Strong Reducibility of Operators

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In this paper all operators are bounded operators on complex Hilbert spaces. An operator is called *strongly reducible* if every similarity of it has a nontrivial reducing subspace. This property of an operator is antipodal to the property of strong irreducibility, in which, every similarity is irreducible. The property of strong irreducibility is possessed by unicellular operators and most weighted shifts [6]. For strong reducibility, one can easily see that any finite rank operator on an infinite dimensional space, as well as any operator on an inseparable space, is strongly reducible. Nontrivial examples on separable spaces are somewhat elusive. We shall show that any operator satisfying a second degree polynomial on a space of dimension greater than two is strongly reducible and that this does not extend to operators satisfying third degree polynomials.

While every operator on an inseparable space is reducible, hence strongly reducible, there are examples of operators on a separable space which are reducible but not strongly reducible. Specifically, we shall show that on a separable space, every normal operator without point spectrum is similar to an irreducible operator. An interesting corollary of this result is that a unitary operator, without point spectrum, is similar to a completely nonunitary operator, however, not necessarily a contraction operator.

The class of binormal operators was introduced by A. Brown and he has given the structure and unitary equivalences for them [3]. Using a result of I. Kaplansky, Brown shows that an operator satisfying a second degree polynomial is binormal. Moreover Brown's structure theorem shows that an operator is binormal if and only if the W^* algebra generated by the operator is of type $I_1 \oplus I_2$, where either summand may be absent. From this we obtain the following result.

Theorem 1. *Let A be an operator on a Hilbert space of dimension greater than two. If A satisfies a monic second degree complex polynomial, then A is strongly reducible.*

Proof. First it is clear that if A satisfies a second degree polynomial, then any similarity of A also satisfies the polynomial. Thus A and every similarity

of A are binormal. Finally, from Brown's structure theorem it is easy to see that all binormal operators on spaces of dimension greater than two have reducing subspaces.

We remark that if A satisfies a third, or higher, degree polynomial, then A need not be strongly reducible. One needs only consider the operator

$$A = \begin{bmatrix} 0 & I & T \\ 0 & 0 & I \\ 0 & 0 & 0 \end{bmatrix}$$

on the space $H = K \oplus K \oplus K$. If T is any irreducible operator, then the operator A is irreducible and since A is nilpotent of order three it satisfies a polynomial of degree three.

The theorem also illustrates a class of operators which remains binormal under every similarity. We shall show later, in a proposition that this is not always the case.

Similarity preserves many properties of an operator including properties of the lattice of invariant subspaces; however it does not necessarily preserve reducibility. We shall show that similarity can destroy the property of reducibility even when the operator has an extensive lattice of reducing subspaces. Before presenting this result, we need a technical lemma concerning normal operators.

If A is an operator on H and x is a vector so that the smallest invariant subspace of A containing x is H , then x is called a *cyclic vector* for A . The smallest invariant subspace of A containing x can be expressed as $\text{span} \{A^n x \mid n = 0, 1, 2, \dots\}$. We shall denote by $R(A)$, the W^* algebra generated by A and I . If A is an operator on H and x is a vector so that the smallest subspace invariant for all the operators in $R(A)$ is H , then x is called a *cyclic vector* for $R(A)$. Whenever N is normal, J. Bram has shown that N has a cyclic vector if and only if $R(N)$ has a cyclic vector [2]. In case x is cyclic for $R(A)$, it is equivalent that H is the smallest reducing space for A that contains x , or that

$$H = \text{span} \{A^{n_1} A^{*m_1} \dots A^{n_k} A^{*m_k} x \mid n_i, m_i, k \in \{0, 1, 2, \dots\}\}.$$

Lemma. *Let N be a normal operator on a separable space H . There is an orthonormal basis $\{y_{i,j}\}$ for H such that, if we let $H_i = \text{span}_i \{y_{i,j}\}$, then*

- a) H_i reduces N , and
- b) $y_{i,1}$ is a cyclic vector for $N \mid H_i$.

Proof. This result follows from the multiplicity theory for normal operators; however an elementary proof can also be given using induction and the result of J. Bram cited above.

We remark that the index i may be finite; however if N has no point spectrum, then for each i the index j must be infinite. Moreover it is clear that this cyclic decomposition, without further conditions, is in general not unique.

Theorem 2. *Let N be a normal operator on a separable Hilbert space. If N has empty point spectrum, then N is similar to an irreducible operator.*

Proof. Using the lemma, we decompose N into a countable sum of cyclic pieces. Thus $H = \Sigma \oplus H_i$ and $N = \Sigma \oplus N_i$, where N_i is N restricted to the reducing space H_i , and moreover N_i has the cyclic vector y_{i1} on H_i , where $\{y_{ii}\}_i$ is an orthonormal basis of H_i . If we let

$$y_n = (n^{1/2})^{-1}(y_{1,n}, y_{2,n-1}, \dots, y_{n,1}, 0, \dots),$$

then the orthonormal sequence $\{y_i\}$ has the property that N and the set $\{y_1, \dots, y_i\}$ generate $H_1 \oplus H_2 \oplus \dots \oplus H_i \oplus 0 \dots$. Let C be the operator defined on H by $Cy_i = 2^{-i}y_i$ and $C \mid (\text{span } \{y_i\})^\perp = 0$. Then C is a compact self-adjoint operator such that $I + C = S$ is invertible on H . Whenever the cyclic decomposition is finite, that is, $H = \sum_{i=1}^m \oplus H_i$, then y_n is defined only for $n \leq m$ and C is a finite rank operator.

We shall show that $SNS^{-1} = T$ is an irreducible operator. If we let M_0 be a reducing subspace of T , we shall show that $M_0 = \{0\}$ or $M_0 = H$. Since $SNS^{-1}M_0 \subset M_0$, we have $NS^{-1}M_0 \subset S^{-1}M_0$ and similarly $NS^{-1}(M_0^\perp) \subset S^{-1}(M_0^\perp)$. Thus $S^{-1}M_0 \equiv M_1$ and $S^{-1}(M_0^\perp) \equiv M_2$ are complementary invariant subspaces of N . It follows, by using the Putnam-Fuglede theorem, that M_1 and M_2 in fact reduce N [5]. Since the algebra $R(N)$ contains no minimal projections, there exists a sequence $\{U_n\}$ of unitary operators in $R(N)$ so that U_n converges to zero weakly [4].

Because M_i , $i = 1, 2$, also reduce each U_n and SM_1 is perpendicular to SM_2 , it follows that $SU_nM_1 \perp SU_nM_2$. This last relationship is equivalent to $U_n^*S^*SU_nM_1 \perp M_2$; thus $U_n^*S^2U_nM_1 \perp M_2$. Now $S^2 = (I + C)(I + C) = I + 2C + C^2$ and since $2C + C^2$ is a compact operator, the operators $(2C + C^2)U_n$ converge in the strong topology to zero. Thus $U_n^*S^2U_nM_1 \perp M_2$, implies that $(U_n^*IU_n + U_n^*(2C + C^2)U_n)M_1 \perp M_2$ and taking the limit as n goes to infinity, we have that $M_1 \perp M_2$. Recall that $SM_1 \perp SM_2$; hence $S^*SM_1 \perp M_2$ or $(I + 2C + C^2)M_1 \perp M_2$. Since $M_1 \perp M_2$, it follows that $(2C + C^2)M_1 \perp M_2$ and M_i , $i = 1, 2$, reduces $2C + C^2$.

The compact self-adjoint operator $2C + C^2$ has simple point spectrum and hence if M_1 reduces it, then M_1 or its orthonormal complement M_2 must contain each eigenvector of $2C + C^2$. This is equivalent to noting that the orthogonal projection on the eigenspace for an eigenvalue of a self-adjoint operator is a central projection in $R(2C + C^2)$. Therefore, either M_1 or M_2 must contain the vector y_i ($i = 1, 2, \dots$). Let us assume that $y_1 \in M_1$ (the following argument will be symmetric if $y_1 \in M_2$). If $\{y_i\} \subset M_1$, then $M_1 = H$, since M_1 is invariant under N and N acting on the set $\{y_i\}$ generates H . Otherwise there is a smallest k such that $\{y_1, \dots, y_k\} \subset M_1$ and $y_{k+1} \notin M_1$. This is impossible since N acting on $\{y_1, \dots, y_k\}$ is $H_1 \oplus \dots \oplus H_k \oplus 0 \oplus \dots$ and is in M_1 , while y_{k+1} has to be perpendicular to $H_1 \oplus \dots \oplus H_k \oplus 0 \oplus \dots$. Therefore either $M_1 = \{0\}$ or $M_1 = H$; hence $M_0 = \{0\}$ or $M_0 = H$, thus completing our proof.

The similarity operator was of the form $I + C$, where C was a compact self-adjoint operator. Moreover, if N could be decomposed into a finite number of cyclic pieces, then C can be taken to be a self-adjoint finite rank operator. Specifically, if N has a cyclic vector, then C can be just the self-adjoint projection onto the subspace it spans.

P. Halmos has shown that the set of irreducible operators is dense in the space of bounded operators with the uniform operator topology. In the proof of Theorem 2, if we define $S_t = I + tC$ ($0 < t \leq 1$), then it follows that $S_t N S_t^{-1}$ converges uniformly to N and each $S_t N S_t^{-1}$ is an irreducible operator. Thus N is not only the uniform limit of irreducible operators, but the uniform limit of irreducible operators each one of which is similar to N . This proves the following corollary.

Corollary 1. *Every normal operator, without point spectrum, on a separable Hilbert space is the uniform limit of irreducible operators each similar to N .*

This cannot of course be generalized to all operators. Moreover the existence of a nontrivial class of strongly reducible operators, via Theorem 1, makes the generalization of Corollary 1 a more interesting question.

An operator is called completely nonunitary (nonnormal, nonself-adjoint), if it has no non-trivial reducing subspaces on which it is unitary (normal, self-adjoint). Since an irreducible operator is completely nonnormal and hence nonunitary and nonself-adjoint, a corollary of Theorem 2 is the following result.

Corollary 2. *Let U be a unitary operator on a separable Hilbert space. If U has no point spectrum, then U is similar to a completely nonunitary operator.*

It must be remarked that the completely nonunitary operator to which U is similar is not in general a contraction. For if that were the case, then we could conclude that $\{U^n\}$ converges to zero weakly whenever U is unitary and has no point spectrum. This would imply that every Borel measure without atoms would have Fourier coefficients which tend to zero as $|n|$ tends to infinity. Counter examples to this assertion are given by Theorem V.7.5 in [7].

Finally, we have the following obvious generalization of Theorem 2.

Corollary 3. *Let A be an operator and f an analytic function on a domain $\mathcal{D} \supset \sigma(A)$. If $f(A) = N$, where N is a normal operator with empty point spectrum, then A is similar to an irreducible operator.*

Theorem 1 describes a class of binormal operators which remain binormal under every similarity, while Theorem 2 states that this is not always the case, as the operators described in Theorem 2 have similarities which are not binormal. The following proposition begins to explore this question for operators in between the antipodal cases already given. H. Behncke has recently shown that if $A^2 + aA = N$, where N is a normal operator, then A is binormal [1]. For this class of binormal operators we can give a sufficient condition for when a similarity is not binormal.

Proposition. *If A is a binormal operator of the form $A^2 + aA = N$, where the span of the eigenspaces for N is not H , then A is similar to an operator which is not binormal.*

Proof. Every binormal operator restricted to a reducing subspace is binormal. Let $I - P$ be the projection on the subspace span by the eigenvectors of N . Thus there is a map S on PH so that $S^{-1}(N/P)S$ is an irreducible operator. If we extend S to H by making it the identity map on $I - P$, we then can form a similarity of A which is not binormal.

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