A PDE in $\mathbf{R}^3$ with Strange Behavior

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Analytic functions of one (or several) complex variables have certain well-known properties; *e.g.*, Liouville's theorem, unique continuation, the maximum modulus principle. It is a natural question to ask what becomes of these properties if, instead of analytic functions, one considers the class of functions annihilated by one (or several) smooth complex vector fields in $\mathbf{R}^n$. The purpose of this note is to discuss a very simple example of a smooth complex vector field in $\mathbf{R}^3$; it illustrates how the behavior of the coefficients in the equation can have a weird influence on the properties mentioned above.

The coordinates in $\mathbf{R}^3$ will be denoted by (t, y_1, y_2) . Let $a(t) \in C^\infty(\mathbf{R}^1)$ be a function $a : \mathbf{R}^1 \rightarrow \mathbf{R}^1$. The operator we consider is

$$L = \frac{\partial}{\partial t} + i \left(\frac{\partial}{\partial y_1} + a(t) \frac{\partial}{\partial y_2} \right), \quad i = (-1)^{1/2}.$$

By a solution of $Lu = 0$ in $\mathbf{R}^3$, we shall mean a sufficiently smooth complex-valued solution, say $u \in C^\infty(\mathbf{R}^3)$.

We use the following terminology: L has the *Liouville property* on $\mathbf{R}^3$ iff every bounded solution of $Lu = 0$ in $\mathbf{R}^3$ is constant. L has the *unique continuation property* on $\mathbf{R}^3$ iff any solution of $Lu = 0$ in $\mathbf{R}^3$, which vanishes on some nonvoid open set, must be identically zero. L satisfies the *maximum modulus principle* on $\mathbf{R}^3$ iff any solution of $Lu = 0$ in $\mathbf{R}^3$, whose modulus has, at some point, a weak maximum relative to $\mathbf{R}^3$, must be constant.

In what follows, let $\psi(t) = \int_0^t a(\tau) d\tau$ and $\Gamma = \text{graph } \psi$. The convex hull of a set A in $\mathbf{R}^2$ will be denoted by $\text{ch } A$.

Theorem. (a) L has the Liouville property on $\mathbf{R}^3 \Leftrightarrow \text{ch } \Gamma = \mathbf{R}^2$.

(b) L has the unique continuation property on $\mathbf{R}^3 \Leftrightarrow a(t)$ is not identically constant.

(c) L satisfies the maximum modulus principle on $\mathbf{R}^3 \Leftrightarrow \Gamma \subset \text{interior ch } \Gamma$.

Remarks. From (a) we see that the Liouville property depends only on the behavior of $a(t)$ at $t = \pm \infty$; *e.g.*, $a(t)$ can be arbitrary in any bounded interval,

and if $\psi(t)$ looks something like $t^2 \cos t$ for $|t|$ sufficiently large, then L will have the Liouville property.

According to (b), the unique continuation property does not hold if $a(t) \equiv a = \text{const}$. But if, in any arbitrarily small interval $|t - t_0| < \epsilon$, the function $a(t)$ is perturbed in any way whatsoever away from the constant value a , then L will have the unique continuation property.

Consider the example where $a(t) = \psi'(t)$, $\psi(t) = t^2(1 + t^2)^{-1} \cos t$. Then $\Gamma \subset \text{interior ch } \Gamma$, so by (c) the maximum modulus principle is valid. But $\text{ch } \Gamma \neq \mathbf{R}^2$. Thus it is possible for L to satisfy the maximum modulus principle on $\mathbf{R}^3$, and yet not have the Liouville property on $\mathbf{R}^3$.

Proof of the theorem. Let $z = x + iy$, $x = (x_1, x_2)$, $y = (y_1, y_2)$ denote the usual coordinates in $\mathbf{C}^2$; we shall identify $t = x_1$ and $\mathbf{R}^2 = \text{Re } \mathbf{C}^2$, so that $\Gamma = \{x \in \mathbf{R}^2 \mid x_2 = \psi(x_1)\}$. For any set $B \subset \mathbf{R}^2$, $\tau B = \{z \in \mathbf{C}^2 \mid x \in B\}$ will denote the tube in $\mathbf{C}^2$ with base B .

The functions $z_1 = t + iy_1$ and $z_2 = \psi(t) + iy_2$ give a maximal set of functionally independent complex characteristic coordinates for L . They provide an imbedding of $\mathbf{R}^3$ into $\mathbf{C}^2$, given by $(t, y_1, y_2) \mapsto (z_1, z_2)$, whose image M is the same as $\tau\Gamma$. The equation $Lu = 0$ on $\mathbf{R}^3$ is just $\bar{\partial}_M u = 0$ on M written out in terms of the coordinates (t, y_1, y_2) for M , where $\bar{\partial}_M$ is the tangential Cauchy-Riemann operator to M (see [1] for a detailed explanation of these concepts). Thus a solution u of $Lu = 0$ in $\mathbf{R}^3$ is what is known as a *CR* function on the tube manifold $M = \tau\Gamma$.

(a) Suppose first that $\text{ch } \Gamma = \mathbf{R}^2$. Introduce the sets $\Gamma^+ = \{x \in \mathbf{R}^2 \mid x_2 \geq \psi(x_1)\}$, $\Gamma^- = \{x \in \mathbf{R}^2 \mid x_2 \leq \psi(x_1)\}$, $U^+ = \tau\Gamma^+$, $U^- = \tau\Gamma^-$. Then $U^+ \cup U^- = \mathbf{C}^2$ and $U^+ \cap U^- = M$. Since $\Gamma^+ \subset \text{ch } \Gamma$, we have $\Gamma^+ \subset \text{ch } \Gamma^-$; invariance of the geometry with respect to translations in the x_2 -directions shows that $\Gamma^+ \subset \text{ch } \Gamma^{-\circ}$ ($\Gamma^{-\circ}$ denotes the interior of Γ^-). Therefore, $U^+ \subset \text{ch } U^{-\circ}$, and similarly $U^- \subset \text{ch } U^{+\circ}$. It follows (from Theorem 2.5.10, p. 41 of [4]) that every holomorphic function in $U^{-\circ}$ extends as a holomorphic function to $\mathbf{C}^2$, and similarly holomorphic functions in $U^{+\circ}$ extend to holomorphic functions in $\mathbf{C}^2$. Hence (see Section 4.5 of [2]) every *CR* function u on M has a unique extension $\tilde{u}$ to $\mathbf{C}^2$ which is an entire function of two complex variables. Using the fact that *CR* functions on M form an algebra, and they all extend to the same set, it is easy to show (see [3] for a proof) that the maximum modulus of $\tilde{u}$ on $\mathbf{C}^2$ cannot exceed the maximum modulus of u on M . Therefore $\tilde{u}$ is bounded if u is bounded, so by the usual Liouville theorem, $\tilde{u}$, and hence also u , is constant.

Next consider the case where $\text{ch } \Gamma \neq \mathbf{R}^2$. Since $\text{ch } \Gamma$ is not void, it must contain a boundary point p , and there is a supporting hyperplane P passing through p . Let the equation for P be $\ell(x) = ax_1 + bx_2 + c = 0$; we may assume that $\text{ch } \Gamma$ is contained in the closed half-space $\ell(x) \leq 0$. Consider the function $\tilde{u}(z) = [az_1 + bz_2 + c - 1]^{-1}$. On M it is holomorphic, non-constant, and has its modulus bounded by one. Therefore the function $u = \tilde{u}|_M$ provides a counterexample to the Liouville property for L on $\mathbf{R}^3$.

(b) Suppose that $a(t)$ is not identically constant. Then Γ contains three dis-

tinuous points that are not collinear, so $\text{ch } \Gamma$ has a nonempty convex connected interior $(\text{ch } \Gamma)^\circ$. Therefore $U = \tau \text{ ch } \Gamma$ is a connected set whose nonempty interior $U^\circ = \tau(\text{ch } \Gamma)^\circ$ is a connected domain of holomorphy. Introduce the sets U^+ and U^- as in part (a). It then follows (by an easy modification of the type of argument used to prove Proposition 4.3 of [2]) that a CR function u on M has an extension $\tilde{u}$ that is holomorphic in U° . But if u vanishes on some open subset of M , it follows (because U° is connected) that $\tilde{u}$, and hence also u , must be identically zero.

If $a(t) \equiv \text{constant} = a$, consider the nonsingular change of variables $x_1 = \xi_1$, $y_1 = \eta_1$, $y_2 = \eta_2 + a\eta_1$ on $\mathbf{R}^3$. In terms of these new coordinates, L becomes $\partial/\partial\xi_1 + i\partial/\partial\eta_1$. Thus solutions of $Lu = 0$ are just holomorphic functions of $\zeta = \xi_1 + i\eta_1$. Take a smooth function u of η_2 only; it can satisfy $Lu = 0$, vanish in some open set, and not be constant.

(c) Suppose that $\Gamma \subset (\text{ch } \Gamma)^\circ$. Then $(\text{ch } \Gamma)^\circ$ is not void. Hence, as in part (b) above, $U^\circ = \tau(\text{ch } \Gamma)^\circ$ is a nonvoid open connected domain of holomorphy. The same argument that was used before shows that a CR function u on M has an extension $\tilde{u}$ that is holomorphic in U° . As before (again, see [3] for a proof), the maximum modulus of $\tilde{u}$ on U° cannot exceed the maximum modulus of u on M . Hence, if $|u|$ has a maximum at a point $p \in M$, then $|\tilde{u}|$ also has a maximum at p . Therefore, since U° is connected, by the usual maximum modulus principle in $\mathbf{C}^2$, the function $\tilde{u}$, and hence also u , must be a constant.

If $\Gamma \not\subset (\text{ch } \Gamma)^\circ$, then Γ must contain some point p which is a boundary point of $\text{ch } \Gamma$. There is a supporting hyperplane P for $\text{ch } \Gamma$ passing through p . Let $\ell(x)$ and the function $\tilde{u}(z)$ be as in part (a). Then $u = \tilde{u}|_M$ is not constant, satisfies $Lu = 0$, and its modulus has a maximum at p . This provides a counterexample to the maximum modulus principle.

The proof of the theorem is complete.

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