Tangent Cones to Real Analytic Varieties

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Introduction. The usual notion of “tangent space” at a simple point of a real or complex analytic variety has been extended to “tangent cone” at an arbitrary point of the variety, simple or not (see, *e.g.*, [7, §§3–7], or [8, §§8–10]); in the complex case, Whitney in [7] and [8] has worked out a number of important properties of these tangent cones. We show in this paper that although these properties may not hold at certain points of a real analytic variety (Examples 1.6a–c), they must hold at each point off a real analytic subvariety of codimension ≥ 2 in the variety (Theorem 1.8).

1. Definitions; examples.

Definition 1.1. Let H be an open set in complex n -space, $\mathbf{C}^n$. A subset W of H is a *complex-analytic variety in H* if W is closed, and if at any point $p \in W$, there is a $\mathbf{C}^n$ -open neighborhood U_p such that $W \cap U_p$ is the set of common zeros of a collection of functions complex-analytic in U_p . A set V of real n -space $\mathbf{R}^n$ is a *real $\mathbf{C}$ -analytic variety in $\mathbf{R}^n$* if it is the real part of some variety W complex-analytic in some $\mathbf{C}^n$ -open neighborhood of $\mathbf{R}^n$.

We shall refer to these simply as “*real-analytic varieties*”, or “*real varieties*”.

Convention. Throughout this paper, the letter “ V ” will generally refer to the real case, the letter “ W ” to the complex case; the letters “ X ” and “ Y ” will be used when consideration extends to both cases.

Notation 1.2. We denote the germ at p of a real or complex variety X by $\tilde{X}_p$.

For basic properties of real $\mathbf{C}$ -analytic varieties and germs, see [11] or [4].

Definition 1.3. Let $\tilde{X}_p$ be a real or complex analytic germ. The *geometric tangent cone to $\tilde{X}_p$* , denoted $T_p(\tilde{X}_p)$, is the set of all vectors v satisfying the following condition:

There is a sequence of points $\{p_i\}$, $p_i \in X \setminus \{p\}$, $p_i \rightarrow p$, and a sequence of real numbers $\{a_i\}$ such that $v = \lim_{i \rightarrow \infty} a_i(p_i - p)$.

Remark. $T_o(\tilde{V}_p)$ is clearly an $\mathbf{R}$ -homogeneous set in $\mathbf{R}^n$, $T_o(\tilde{W}_p)$, a $\mathbf{C}$ -homogeneous set in $\mathbf{C}^n$; $T_o(\tilde{X}_p)$ coincides with the usual "tangent space" if X is a manifold in a neighborhood of p .

For $\tilde{W}_p (\subseteq (\mathbf{C}^n)^{\sim}_p)$, Whitney has proved the following basic facts (see [8, §§8–10]):

(1.4.1) $T_o(\tilde{W}_p)$ is a (homogeneous) complex-algebraic variety in $\mathbf{C}^n$.

(1.4.2) Let $R_p = \mathbf{C}\{x_1, \dots, x_n\}$ be the local ring of all functions analytic at $p \in \mathbf{C}^n$, and let $I(\tilde{W}_p) = \{f_{\alpha,p}\}$ be the ideal of functions of R_p vanishing on $\tilde{W}_p$. Let $f_{\alpha,p}^t$ be the homogeneous form of lowest degree of $f_{\alpha,p}$ when $f_{\alpha,p}$ is expanded about p . Then the homogeneous ideal $I(\tilde{W}_p)^t$ of R_p generated by the $f_{\alpha,p}^t$ defines the homogeneous variety $T_o(\tilde{W}_p)$. (In particular, if $W(f)$ is the complex hypersurface defined by f , then $T_o(W(f)^{\sim}_p) = W(f_p^t)$.)

(1.4.3) With dimension "dim" as defined below,

$$\dim T_o(\tilde{W}_p) = \dim (\tilde{W}_p);$$

for any p , if $\tilde{W}_p$ has pure dimension, so does $T_o(\tilde{W}_p)$.

Definition 1.5. For any real or complex variety X , $\dim \tilde{X}_p$ is the highest dimension of manifold points of X arbitrarily near p . $\dim X = \max_{p \in X} \dim \tilde{X}_p$.

Notation 1.6. For any real (or complex) germ $\tilde{X}_p$, the initial-form ideal $I(\tilde{X}_p)^t$ defines a real (or complex) homogeneous algebraic variety. We call this the *algebraic tangent cone* to $\tilde{X}_p$, and denote it by $T_a(\tilde{X}_p)$. If it is clear from context that $T_a(\tilde{X}_p) = T_o(\tilde{X}_p)$, we shall write $T(\tilde{X}_p)$. (This of course always occurs in the complex case, by (1.4.2).)

The tangent cones to real-analytic germs can have quite pathological behavior:

- (a) $T_o(\tilde{V}_p)$ may not be real algebraic (or analytic) even if $\tilde{V}_p$ is algebraic;
- (b) $T_o(\tilde{V}_p)$ may be algebraic, but not defined by $I(\tilde{V}_p)^t$;
- (c) $T_o(\tilde{V}_p)$ may be algebraic, defined by $I(\tilde{V}_p)^t$, but $\dim T(\tilde{V}_p) \neq \dim \tilde{V}_p$.

Example 1.7(a). Let $V \subseteq \mathbf{R}^3$ be defined by $f(x, y, z) = y^2z - ((x^2 + z^4) \cdot (x - z)) = 0$. It is of pure dimension 2 in $\mathbf{R}^3$; the part of this surface in any plane $z = c$ (c , a nonzero constant), consists of a single branch; it opens to the right for $c > 0$, to the left for $c < 0$. Since z^4 becomes small compared to z as z approaches 0, for $|c|$ small, the shapes of these branches approach that of the branch in $y^2 - x^2(x - 1) = 0$. (The origin is an isolated point of this last variety.) One easily sees that $T_o(\tilde{V}_0)$ consists of the two-dimensional part of the homogeneous surface V' : $y^2z - x^2(x - z) = 0$. However this two-dimensional part is *not* a real analytic variety, since V' (which is real-irreducible—see Definition 2.1) contains also the z -axis.

Example 1.7(b). Let $V \subseteq \mathbf{R}^3$ be defined by $g(x, y, z) = x^2 + y^2z^2 - z^5 = 0$; it consists of the y -axis together with a funnel-shaped part symmetric about the z -axis. The sections of the funnel-part lying in the planes $z =$ positive constant, are ellipses whose shapes become very elongated as $z \rightarrow 0$; their sizes

decrease so fast as $z \rightarrow 0$, that the z -axis is the only real line geometrically tangent to the funnel at the origin 0. (The funnel lies in a region bounded by the real "cusp of revolution" $x^2 + y^2 - z^3 = 0$, and clearly the geometric tangent cone of this cusp of revolution is just the z -axis.) Hence $T_o(\tilde{V}_0) = (y\text{-axis}) \cup (z\text{-axis})$, while $T_a(\tilde{V}_0) = (y, z)\text{-plane}$.

Example 1.7(c). The above "cusp of revolution" $h(x, y, z) = x^2 + y^2 - z^3 = 0$; the geometric tangent cone at 0 is the z -axis and is defined by $I(\tilde{V}(h))_0^\dagger = (x^2 + y^2)$, yet the dimensions of the variety and tangent cone do not agree at 0.

Remark. As we shall see in Lemma 2.5, we always have $T_o(\tilde{V}_v) \subseteq T_a(\tilde{V}_v)$. In Example 1.7(a), $\dim (T_a(\tilde{V}_0) \setminus T_o(\tilde{V}_0)) < \dim \tilde{V}_0$. However, T_a and T_o can differ by much more than this; for instance let V be defined by $(xy - z^2)^2 - xz^5 = 0$. $T_a(\tilde{V}_0)$ is the "homogenized hyperbola" $xy = z^2$; $T_o(\tilde{V}_0)$ is contained in $T_a(\tilde{V}_0)$, but since within V , x and z must have the same sign, at least "half of the points" of $T_a(\tilde{V}_0)$ (i.e., where x and z have opposite signs), cannot be in $T_o(\tilde{V}_0)$. On the other hand, the part of V lying in any plane of $\mathbf{R}^3$ defined by $ax + by = 0$ ($ab > 0$) is a curve, and its tangent line at $(0, 0)$ lies in $T_o(\tilde{V}_0)$. Hence the part of V near the origin looks like, and $T_o(\tilde{V}_0)$ is equal to, the homogenization of only one branch of a hyperbola. Hence $\dim (T_a(\tilde{V}_0) \setminus T_o(\tilde{V}_0)) = \dim \tilde{V}_0$.

Note that in all these examples the pathological points form a set of co-dimension 2 in the variety, since at all points away from the origin, each variety is a real analytic manifold.

We now state the main result to be proved in this paper.

Theorem 1.8. *Let V be a real analytic variety in $\mathbf{R}^n$. There is a real analytic variety A_V of codimension ≥ 2 in V such that at each point $p \in V \setminus A_V$,*

(1.8.1) $T_o(\tilde{V}_p)$ is a real homogeneous algebraic variety in $\mathbf{R}^n$;

(1.8.2) Let $\mathbf{R}\{x_1, \dots, x_n\}_p$ be the ring of functions real-analytic at $p \in \mathbf{R}^n$; $T_o(\tilde{V}_p)$ is defined by the ideal $I(\tilde{V}_p) \subseteq \mathbf{R}\{x_1, \dots, x_n\}_p$ —i.e., $T_o(\tilde{V}_p) = T_a(\tilde{V}_p) = T(\tilde{V}_p)$;

(1.8.3) $\dim T(\tilde{V}_p) = \dim \tilde{V}_p$.

The examples of (1.6) show we cannot in general improve the above to higher codimension.

In the next section we prove this theorem for real analytic space curves (i.e., that (1.8.1)–(1.8.3) hold at each point of a curve). In §3 we show that for a complex-analytic r -dimensional variety $W \subseteq H \subseteq \mathbf{C}^n$, at each point $p \in W$ off a subvariety of codimension ≥ 2 in W , there is an analytic $(r - 1)$ -submanifold M and a subcurve C , so that

$$T(\tilde{W}_p) = T(\tilde{M}_p) \times T(\tilde{C}_p).$$

Then in §§4 and 5 we refine the method of proof of this to obtain an analogous statement for arbitrary real r -varieties; from this we will obtain (1.8.1)–(1.8.3).

2. $T_a = T_o$ for real curves. We briefly recall a few definitions and basic facts about analytic germs which will be used in the sequel (see [4, Chap. V]).

Definition 2.1. Let $\tilde{X}_p$ be real (resp. complex) analytic; it is *real* (resp. *complex*) *irreducible* if $\tilde{X}_p = (Y_1)_p \cup (Y_2)_p \Rightarrow \tilde{X}_p = (Y_1)_p$ or $\tilde{X}_p = (Y_2)_p$.

Lemma 2.2. Any real (resp. complex) $\tilde{X}_p$ is the finite union $(X_1)_p \cup \dots \cup (X_i)_p$ of real (resp. complex) irreducible germs, where no containment relations $(X_i)_p \subseteq (X_j)_p$ hold (i.e., the decomposition is irredundant); such a union is unique up to order.

Lemma 2.3. Let $\tilde{V}_p$ be real analytic. There exists a unique smallest complex-analytic germ $(\tilde{V}_p)^*$ containing $\tilde{V}_p$. Let $\mathbf{R}\{x_1, \dots, x_n\}_p$ (resp. $\mathbf{C}\{x_1, \dots, x_n\}_p$) be the ring of functions real (resp. complex) analytic at $p \in \mathbf{R}^n$ (resp. $p \in \mathbf{C}^n$). If $I(\tilde{V}_p) \subseteq \mathbf{R}\{x_1, \dots, x_n\}_p$ is the ideal of $\tilde{V}_p$, then the set $I(\tilde{V}_p)$ in $\mathbf{C}\{x_1, \dots, x_n\}_p$ defines $(\tilde{V}_p)^*$. (It is therefore easily seen that $\tilde{V}_p = \text{Re}((\tilde{V}_p)^*)$. $\tilde{V}_p$ is irreducible iff $(\tilde{V}_p)^*$ is. However $\tilde{W}_p \subseteq (\mathbf{C}^n)_p$ may be irreducible, but not $\text{Re}(\tilde{W}_p)$.)

We also note the following global form of part of the above.

Lemma 2.4. Let V be real-analytic in $\mathbf{R}^n$. Then there is a unique smallest variety V^* ($V \subseteq V^*$) complex-analytic in a neighborhood of $\mathbf{R}^n$ in $\mathbf{C}^n$; also, (real) $\dim V = (\text{complex}) \dim V^*$. We call V^* the complexification of V (see [11, p. 132–3]).

Lemma 2.5. Let $\tilde{V}_p \subseteq (\mathbf{R}^n)_p$ be real analytic. Then

- (a) $T_a(\tilde{V}_p) = \text{Re}(T_a(\tilde{V}_p)^*)$,
- (b) $T_o(\tilde{V}_p) \subseteq T_a(\tilde{V}_p)$.

Proof. (a) follows easily from Theorem 2.3;

(b) Proof of the complex case (in [7, Theorem 5.8]) works also for the real case.

Lemma 2.6. In both the real and complex case:

- (a) $T_o(\tilde{X}_p \cup \tilde{Y}_p) = T_o(\tilde{X}_p) \cup T_o(\tilde{Y}_p)$;
- (b) $T_a(\tilde{X}_p \cup \tilde{Y}_p) = T_a(\tilde{X}_p) \cup T_a(\tilde{Y}_p)$.

Proof. (a) is clear from the definitions;

(b) in the complex case is immediate since $T_o = T_a$ (from (1.4.2)), and in the real case, it follows from Lemma 2.5(a).

Lemma 2.7. Let $\tilde{W}_p \subseteq (\mathbf{C}^n)_p$ be complex-analytic, and suppose $\tilde{L}_p, \tilde{N}_p$ ($\subseteq (\mathbf{C}^n)_p$) are complex-linear of dimensions s and $n - s$ respectively, and such that $\tilde{L}_p \cap \tilde{N}_p = \{0\}$. Let π_L be the natural projection of $(\mathbf{C}^n)_p$ onto $\tilde{N}_p$ along $\tilde{L}_p$. If $\tilde{V}_p \cap \tilde{N}_p = \{p\}$, then

- (a) $\pi_L(\tilde{W}_p)$ is analytic in $\tilde{N}_p$;
- (b) if $\tilde{W}_p$ is irreducible, so is $\pi_L(\tilde{W}_p)$.

Proof. (a) is just the “projection theorem” (see, e.g., [8, Lemma 3.4(f)]).

For (b), if $\pi(\tilde{W}_p)$ were the proper union of subgerms $(W_1)_p^\sim$ and $(W_2)_p^\sim$, the union $\tilde{W}_p = (\pi_L^{-1}((W_1)_p^\sim) \cap \tilde{W}_p) \cup (\pi_L^{-1}((W_2)_p^\sim) \cap \tilde{W}_p)$ would be proper, too.

The main result we prove in this section is

Theorem 2.8. *Let $\tilde{C}_p \subseteq (\mathbb{R}^n)_p^\sim$ be real analytic, $\dim \tilde{C}_p = 1$. Then*

$$T_p(\tilde{C}_p) = T_a(\tilde{C}_p).$$

Proof. Without loss of generality, take $p = 0$. We may assume $\tilde{C}_0$ is irreducible, for if the theorem holds for irreducibles, then the existence of a decomposition $\tilde{C}_0 = \bigcup_i (C_i)_0^\sim$ into irreducibles implies (using Lemma 2.6)

$$T_p(\bigcup_i (C_i)_0^\sim) = \bigcup_i T_p((C_i)_0^\sim) = \bigcup_i T_a((C_i)_0^\sim) = T_a(\bigcup_i (C_i)_0^\sim).$$

We begin with $n = 2$ —i.e., $\tilde{C}_0 \subseteq (\mathbb{R}^2)_0^\sim$. Certainly $T_p(\tilde{C}_0) \subseteq T_a(\tilde{C}_0)$ by Lemma 2.5(b). For the reverse inclusion, note that $(\tilde{C}_0)^*$ is irreducible of complex dimension one. There is therefore exactly one complex line tangent to this place ([6, Theorem 5.7]). But since also $T_p(\tilde{C}_0)$ is at least a real line, Lemma 2.5(a) then implies $T_a(\tilde{C}_0) \subseteq T_p(\tilde{C}_0)$.

We now prove the theorem for arbitrary n . First note that for $\tilde{C}_0$ real-irreducible, $T_p(\tilde{C}_0)$ is a real line since $T((\tilde{C}_0)^*)$ is a complex line. (If $T((\tilde{C}_0)^*)$ consisted of at least two complex lines, one could choose transverse $(n-2)$ - and 2-dimensional linear germs $\tilde{L}_0$ and $\tilde{N}_0$ respectively, so that the plane germ $\pi_L((\tilde{C}_0)^*) \subset \tilde{N}_0$ would be irreducible with at least two distinct tangent lines. As we have just seen, this is not possible.)

Therefore let $T_p(\tilde{C}_0) = \ell$, and let ℓ' be any real line in $T_a(\tilde{C}_0)$; we show $\ell = \ell'$. If $\ell \neq \ell'$, let P be the real 2-space containing ℓ and ℓ' , and Q , an $(n-2)$ -space transverse to P , so that $(\tilde{C}_0)^* \cap (Q)_0^* = 0$. Then $\pi_{Q*}((\tilde{C}_0)^*)$ is a plane germ which is irreducible by Lemma 2.7(b). Therefore

$$T_a(\pi_{Q*}(\tilde{C}_0)^*) = T_p(\pi_{Q*}(\tilde{C}_0)^*) = \pi_{Q*}(\ell^*) = \ell^*$$

from which one gets

$$T_a(\pi_Q(\tilde{C}_0)) = \ell.$$

Now with respect to appropriate co-ordinates, the function f , where $(f) = I_p(\pi_Q(\tilde{C}_0)) \subset \mathbb{C}\{x_1, x_2\}$, defines the cylindrization $\tilde{H}_0 \subseteq (\mathbb{C}^n)_0^\sim$ of $\pi_Q(\tilde{C}_0)$ along Q (and therefore also the cylindrization of $\tilde{C}_0$ along Q). The algebraic tangent cone of this “hypersurface germ” is clearly just the corresponding cylindrization of $T_a(\pi_Q(\tilde{C}_0))$; that is, $T_a(\tilde{H}_0) = (f_0^*)$ is just the hyperplane containing ℓ and Q . Since $\tilde{C}_0 \subseteq \tilde{H}_0$, we have $T_a(\tilde{C}_0) \subseteq T_a(\tilde{H}_0)$; therefore $\ell' \notin T_a(\tilde{C}_0)$, as required.

3. A product theorem for tangent cones. We now consider real varieties of higher dimension. One can show in a relatively straightforward way that for germs $(V_1)_p^\sim \subseteq (\mathbb{R}^m)_p^\sim$ and $(V_2)_q^\sim \subseteq (\mathbb{R}^n)_q^\sim$,

$$T_a((V_1)_p^\sim \times (V_2)_q^\sim) = T_a((V_1)_p^\sim) \times T_a((V_2)_q^\sim);$$

similarly for T_p . If we knew $T_a = T_p$ for $(V_1)_p^\sim$ and $(V_2)_q^\sim$, we would then have $T_a = T_p$ for the product germ. In particular, using the results of the last section, if $\tilde{C}_p$ is one-dimensional and $\tilde{M}_p$ is a real manifold of dimension $r - 1$, if $\tilde{V}_p = \tilde{C}_p \times \tilde{M}_p$, then $T_a(\tilde{V}_p) = T_p(\tilde{V}_p)$, thus proving in this case both (1.8.1) and (1.8.2).

Now surely such a “1 by $r - 1$ ” analytic decomposition cannot in general be made about each point, as shown by the examples of (1.7). In fact, the set of points where it cannot be made may form a subvariety of codimension one. This occurs, for instance, on any analytic r -variety having a local analytic invariant not constant on any analytic sub- $(r - 1)$ -manifold. The hypersurface $V(xy(y - x)(y - (3 + z)x)) \subseteq \mathbf{R}^3$ is an example, due to Whitney. The set of non-manifold points is the z -axis, having codimension one in V . The six cross-ratios of the four real 2-planes comprising $T_a(\tilde{V}_z)$ ($|z|$ small), are analytic invariants defined on the z -axis, but some of them vary with z . (For details, see [7, Example 13.2].)

The main point is this: by broadening the notion of “analytic product”, we will be able to find conditions holding off a subvariety of codimension ≥ 2 in V so the above line of reasoning can be made to work.

In this section we prove a preliminary result holding in the complex case: we find a “bad” subvariety B'_W of codimension ≥ 2 in any complex variety W so that at each point of $W \setminus B'_W$ there is an analytic product structure on the tangent cone there. (In general $T_a \neq T_p$ at points of $\text{Re}(W \setminus B'_W)$.) In §4 we strengthen this condition by increasing the size of B'_W to B_W , arriving at a kind of local product at points of $W \setminus B_W$, and then correspondingly for the real case. From this we prove (in §5) our main contention, Theorem 1.8.

The main result of this section is

Theorem 3.1. *Let $W \subseteq H \subseteq \mathbf{C}^n$ be complex analytic of dimension r . Then there is an analytic subvariety B'_W of codimension ≥ 2 in W such that for each point $p \in W \setminus B'_W$, there is a one-dimensional analytic germ $\tilde{C}_p \subseteq \tilde{W}_p$ and an $(r - 1)$ -dimensional nonsingular germ $\tilde{M}_p \subseteq \tilde{W}_p$ such that*

$$T(\tilde{W}_p) = T(\tilde{C}_p) \times T(\tilde{M}_p).$$

Before giving the proof, we recall some facts:

Definition 3.2. Let W' be any complex-analytic variety having pure dimension r at a point p . Then there are $\mathbf{C}^n$ -open neighborhoods U_p, U'_p of p ($U_p \subseteq U'_p$), and an integer $m(\tilde{W}'_p)$ such that at each point $q \in W' \cap U_p$ off a proper subvariety of $W' \cap U_p$, “almost every” analytic $(n - r)$ -plane P_{n-r} in $\mathbf{C}^n$ through q , intersects $W' \cap U'_p$ in exactly $m(\tilde{W}'_p)$ points. We call $m(\tilde{W}'_p)$ the *multiplicity of $\tilde{W}'_p$* , or the *multiplicity of W' at p* . (By “almost every” we mean off a proper algebraic subvariety of the algebraic variety $G_{r,n}$ = Grassmannian variety of complex r -planes in $\mathbf{C}^n$ through q .)

Lemma 3.3. *The set of points of W' having multiplicity greater than any given integer forms an analytic subvariety of W' .*

(A proof will appear in Whitney's forthcoming book on analytic varieties, [10]. A general ring-theoretic proof is given in [3, Theorem 40.3].)

Lemma 3.4. *If a germ $\tilde{H}_p$ is of codimension one in $(\mathbf{C}^n)_p$ and is defined by a function h analytic at p , then $m(\tilde{H}_p)$ is equal to the order of h when expanded about p .*

(This is an easy consequence of [1, p. 13, Lemmas 2 and 3].)

We now describe the B'_W , $\tilde{C}_p$ and $\tilde{M}_p$ used in proving (3.1).

B'_W : First, we may assume W in (3.1) is of pure dimension. Now let $S(W)$ be the singular subvariety of W ; let $S_1(W)$, $S_2(W)$, $\dots$ be the irreducible components of $S(W)$ having codimension one in W ; for each $S_i(W)$, let m_i be the smallest multiplicity $m(\tilde{W}_{p_i})$ at points p_i on $S_i(W)$. Let $T_i(W)$ be the subvariety of $S_i(W)$ of points having multiplicity greater than m_i . Then (denoting topological closure by "Cl"), let

$$B'_W = \bigcup_i (T_i(W)) \cup \text{Cl} (S(W) \setminus \bigcup_i S_i(W)) \cup \left(\bigcup_i S(S_i(W)) \right).$$

Clearly the codimension of B'_W in W is ≥ 2 .

$\tilde{M}_p$: Let $p \notin W \setminus B'_W$. Since Theorem 3.1 is trivial at nonsingular points of W , assume $p \in S(W)$. $(S(W))_p$ is nonsingular of dimension $r - 1$ since $p \notin B'_W$; let $\tilde{M}_p$ be this germ.

$\tilde{C}_p$: Select analytic co-ordinates $(x_1, \dots, x_n)$ of $\mathbf{C}^n$ about p so that a representative M_p of $\tilde{M}_p$ is part of the $(x_1, \dots, x_{r-1})$ -plane $\mathbf{C}_{x_1, \dots, x_{r-1}}$; then let

$$\tilde{C}_p = \tilde{W}_p \cap (\mathbf{C}_{x_r, \dots, x_n})_p \quad (\mathbf{C}_{x_r, \dots, x_n} = (x_r, \dots, x_n)\text{-plane}).$$

Notation 3.5. From now on L_s will mean an s -dimensional subspace of $\mathbf{C}^n$.

We now prove Theorem 3.1 with the above B'_W , $\tilde{M}_p$, and $\tilde{C}_p$. Without loss of generality, assume $p = 0$. Let $L_{n-r-1} \subset \mathbf{C}_{x_r, \dots, x_n}$ be chosen so $(L_{n-r-1})_0 \cap \tilde{C}_0 = \{0\}$, and so that L_{n-r-1} is transverse to $\mathbf{C}_{x_1, \dots, x_{r+1}}$. Let $\pi_{L_{n-r-1}}$ be the natural projection of $\mathbf{C}^n$ onto $\mathbf{C}_{x_1, \dots, x_{r+1}}$ along L_{n-r-1} . By the "projection theorem" (Lemma 2.7(a)), $\pi_{L_{n-r-1}}(\tilde{W}_0)$ is analytic in $(\mathbf{C}_{x_1, \dots, x_{r+1}})_0$ and is of codimension one there. From Definition 3.2 it is easily seen that for almost every subspace L_{n-r-1} ("almost every" having the obvious meaning), $\pi_{L_{n-r-1}}$ will not only preserve analyticity, but multiplicity as well—i.e., for almost every L_{n-r-1} , $m(\tilde{W}_0) = m(\pi_{L_{n-r-1}}(\tilde{W}_0))$. In fact, one can easily choose L_{n-r-1} so $\pi_{L_{n-r-1}}$ is one-to-one on some representative W_0 of $\tilde{W}_0$ except off a proper subvariety of W_0 . One sees from Definition 3.2 that $\pi_{L_{n-r-1}}$ cannot decrease multiplicity at points of W_0 near 0; then using Lemma 3.3, since it preserves multiplicity at 0, the multiplicity of $\pi_{L_{n-r-1}}(\tilde{W}_0)$ must still be constant at each point of some representative M_0 of $\tilde{M}_0$. (From the definition of B'_W , the multiplicity of W_0 is constant at all points of some M_0 . We assume any L_{n-r-1} occurring henceforth has been so chosen.)

$\pi_{L_{n-r-1}}(\tilde{W}_0)$ has codimension one in $(\mathbf{C}_{x_1, \dots, x_{r+1}})_0^\sim$; let it be defined by f , analytic at $0 \in \mathbf{C}_{x_1, \dots, x_{r+1}}$. Since $m(\pi_{L_{n-r-1}}(\tilde{W}_p))$ is constant for all $p \in \mathbf{C}_{x_1, \dots, x_{r+1}}$ near 0, one can easily see from Lemma 3.4 that $f_0^\dagger$ cannot involve any of the variables $x_1, \dots, x_{r-1}$. Now $\pi_{L_{n-r-1}}(\tilde{C}_0) (\subseteq (\mathbf{C}_{x_r, x_{r+1}})_0^\sim)$ is defined by $f(0, \dots, 0, x_r, x_{r+1})$, and $f^\dagger(x_1, \dots, x_{r+1}) = f^\dagger(0, \dots, 0, x_r, x_{r+1})$. We therefore conclude that

$$T(\pi_{L_{n-r-1}}(\tilde{W}_0)) = T(\pi_{L_{n-r-1}}(\tilde{C}_0)) \times \mathbf{C}_{x_1, \dots, x_{r-1}}.$$

Now the cylindrization $\pi_{L_{n-r-1}}^{-1}(\pi_{L_{n-r-1}}(\tilde{W}_0))$ of $\pi_{L_{n-r-1}}(\tilde{W}_0)$ along L_{n-r-1} is defined at $0 \in \mathbf{C}^n$ by the same function f , relative to co-ordinates in $\mathbf{C}^n$ in which L_{n-r-1} is a co-ordinate subspace. It is clear that

$$T(\pi_{L_{n-r-1}}^{-1}(\pi_{L_{n-r-1}}(\tilde{W}_0))) = T(\pi_{L_{n-r-1}}(\tilde{C}_0)) \times \mathbf{C}_{x_1, \dots, x_{r-1}} \times L_{n-r-1}.$$

Each of the parallel translates $p + (\mathbf{C}_{x_1, \dots, x_{r-1}} \times L_{n-r-1})$ (p near 0 and in a representative of $\pi_{L_{n-r-1}}(\tilde{C}_0)$) contains a point q in a representative of $\tilde{C}_0$; it is then clear that also

$$T(\pi_{L_{n-r-1}}^{-1}(\pi_{L_{n-r-1}}(\tilde{W}_0))) = T(\tilde{C}_0) \times \mathbf{C}_{x_1, \dots, x_{r-1}} \times L_{n-r-1}.$$

Now $\tilde{W}_0 \subseteq \pi_{L_{n-r-1}}^{-1}(\pi_{L_{n-r-1}}(\tilde{W}_0))$, so

$$T(\tilde{W}_0) \subseteq T(\tilde{C}_0) \times \mathbf{C}_{x_1, \dots, x_{r-1}} \times L_{n-r-1},$$

and therefore

$$T(\tilde{W}_0) \subseteq \cap (T(\tilde{C}_0) \times \mathbf{C}_{x_1, \dots, x_{r-1}} \times L_{n-r-1}),$$

this intersection being taken over all those L_{n-r-1} 's in $\mathbf{C}_{x_r, \dots, x_n}$ transverse to $\mathbf{C}_{x_1, \dots, x_{r+1}}$ and preserving multiplicity as above. The intersection of these L_{n-r-1} 's is clearly 0, so

$$T(\tilde{W}_0) \subseteq T(\tilde{C}_0) \times \mathbf{C}_{x_1, \dots, x_{r-1}}.$$

To prove equality, let v be a vector in $T(\tilde{C}_0) \times \mathbf{C}_{x_1, \dots, x_{r-1}}$. Since $T(\tilde{W}_0)$ has pure dimension r , and since $T(\tilde{C}_0) \times \mathbf{C}_{x_1, \dots, x_{r-1}}$ is a finite union of r -planes, $T(\tilde{W}_0)$ is itself a finite union of r -planes (each plane containing $\mathbf{C}_{x_1, \dots, x_{r-1}}$). Now $\tilde{C}_0 \subseteq \tilde{W}_0$, so $T(\tilde{C}_0) \subseteq T(\tilde{W}_0)$; therefore, if v is nonzero, then $v \in T(\tilde{C}_0)$ implies that the r -plane containing v and $\mathbf{C}_{x_1, \dots, x_{r-1}}$ is in $T(\tilde{W}_0)$. From this we immediately get the reverse inclusion, hence equality. Since $\mathbf{C}_{x_1, \dots, x_{r-1}} = T(\tilde{M}_0)$, we have proved Theorem 3.1.

4. A local product theorem for varieties; definition of A_V . Let V be a real variety. Suppose that at each point p off some real subvariety of codimension ≥ 2 in V , there were a real 1-germ $\tilde{C}_p$ and an $(r-1)$ -dimensional manifold germ $\tilde{M}_p$ analogous to the complex ones of Theorem 3.1. Then if the T_a and T_σ analogues of Theorem 3.1 held, we would clearly have $T_a = T_\sigma$, thus proving (1.8.1) and (1.8.2). In light of the last section, one might be led to hope that removing the real part of B'_{V*} would do this for us; however Example 1.7(b) rules this out. (As we have seen, $T_a(V_0) = T_\sigma(V_0)$ in this example. The singular

locus of V^* is the y -axis $\mathbf{C}_y$ and, using Lemma 3.4, we see that the multiplicity of V^* at each point of $\mathbf{C}_y$ is two; hence $B'_{V^*} = \emptyset$.)

To prove our main theorem, we clearly must somehow strengthen our condition on p . We shall do this by increasing B'_{V^*} first to a larger variety B_{V^*} so that at $p \in V^* \setminus B_{V^*}$ there will exist a local product decomposition of the complex variety (Definition 4.1, Lemma 4.3); then we increase $\text{Re } (B_{V^*})$ to A_V so that at $p \in V \setminus A_V$ there will exist a local product decomposition of the real variety (Definition 4.6). From this it will not be hard to prove Theorem 1.8 (in §5).

We divide this section into two parts, I and II; in I we consider B_{V^*} , and in II, A_V .

I. Definition of B_{V^*} .

In answering a conjecture of H. Whitney concerning singularities of analytic varieties, the author ([2], Theorem 1.5) has shown that for a complex-analytic variety W in an open set H of $\mathbf{C}^n$, there exists a certain "bad" subvariety (call it B_W) of codimension ≥ 2 in W , and such that at any $p \in W \setminus B_W$, W locally factors into a " $1 \times (r - 1)$ -semi-analytic product" (see Definition 4.1). This semi-analytic product will behave enough like the ordinary "analytic product" to allow us to use the basic line of argument indicated in §3 to prove our main theorem.

Definition 4.1. Let H, K, L be open sets in $\mathbf{C}^s, \mathbf{C}^t$, and $\mathbf{C}^{s+t}$ resp.; let $W_1 \subseteq H \subseteq \mathbf{C}^s, W_2 \subseteq K \subseteq \mathbf{C}^t, W \subseteq L \subseteq \mathbf{C}^{s+t}$ be analytic varieties. Then W is a *semi-analytic product* of W_1 and W_2 (denoted $W = W_1 \times_s W_2$) if there is a homeomorphism φ from $H \times K$ to L carrying $W_1 \times W_2$ to W and such that for each $h_0 \in H$, $\varphi(h_0, y)$ is an analytic map from K to L (or for each $k_0 \in K$, $\varphi(x, k_0)$ is analytic from H to L). If $p \in H, q \in K$, we write $W_{(p,q)}^{\sim} = (W_1)_{\tilde{p}} \times_s (W_2)_{\tilde{q}}$.

Hence a topological product is semi-analytic if it is "analytic in at least one of the factors".

Definition 4.2. A *polycylindrical neighborhood* $\Delta^n(a, \epsilon)$ about $a = (a_1, \dots, a_n) \in \mathbf{C}^n$ is any product $\Delta(a_1, \epsilon_1) \times \dots \times \Delta(a_n, \epsilon_n)$, where $\Delta(a_i, \epsilon_i) = \{z \in \mathbf{C}_{x_i} : |z - a_i| < \epsilon_i > 0\}$.

The following lemma follows at once from [2], Theorem 1.5, and will give us the existence of B_{V^*} .

Lemma 4.3. Let $W \subseteq K$ (K open in $\mathbf{C}^n$), and suppose $\dim_p W = r$. Then there is an analytic subvariety B_W of codimension ≥ 2 in W such that in some neighborhood of any $p \in W \setminus B_W$, W decomposes into a semi-analytic product of a 1-dimension variety by an $(r - 1)$ -dimension variety; specifically, at each $p \in W \setminus B_W$ the following holds:

In an appropriate (analytic) co-ordinate system about p , there is a polycylindrical neighborhood

$$\Delta^n = \Delta^n(p, \epsilon) = \Delta^{r-1} \times \Delta^{n-r+1} = (\Delta(p_1, \epsilon_1) \times \dots \times \Delta(p_{r-1}, \epsilon_{r-1})) \times (\Delta(p_r, \epsilon_r) \times \dots \times \Delta(p_n, \epsilon_n))$$

such that

- (a) $W \cap \Delta^{r-1} = \Delta^{r-1}$;
- (b) $W \cap \Delta^{n-r+1}$ is an analytic curve in Δ^{n-r+1} ;
- (c) $W \cap \Delta^n = C \times_s \Delta^{r-1}$.

Let 0 be a typical point of $W \setminus B_W$. Since Δ^{r-1} above is a local manifold, we may suggestively write

$$\tilde{W}_0 = \tilde{C}_0 \times_s \tilde{M}_0.$$

Definition 4.4. Let B_{V^*} be any subvariety of V^* satisfying Lemma 4.3, with V^* as W .

II. Definition of A_V .

We now consider a real variety V ; we define A_V and show that at a point $p \in V \setminus A_V$ there exists a real product decomposition analogous to the complex one in I . We first note that, of course, generally a product in the complex sense does not induce a corresponding real one on the real parts. For instance $y + ix + t$ ($t \in \mathbf{C}$) decomposes $\mathbf{C}_{xy}$ into $\mathbf{C} \times \mathbf{C}$ by the parallel complex lines F_t , but this does not induce the corresponding decomposition $\mathbf{R} \times \mathbf{R}$ in $\mathbf{R}_{xy}$; the real part of each F_t is a single point in $\mathbf{R}_{xy}$, not a real line.

In the decomposition of I we are in a more fortunate position since each fiber lies in a parallel translate of a co-ordinate subcylinder of Δ^n . Still, difficulties can arise if any part of $\text{Re } (\tilde{V}_p)$ has dimension different from $\tilde{V}_p$.

Example. The two-dimensional surface $S: xz + iy = 0$ in $\mathbf{C}_{xyz}$ is fibered into complex lines by intersecting S with the planes $x = \text{constant}$. But $\dim (\text{Re } (S)) = \dim (\mathbf{R}_x \cup \mathbf{R}_y) = 1$ and the induced fibering splits up part of $\text{Re } (S)$, viz. $\mathbf{R}_x$, into points but leaves $\mathbf{R}_z \setminus \{0\}$ alone.

If, in the decomposition of I we have at a real point $0 \in V \setminus B_{V^*}$, pure and equal dimension, the complex fibering there does induce an analogous real one because the $F_i(x_r)$, for small $|x_r|$, are disjoint $(r-1)$ -dimensional complex holomorphs in $\mathbf{C}_{x_1, \dots, x_{r-1}, x_{r+1}, \dots, x_n} \cap \Delta^n(0, \epsilon)$ of $\mathbf{C}_{x_1, \dots, x_{r-1}} \cap \Delta^n(0, \epsilon)$, which implies that $\text{Re } (F_i(x_r))$ is of real dimension $r-1$, so the $\text{Re } F_i(x_r)$ do give us an analogous real fibering.

We therefore will have attained the desired real product structure off a variety of codimension ≥ 2 in V as soon as we show

Lemma 4.5. *The points of real impure dimension are contained in an analytic variety A'_V of codimension ≥ 2 in V .*

Definition 4.6. Let A_V be $\text{Re } (B_{V^*}) \cup A'_V$ (note that $\dim V = \dim V^*$ easily implies that $\text{cod } (\text{Re } (B_{V^*})) \geq 2$ in V).

Before giving the proof, we note

Lemma 4.7. *Any real (resp. complex) analytic variety is the union of a disjoint, locally finite collection of connected analytic real (resp. complex) manifolds*

$\{M_i\}$, in such a way that the boundary $\text{Cl}(M_i) \setminus M_i$ of any M_i is a union of lower-dimensional M_i 's of $\{M_i\}$; further, in the real case, each $\text{Cl}(M_i)$ is a semi-analytic set in the sense of the following

Definition 4.8. A semi-analytic set in $\mathbf{R}^n$ is a closed set S such that at each point $p \in S$, there is an $\mathbf{R}^n$ -open U_p and a corresponding finite set of inequalities $\{f_i(x_1, \dots, x_n) \geq 0\}$ (each f_i analytic in U_p) such that $S \cap U_p$ is defined by the f_i .

Remark. A decomposition as in Lemma 4.7 is called a *stratification* of V . Proof of existence appears in [7, §8] in the complex case; for the real case, see [5].

Remark. Note that each M_i is of pure dimension; also, the real stratification of V may be made a refinement of a complex stratification in the sense that any real M_i is contained in the real part of some stratum of V^* . It is then clear that any locally finite collection of M_i 's is contained in a (real) analytic subvariety of the same dimension.

Proof of Lemma 4.5. Let $p \in V$. Suppose every M_i whose closure contains p , is contained in the closure of some M_j of dimension r . Then clearly, from Definition 1.5, p has pure dimension r in V .

Now suppose p has impure dimension r in V . Then, in some $M_k : p \in \text{Cl}(M_k)$, and M_k is not in the closure of any M_i of dimension r . Then certainly $\dim M_k < r$. Now $p \notin M_k$; otherwise $\dim_p V < r$. Therefore $p \in \text{Cl}(M_k) \setminus M_k$, and this has dimension less than M_k . Therefore any point of impure dimension r is contained in a locally finite collection of analytic manifolds of codimension ≥ 2 in V , and therefore, by the above remark, in an analytic subvariety of codimension ≥ 2 in V . We have thus proved Lemma 4.5.

5. Proof of the main result. Since (1.8.1) follows from (1.8.2), we now prove (1.8.2), which we state here as

Theorem 5.1. Let $V \subseteq \mathbf{R}^n$ be real analytic; there is a subvariety A_V of codimension ≥ 2 in V such that for $p \in V \setminus A_V$,

$$T_p(\tilde{V}_p) = T_p(\tilde{V}_p).$$

In proving this theorem, we take $p = 0$; also, we may clearly assume $\dim \tilde{V}_0 = \dim V (= r)$. To reiterate, at the beginning of §3, we indicated that by using a product more general than Cartesian product, we could take our cue from the general scheme of the argument there, to prove Theorem 5.1. We continue to write " $\times$ " for Cartesian product, and " $\times_s$ " for semi-analytic product. We now show

$$(5.2) \quad T_a(\tilde{C}_0 \times_s \tilde{M}_0) \subseteq T_a(\tilde{C}_0) \times T_a(\tilde{M}_0),$$

and

$$(5.3) \quad T_s(\tilde{C}_0 \times_s \tilde{M}_0) \supseteq T_s(\tilde{C}_0) \times T_s(\tilde{M}_0).$$

Since by Theorem 2.8, the right hand sides are the same, we get $T_a(\tilde{V}_0) \subseteq T_\sigma(\tilde{V}_0)$. But in general $T_\sigma(\tilde{V}_0) \subseteq T_a(\tilde{V}_0)$ (Lemma 2.5(b)); hence we have equality. Therefore we will have proved Theorem 5.1 as soon as we have proved (5.2) and (5.3).

In proving (5.2) and (5.3), one may assume $\tilde{V}_0$ is irreducible, since at any point $q \in V$ we have

$$(5.4) \quad T_\sigma((V_1)_q \cup (V_2)_q) = T_\sigma((V_1)_q) \cup T_\sigma((V_2)_q)$$

(this is obvious from the definition of T_σ); and

$$(5.5) \quad T_a((V_1)_q \cup (V_2)_q) = T_a((V_1)_q) \cup T_a((V_2)_q).$$

((5.5) holds using the complexifications of the two germs since then $T_a = T_\sigma$; so it holds also for their real parts since, by Lemma 2.5(a), $T_a(\tilde{V}_q) = \text{Re } T((\tilde{V}_q)^*)$.)

Proof of (5.2). Recall that any complex-analytic germ $\tilde{W}_q$ is irreducible iff for every complex analytic $\tilde{Z}_q$, $\tilde{W}_q \setminus \tilde{Z}_q$ is topologically connected. Then since $(\tilde{V}_0)^*$ is irreducible of dimension r , $(\tilde{V}_0)^* \setminus (\tilde{M}_0)^*$ is connected; by the product decomposition result of §4, $(\tilde{V}_0)^* \setminus (\tilde{M}_0)^*$ is homeomorphic to $((\tilde{C}_0)^* \setminus 0) \times (\tilde{M}_0)^*$; hence $((\tilde{C}_0)^* \setminus 0) \times (\tilde{M}_0)^*$, and therefore $(\tilde{C}_0)^*$, is connected. This easily implies $(\tilde{C}_0)^*$ is irreducible (since it is of dimension one). By [6, Theorem 5.7] $T((\tilde{C}_0)^*)$ is a single complex line; by Theorem 3.1, $T((\tilde{V}_0)^*)$ is an analytic r -plane. Since $T_a(\tilde{V}_0) = \text{Re } T((\tilde{V}_0)^*)$, $T_a(\tilde{V}_0)$ is no more than a real r -plane; but $T_a(\tilde{C}_0) \times T_a(\tilde{M}_0)$ is at least a real r -plane by Theorem 2.8, so (5.2) holds.

Proof of (5.3). Let $v = \lim_{i \rightarrow \infty} p_i/|p_i|$, p_i in a representative of the real germ $\text{Re } ((\tilde{C}_0)^*)$, be a nonzero vector tangent to this germ; let P be the real r -plane containing $\mathbf{R}_{x_1, \dots, x_{r-1}}$ and v . Let w be any vector of P . We show $w \in T_\sigma(\tilde{V}_0)$.

For this purpose assume without loss of generality that $w \notin \mathbf{R}_{x_1, \dots, x_{r-1}}$, and let L be any $(n - r + 1)$ -space transverse to $\mathbf{R}_{x_1, \dots, x_{r-1}}$ and containing w . Now a standard topological argument shows that the fibers $\text{Re } F_i(x_r)$ are uniformly close to $\Delta^n \cap \mathbf{R}_{x_1, \dots, x_{r-1}}$ for $|x_r|$ sufficiently small. Hence, for p_i sufficiently large, these real fibers through p_i clearly intersect L ; let q_i be a point of this intersection, and let $w' = \lim_{i \rightarrow \infty} q_i/|q_i| \in T_\sigma(\tilde{V}_0)$. Now since each $q_i \in L$, clearly $w' \in L$. And since each $q_i \in V$, $w' \in \text{Re } T((\tilde{V}_0)^*)$. Since $\text{Re } T((\tilde{V}_0)^*)$ is no more than a real r -plane, $w' \in P$; therefore the nonzero vector w' is in $P \cap L$ (= the 1-space containing w). Hence $w = (|w|/|w'|)w' \in T_\sigma(\tilde{V}_0)$, as required.

Hence we have proved Theorem 5.1; therefore (1.8.1) and (1.8.2).

From the above proof, it is obvious (using (5.4) and (5.5)) that $T(\tilde{V}_0)$ is a finite union of real r -planes, so we have also shown (1.8.3),

Q.E.D.

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