcal{L}(\mathcal{H})$ we have

$$(4.1) \quad \overline{\text{co}(\sigma(T))} = \bigcap_{B \in \mathfrak{B}} \overline{W(BTB^{-1})}$$

where $\mathfrak{B} = \{\text{all invertible operators in } \mathcal{L}(\mathcal{H})\}$. This theorem prompts the next definition.

Definition 4.1. If $T \in \mathcal{L}(\mathcal{H})$, then $V_n(T)$ is defined by

$$V_n(T) = \bigcap_{B \in \mathfrak{B}} W_n(BTB^{-1}).$$

Recall that for an operator A the spectral radius of A , denoted by $r(A)$, satisfies $r(A) \leq \|A\|$ and that if A is normal equality holds.

Theorem 4.1. *If T is normal, then $V_n(T) = W_n(T)$ for $n = 1, 2, \dots$.*

Proof. Since the inclusion $V_n(T) \subset W_n(T)$ always holds, it suffices to show that $W_n(T) \subset W_n(BTB^{-1})$ for any $B \in \mathfrak{B}$. Define a linear map ψ from $[BTB^{-1}, I]$ by $\psi : \alpha(BTB^{-1}) + \beta I \rightarrow \alpha T + \beta I$ for all $\alpha, \beta \in \mathbb{C}$. If $\gamma \in \mathbb{C}$ we have $\|T + \gamma I\| = r(T + \gamma I) = r(B(T + \gamma I)B^{-1}) \leq \|BTB^{-1} + \gamma I\|$ and hence ψ is contractive. By Lemma 2.1, ψ has a completely positive extension, ψ_1 , to $C^*(BTB^{-1})$. If $S \in W_n(T)$, then there exists $\varphi \in CP(C^*(T), \mathcal{H}; I)$ such that $\varphi(T) = S$ and letting $\omega = \varphi \circ \psi_1$ we see that $S \in W_n(BTB^{-1})$. This completes the proof.

Corollary 4.1.1. *If T is normal, then $\overline{\text{co}(\sigma(T))} = \overline{W(T)}$.*

Proof. By (4.1) and the above theorem we have

$$\overline{\text{co}(\sigma(T))} = \bigcap_{B \in \mathfrak{B}} \overline{W(BTB^{-1})} = V_1(T) = W_1(T) = \overline{W(T)}.$$

For an arbitrary $T \in \mathcal{L}(\mathcal{H})$ and a positive integer n , we define $S_n(T)$ by $S_n(T) = CL\{\sum_{i=1}^r \lambda_i K_i : r \geq 1, \lambda_i \in \sigma(T), K_i \geq 0, K_i \in \mathcal{L}(\mathcal{H}_n), \sum_{i=1}^r K_i = I\}$ where “CL” denotes closure in the norm topology. $S_n(T)$ is an n -dimensional analog of the convex hull of the spectrum of T . Since $\overline{\text{co}(\sigma(T))} \subset \overline{W(T)}$ holds for any

$T \in \mathcal{L}(\mathcal{H})$, we can ask whether the n -dimensional analog of this inclusion is also valid.

Theorem 4.2. *If $T \in \mathcal{L}(\mathcal{H})$, then $S_n(T) \subset W_n(T)$ for all n .*

Proof. Let A be a normal operator such that $\sigma(A) = \sigma(T)$. If $\lambda \in \mathbb{C}$, then $\|A + \lambda I\| = r(A + \lambda I) = r(T + \lambda I) \leq \|T + \lambda I\|$. Thus the linear map φ defined by $\varphi : \alpha T + \beta I \rightarrow \alpha A + \beta I$ is contractive and hence by Lemma 2.1 φ has a completely positive extension to $C^*(T)$. Arguing as in the last part of the proof of Theorem 4.1 we obtain $W_n(A) \subset W_n(T)$ for all n . But since A is normal and $\sigma(A) = \sigma(T)$ we have $S_n(T) = S_n(A) = W_n(A) \subset W_n(T)$ and the proof is complete.

Theorem 3.1 shows that $S_n(T) = W_n(T)$ if T is normal. We will show that equality also holds for other classes of operators.

5. Subnormal and Toeplitz operators. Recall that $T \in \mathcal{L}(\mathcal{H})$ is subnormal if there exists a normal operator A on a Hilbert space $\mathcal{K}$ such that $\mathcal{H}$ is a subspace of $\mathcal{K}$ which is invariant under A and the restriction of A to $\mathcal{H}$ coincides with T . In [10] it is shown that a subnormal operator has an essentially unique minimal normal extension which we denote by A and that $\sigma(A) = \sigma(T)$.

Theorem 5.1. *If T is subnormal with minimal normal extension A , then we have $S_n(T) = W_n(T) = W_n(A)$ for all n .*

Proof. Let P denote the projection of $\mathcal{K}$ onto $\mathcal{H}$. The map φ from $C^*(A)$ to $C^*(T)$ defined by $\varphi : B \rightarrow PBP$ is easily seen to be completely positive and carries A to T . Arguing as in Theorem 4.1 we have $W_n(T) \subset W_n(A)$ for all n . Now by Theorems 4.2 and 3.1 and since A is normal we have

$$(5.1) \quad S_n(T) \subset W_n(T) \subset W_n(A) = S_n(A).$$

Since $\sigma(A) \subset \sigma(T)$ we have $S_n(A) \subset S_n(T)$ and thus all the inclusions in (5.1) become equalities and the conclusion follows.

Remark. An operator T satisfying $T^*T \geq TT^*$ is said to be hyponormal. All subnormal operators are hyponormal but there are hyponormal operators which are not subnormal [10]. We conjecture that if T is hyponormal, then $W_n(T) = S_n(T)$ for all n . If T is a hyponormal weighted shift [10], then it is shown in [14] that $S_n(T) = W_n(T)$.

Let $D = \{z : |z| = 1\}$. If $\varphi \in \mathcal{L}_\infty(D)$, then φ induces an operator L_φ on $\mathcal{L}_2(D)$ defined by $L_\varphi : f \rightarrow \varphi \cdot f$. L_φ is called the Laurent operator induced by φ and it is clearly normal. Denote by $\mathcal{H}^2(D)$ the subspace of $\mathcal{L}_2(D)$ which consists of all functions whose negative Fourier coefficients vanish (where the Fourier coefficients are taken with respect to the standard orthonormal basis $e_n(z) = z^n$, $n = 0, \pm 1, \pm 2, \dots$) and let P be the projection of $\mathcal{L}_2(D)$ onto $\mathcal{H}^2(D)$. If $\varphi \in \mathcal{L}_\infty(D)$, then φ induces an operator T_φ on $\mathcal{H}^2(D)$ defined by $T_\varphi : f \rightarrow P(\varphi \cdot f)$. T_φ is called the Toeplitz operator induced by φ (see [5] for further properties).

Theorem 5.2. *With the above notation we have*

$$W_n(T_\varphi) = W_n(L_\varphi) = S_n(T_\varphi).$$

Proof. The map ψ from $C^*(L_\varphi)$ to $C^*(T_\varphi)$ defined by $\psi : B \rightarrow PBP$ is easily seen to be completely positive and carries L_φ to T_φ . Arguing as in the proof of Theorem 4.1 we have $W_n(T_\varphi) \subset W_n(L_\varphi)$. By the Hartman–Wintner spectral inclusion theorem [11], $\sigma(L_\varphi) \subset \sigma(T_\varphi)$ and thus $S_n(L_\varphi) \subset S_n(T_\varphi)$. Thus $W_n(L_\varphi) = S_n(L_\varphi) \subset S_n(T_\varphi) \subset W_n(T_\varphi) \subset W_n(L_\varphi)$ and the theorem is proved.

6. Matrix-normal operators. Theorems 5.1 and 5.2 enable us to reduce the problem of finding $W_n(T)$ for the not necessarily normal operator T , to the more tractable problem of finding the n -dimensional matrix range of a normal operator for which we can use Theorem 3.1. Therefore, given a non-normal operator T we can ask when there exists a normal operator S such that $W_n(S) = W_n(T)$ for $n = 1, 2, \dots$.

Definition 6.1. An operator T is matrix-normal if there exists a normal operator S such that $W_n(T) = W_n(S)$ for all n .

Given $T \in \mathcal{L}(\mathcal{H})$, recall that a nonempty closed proper subset of the plane K , which contains $\sigma(T)$, is called a spectral set for the operator T if whenever f is a rational function on K with no poles on K , then the inequality $\|f(T)\| \leq \sup \{|f(z)| : z \in K\}$ holds. We will need a dilation theorem which was proved independently by C. Foias, C. Berger and A. Lebow [13].

Theorem 6.1. *If K is a convex spectral set for $T \in \mathcal{L}(\mathcal{H})$, then there exists a normal operator S defined on a larger Hilbert space $\mathcal{H}_1 \supset \mathcal{H}$ such that*

- (i) $\sigma(S) \subset \partial K$,
- (ii) $T^n x = PS^n x$ ($x \in \mathcal{H}$, $n = 0, 1, 2, \dots$),

where P is the projection of $\mathcal{H}$ onto $\mathcal{H}_1$.

The next theorem gives a sufficient condition for an operator to be matrix-normal.

Theorem 6.2. *Let $T \in \mathcal{L}(\mathcal{H})$. If $W(T)$ is a spectral set for T , then there exists a normal operator S such that $W_n(S) = W_n(T)$ for all n . Moreover we can choose S so that $\sigma(S) \subset \partial W(T)$.*

Proof. Since $\overline{W(T)}$ is compact and convex, we can apply Theorem 5.1 to $\overline{W(T)}$ to obtain a normal operator S with properties (i) and (ii). The map φ from $C^*(S)$ to $C^*(T)$ defined by $\varphi(B) = PBP$ is completely positive and by taking $n = 1$ in (ii) of Theorem 6.1 we see that $\varphi(S) = T$. Arguing as in the proof of Theorem 4.1 we have $W_n(T) \subset W_n(S)$ for all n . Since S is normal and $\sigma(S) \subset \partial(\overline{W(T)})$ we have $\text{co}(\sigma(S)) = \overline{W(S)} \subset \overline{W(T)}$ from which we see that $\overline{W(S + \lambda I)} \subset \overline{W(T + \lambda I)}$ for $\lambda \in \mathbb{C}$ and hence $\|S + \lambda I\| = \sup \{|z + \lambda| : z \in \overline{W(S)}\} \leq \sup \{|z + \lambda| : z \in \overline{W(T)}\} = \|T + \lambda I\|$. This last inequality shows that the map ψ defined by $\alpha T + \beta I \rightarrow \alpha S + \beta I$ is contractive and hence by

Lemma 2.1 ψ has a completely positive extension to $C^*(T)$. Arguing as in Theorem 4.1 we have $W_n(S) \subset W_n(T)$ for all n , and the proof is complete.

Is the converse of Theorem 6.2 true? Explicitly, if T is matrix-normal must $\overline{W(T)}$ be a spectral set? Our previous results have shown that subnormal and Toeplitz operators are matrix-normal and it is easy to see that $\overline{W(T)}$ is a spectral set if T is subnormal. However, for arbitrary Toeplitz operators the answer is not known. Although we cannot show that $\overline{W(T)}$ is a spectral set if T is matrix-normal, we can ask what conditions are imposed on an operator if it is matrix-normal.

Recall that the numerical radius, $|W(A)|$, of an operator A is defined by $|W(A)| = \sup \{|\lambda| : \lambda \in W(A)\}$ and that A is *normaloid* if $|W(A)| = \|A\|$. Also A is *spectraloid* if $r(A) = |W(A)|$. Every normaloid operator is spectraloid [10]. We will need the next lemma which is proved in [14].

Lemma 6.1. *If B and C are compact convex nonempty plane sets and $B \subseteq C$, then there exists a complex number λ such that*

$$(6.1) \quad \sup \{|z + \lambda| : z \in B\} < \sup \{|z + \lambda| : z \in C\}.$$

Theorem 6.3. *If T is matrix-normal, then*

- (i) $T + \lambda I$ is normaloid for all $\lambda \in \mathbf{C}$,
- (ii) $\overline{\text{co}}(\sigma(T)) = \overline{W(T)}$,
- (iii) there exists a normal operator S_1 such that $W_n(S_1) = W_n(T)$ for all n and $\sigma(S_1) \subset \partial \overline{W(T)}$. Moreover there exists a representation π of $C^*(T)$ such that $\pi(T) = S_1$.

Proof. Since T is matrix-normal there exists a normal operator S such that $W_n(T) = W_n(S)$ for all n and therefore for any $\lambda \in \mathbf{C}$ we have $W_n(T + \lambda I) = W_n(S + \lambda I)$ for all n . Thus by Theorem 2.2 we have $\|T + \lambda I\| = \|S + \lambda I\|$ for $\lambda \in \mathbf{C}$. Since $\overline{W(T)} = \overline{W(S)}$ we have

$$(6.2) \quad \|T + \lambda I\| = \|S + \lambda I\| = |W(S + \lambda I)| = |W(T + \lambda I)| \quad \text{for } \lambda \in \mathbf{C},$$

and therefore $T + \lambda I$ is normaloid. Since a normaloid operator is spectraloid we obtain for $\lambda \in \mathbf{C}$

$$(6.3) \quad \|T + \lambda I\| = r(T + \lambda I) = \sup \{|z + \lambda| : z \in \overline{\text{co}}(\sigma(T))\}.$$

However, from (6.2) we have

$$(6.4) \quad \|T + \lambda I\| = \|W(S + \lambda I)\| = \sup \{|z + \lambda| : z \in \overline{W(S)}\}.$$

Thus for $\lambda \in \mathbf{C}$ we obtain

$$(6.5) \quad \sup \{|z + \lambda| : z \in \overline{\text{co}}(\sigma(T))\} = \sup \{|z + \lambda| : z \in \overline{W(S)}\}.$$

We know that $\overline{\text{co}}(\sigma(T)) = \overline{W(T)} = \overline{W(S)}$ and thus by Lemma 6.1 we must have $\overline{\text{co}}(\sigma(T)) = \overline{W(S)} = \overline{W(T)}$, otherwise (5.5) would not hold for all $\lambda \in \mathbf{C}$.

Letting S_1 be a diagonal (and therefore normal) operator such that $\sigma(S_1) = EX(\overline{W(S)})$ (the extreme points of $\overline{W(S)}$) we obtain for $\lambda \in \mathbf{C}$,

$$(6.6) \quad \begin{aligned} \|S_1 + \lambda I\| &= \sup \{ |z + \lambda| : z \in EX(\overline{W(S)}) \} \\ &= \sup \{ |z + \lambda| : z \in \overline{\text{co}}(EX(\overline{W(S)})) = \overline{W(S)} \} = \|S + \lambda I\|. \end{aligned}$$

Since S_1 and S are both normal (6.6) is easily seen to imply that $W_n(S_1) = W_n(S)$ for all n and thus $W_n(T) = W_n(S_1)$ for all n . Also, $\sigma(S_1) = EX(\overline{W(S)}) = EX(\overline{W(T)}) \subset \partial(\overline{W(T)})$.

If [1] Arveson proved that if $\lambda \in \sigma(T) \cap \partial W(T)$, then there exists a representation π of $C^*(T)$ such that $\pi(T) = \lambda$. Now $\overline{\text{co}}(\sigma(T)) = \overline{W(T)} = \overline{W(S)}$ and therefore by Milman's Lemma [8] $EX(\overline{W(S)}) \subset \sigma(T)$ and thus $\sigma(S_1) = EX(\overline{W(S)}) \subset \sigma(T) \cap \partial W(T)$. If $\lambda \in \sigma(S_1)$, denote by π_λ the representation of $C^*(T)$ such that $\pi_\lambda(T) = \lambda$ and let $\pi = \sum_{\lambda \in \sigma(S_1)} \oplus \pi_\lambda$. It is clear from our construction of S_1 that $\pi(T) = S_1$.

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