Cauchy Integrals and Spectral Measures

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I. Introduction. In recent years the Cauchy integral relative to a vector valued measure has been adopted as a fundamental tool for the study of problems in non-self-adjoint spectral theory [2], [6], [7] stochastic processes [1], [8], [10] and mathematical system theory [3], [12], [13], [14]. The advantage of the Cauchy integral in these problems over the more classical Lebesgue integral is that it is well defined for measures which do not have finite variation or semivariation [4], [5], and in the context of certain applications [10], [12] it may have a desirable "physical" interpretation which is not shared by the Lebesgue integral. Unlike the Cauchy integral defined over Lebesgue measure, which is equivalent to the Riemann integral [9] independently of the "Cauchy valuation" for continuous functions on compact sets, the Cauchy integral relative to a vector valued measure is not equivalent to any of the classical integrals, and, in fact, may exhibit anomalous properties quite unlike those of the classical vector valued integrals. In particular, the integral of the indicator function of a set from the measure space on which the vector valued measure is defined may not exist. Moreover, when the integral exists its value may be highly dependent on the choice of the "Cauchy valuation".

In the present work we consider the Cauchy integral of an operator valued function over a spectral measure such as encountered in the spectral theoretic and system theoretic contexts described above, our main result being concerned with the measurability of Cauchy integrable functions.

Throughout the sequel X and Y are fixed sets, Σ is a semialgebra [4] over Y , i.e., a semiring of subsets of Y such that $Y \in \Sigma$, and E is a spectral measure in a Hilbert space H defined on Σ . That is, E is a finitely additive set function whose values are projections on H satisfying $E(\emptyset) = 0$, $E(Y) = I$, $E(\sigma_1) - E(\sigma_2) \geq 0$ if $\sigma_2 \subset \sigma_1$ and $E(\sigma_1)E(\sigma_2) = E(\sigma_1 \cap \sigma_2)$ where $\sigma_1, \sigma_2 \in \Sigma$.

1.1. Definition. A *Cauchy valuation* is a mapping, c , from $\Sigma \setminus \emptyset$ onto X such that:

- (i) the set of subsets $\Sigma(x) \triangleq c^{-1}(x)$ is directed by containment; i.e., if σ_1 and σ_2 are in $\Sigma(x)$, then there exists a $\sigma \in \Sigma(x)$ such that $\sigma \subset \sigma_1 \cap \sigma_2$.
- (ii) for each finite Σ -partition of Y , $\rho = \{\rho_i\}_{i=1}^{n(\rho)}$, and each $x \in X$; $\Sigma(x)$ is

eventually contained in a single element of ρ ; i.e., there exists $j \leq n(\rho)$ and $\sigma_0 \in \Sigma(x)$ such that if $\sigma \in \Sigma(x)$ and $\sigma \subset \sigma_0$, then $\sigma \subset \rho_j$.

1.2. Definition. Let X, Y, Σ and E be as above and c be a Cauchy valuation. Then the *Cauchy integral* (*c-integral*) of an H -to- H bounded operator-valued function T , defined on X is the uniform limit of the partial sums:

$$(c) \int_{\nu} T(x) dE(y) \triangleq \lim_{\rho} \Sigma_{\rho} T \triangleq \lim_{\rho} \left(\sum_{i=1}^{n(\rho)} T(c(\rho_i)) E(\rho_i) \right)$$

where the limit is taken over the net of finite Σ -partitions, $\rho = \{\rho_i\}_{i=1}^{n(\rho)}$, of Y directed by refinement.

Note that the above c -integral is well defined if one takes the limit in the strong or weak operator topologies rather than the uniform topology and many of the results presented in the sequel remain valid in these topologies, though the results are stated explicitly for the uniform topology. The constituents making up a Cauchy integral are denoted by a 5-tuple (X, Y, Σ, c, E) .

The classical examples of the Cauchy integral are the left and right Cauchy integrals which are the special cases of the above general integral which yield our most interesting examples.

1.3. Definition. The *left Cauchy integral* (denoted by LCf) is the Cauchy integral with X, Y, Σ and c defined as follows:

- (i) $Y = \mathbf{R}$.
 - (ii) Σ is the set of (finite, semi-infinite and infinite) intervals of the form $[a, b)$, $[a, \infty)$, $(-\infty, b)$ and $(-\infty, \infty)$ in $\mathbf{R}$.
 - (iii) $X = \mathbf{R}_+ = \mathbf{R} \cup \{-\infty\}$.
 - (iv) c is the valuation defined on the four types of interval via $c([a, b)) = a$, $c([a, \infty)) = a$, $c((-\infty, b)) = -\infty$ and $c((-\infty, \infty)) = -\infty$.
- and E taken to be any spectral measure on Σ .

1.4. Definition. The *right Cauchy integral* (denoted by RCf) is the Cauchy integral with X, Y, Σ and c defined as follows:

- (i) $Y = \mathbf{R}$.
 - (ii) Σ is the set of (finite, semi-infinite and infinite) intervals of the form $[a, b)$, $[a, \infty)$, $(-\infty, b)$ and $(-\infty, \infty)$ in $\mathbf{R}$.
 - (iii) $X = \mathbf{R}^+ = \mathbf{R} \cup \{\infty\}$.
 - (iv) c is the valuation defined on the four types of interval via $c([a, b)) = b$, $c([a, \infty)) = \infty$, $c((-\infty, b)) = b$ and $c((-\infty, \infty)) = \infty$.
- and E is taken to be any spectral measure on Σ .

We note that even for the classical integrals defined above we require $X \neq Y$; hence, this assumption in our general definition is not a vacuous generalization. We will, however, show that every Cauchy integral is equivalent, in an appropriate sense, to a Cauchy integral in which $X = Y$. A natural generalization of the above right and left Cauchy integral corresponds to the case where c

is taken to be the corner of an n -cube in $\mathbf{R}^n$, which also fits into the axiomatics of the general Cauchy integral of Definition 1.2. Of course, one can also replace $\mathbf{R}$ by an ordered set in the above examples.

In the following section the elementary properties of the Cauchy integral are formulated and a number of examples which illustrate the “apparently” pathological characteristics of the Cauchy integral over a spectral measure are formulated. We then state and prove our main theorem concerning the measurability of c -integrable functions and a number of corollaries wherein the properties of the integral are strengthened upon appropriately restricting the range of T .

II. Elementary properties and examples. Viewed as an operator from the space of c -integrable functions into the bounded linear operators on H , the c -integral over a spectral measure is linear and onto.

2.1. Proposition. *For any Cauchy integral over a spectral measure E the following are true:*

(i) *If T and S are c -integrable functions, then $T + S$ is c -integrable and*

$$(c) \int_Y (T + S)(x) dE(y) = (c) \int_Y T(x) dE(y) + (c) \int_Y S(x) dE(y).$$

(ii) *If T is a c -integrable function and A is a bounded linear operator on H , then AT (i.e., $(AT)(x) = AT(x)$) is c -integrable and*

$$(c) \int_Y (AT)(x) dE(y) = A \left[(c) \int_Y T(x) dE(y) \right].$$

(iii) *For any bounded linear operator, A , on H , the constant function $\hat{A}$ ($= A$ for all $x \in X$) is c -integrable and*

$$(c) \int_Y \hat{A}(x) dE(y) = A.$$

The proofs for the above statements are straightforward, resulting from the linearity of the mapping from T to the partial sum $\Sigma_\rho T$, its preservation under uniform limits, the continuity of A in the uniform operator topology and the fact that $E(Y) = I$, the details being deleted for brevity.

Although the above properties of the c -integral are well behaved, the integral exhibits a number of “apparently” pathological characteristics which are quite unlike those of the classical integrals. In fact, the c -integral of the indicator function of a set in Σ may fail to exist.

2.2. Example. The right Cauchy integral

$$\text{RC} \int_{\mathbf{R}} \chi_{[0,1)}(x) dE(y)$$

is defined to be the uniform limit of the partial sums

$$(2.1) \quad \Sigma_{\rho} \chi_{(0,1)} = \sum_{i=1}^n \chi_{(0,1)}(t_i) E(t_{i-1}, t_i)$$

where ρ is the partition of $\mathbf{R}$ into the intervals $(t_0, t_1), [t_1, t_2), \dots, [t_{n-1}, t_n)$ where $t_0 = -\infty$ and $t_n = \infty$. Now if k is the largest index such that $t_k < 0$ and m is the largest index such that $t_m < 1$ then we have

$$(2.2) \quad \chi_{(0,1)}(t_i) = \begin{cases} 0 & : i \leq k, \\ 1 & : k+1 \leq i \leq m, \\ 0 & : m+1 \leq i. \end{cases}$$

As such

$$(2.3) \quad \Sigma_{\rho} \chi_{(0,1)} = \sum_{i=k+1}^m E(t_{i-1}, t_i) = E(t_k, t_m)$$

and, similarly, for any other partition $\tilde{\rho}$ we have

$$(2.4) \quad \Sigma_{\tilde{\rho}} \chi_{(0,1)} = E(t_{\tilde{k}}, t_{\tilde{m}}).$$

Now, if $t_k \neq t_{\tilde{k}}$ or $t_m \neq t_{\tilde{m}}$

$$(2.5) \quad \|\Sigma_{\rho} \chi_{(0,1)} - \Sigma_{\tilde{\rho}} \chi_{(0,1)}\| = 1$$

since the norm of the difference of any two distinct projections is one. Thus, the limit of the partial sums $\Sigma_{\rho} \chi_{(0,1)}$ fails to converge (unless both zero and one have neighborhoods which are sets of E -measure zero) and we have verified our contention that the indicator function of a set in Σ may not be c -integrable. Note, however, that the failure of the integral to exist is highly dependent on our use of the uniform operator topology. In fact, in the strong operator topology

$$(2.6) \quad s\text{-}\lim_{\rho} \Sigma_{\rho} \chi_{(0,1)} = s\text{-}\lim_{\rho} E(t_k, t_m) = E(0, 1)$$

in which case the right Cauchy integral exists.

Another “apparently” pathological characteristic of the c -integral is that, unlike the case of a real valued Cauchy integral relative to Lebesgue measure, [9] the operator valued Cauchy integral relative to a spectral measure is not independent of the valuation, c .

2.3. Example. We denote by F the operator valued function equal to $E(-\infty, x)$ for all real x and set $F(-\infty) = 0$ and $F(\infty) = I$. Now, for any partition, ρ , of $\mathbf{R}$ into intervals $(t_0, t_1), [t_1, t_2), \dots, [t_{n-1}, t_n)$ where $t_0 = -\infty$ and $t_n = \infty$ the partial sums for the right and left Cauchy integrals of F are given by

$$(2.7) \quad \sum_{i=1}^n E(-\infty, t_i) E(t_{i-1}, t_i) = \sum_{i=1}^n E(t_{i-1}, t_i) = I$$

and

$$(2.8) \quad \sum_{i=1}^n E(-\infty, t_{i-1})E(t_{i-1}, t_i) = \sum_{i=1}^n 0 = 0,$$

respectively, where the equalities result from the multiplicative character of the spectral measure, its finite additivity and the fact that $E(\mathbf{R}) = I$. Consistent with 2.7 and 2.8 the limits of the partial sums trivially exist and we have

$$(2.9) \quad \text{RC} \int_{\mathbf{R}} F(x) dE(y) = I \neq 0 = \text{LC} \int_{\mathbf{R}} F(x) dE(y).$$

As such, it is indeed possible for the right and left Cauchy integrals to exist simultaneously but not coincide.

The source of the above pathological properties of the c -integral is its unboundedness when viewed as a linear operator mapping the c -integrable functions into the bounded linear operators on H . In fact, the operator induced by the c -integral is not even closed.

2.4. Example. To verify that the Cauchy integral may fail to define a bounded linear operator we observe that the *integrals of triangular truncation*

$$\text{LC} \int_{\mathbf{R}} F(x)A dE(y)$$

and

$$\text{RC} \int_{\mathbf{R}} F(x)A dE(y),$$

where F is as in Example 2.3 and A is any fixed operator, coincide precisely with the integrals $m \int PA dP$ and $M \int PA dP$ used by Gohberg and Krein [7] which are known ([7], 101) to be unbounded linear operators when viewed as a mapping from A into $m \int PA dP$ (or $M \int PA dP$). As such, since

$$(2.10) \quad \|FA\|_{\infty} \triangleq \sup_x \|F(x)A\| = \|A\|$$

the Cauchy integrals define unbounded operators from the space of c -integrable functions normed by $\|\cdot\|_{\infty}$ onto the bounded linear operators. Moreover, since the operator defined by the c -integral is onto, the open mapping theorem would imply that it is bounded if it is closed [12]; hence the mapping defined by the Cauchy integral from the c -integrable functions onto the bounded linear operators is not closed. Of course, under various assumptions on A (for instance if A is Hilbert-Schmidt [3] or if A is in S_{∞} [7]) the above integrals of triangular truncation may be bounded. Similarly, if one restricts the class of functions under consideration either by assuming a smoothness condition [2] or restricting its range (c.f. Section 3) the “general” Cauchy integral may be bounded.

III. A measurability theorem. Since the semialgebra Σ is defined on Y and the c -integrable functions are defined on X , we cannot define the measurability of such functions with respect to Σ . In fact even when $X = Y$, Σ may not be the appropriate semialgebra with which to define the measurability of c -integrable functions. Rather, we define a new semialgebra on X induced by Σ and c . Here and throughout this section X , Y , Σ , c and E denote the constituents of an arbitrarily specified Cauchy integral.

3.1. Lemma. For each $\sigma \in \Sigma$ let

$$\bar{\sigma} = \{x \in X; x = c(\mu); \text{ for some } \mu \subset \sigma, \mu \in \Sigma\}$$

and denote the set of all such $\bar{\sigma}$ by $\bar{\Sigma}$. Then the mapping $\sigma \rightarrow \bar{\sigma}$ is a semialgebra isomorphism between Σ and $\bar{\Sigma}$ (i.e., the mapping is a one-to-one onto transformation preserving intersections, complements, partitions and refinements of partitions).

The straightforward but rather detailed proof of the lemma appears in the appendix.

Unlike Example 2.2 where we found that the indicator function of a set in Σ did not have a right Cauchy integral, the indicator functions of sets in $\bar{\Sigma}$ are always c -integrable.

3.2. Proposition. Let

$$S = \sum_{i=1}^{n(\rho)} S_i \chi_{\bar{\rho}_i}$$

be an operator valued $\bar{\Sigma}$ -simple function. Then S is c -integrable and

$$(c) \int_y S(x) dE(y) = \sum_{i=1}^{n(\rho)} S_i E(\rho_i)$$

where the Σ -partition, ρ , of Y is the unique preimage of the $\bar{\Sigma}$ -partition $\bar{\rho}$ of X under the mapping of Lemma 3.1.

Proof. If $S = \sum_{i=1}^{n(\bar{\rho})} S_i \chi_{\bar{\rho}_i}$ is a $\bar{\Sigma}$ -simple function, the lemma guarantees the existence of a unique Σ -partition ρ such that $c(\mu) \in \bar{\rho}_i$ for all $\mu \subset \rho_i$. Thus for any Σ -partition $\pi \geq \rho$ (i.e., π refines ρ) we have

$$(3.1) \quad \Sigma_{\pi} S = \sum_{j=1}^{n(\pi)} T(c(\pi_j)) E(\pi_j) = \sum_{i=1}^{n(\rho)} S_i \left[\sum_{\{j: \pi_j \subset \rho_i\}} E(\pi_j) \right] = \sum_{i=1}^{n(\rho)} S_i E(\rho_i).$$

Since this is independent of π ,

$$(3.2) \quad (c) \int S(x) dE(y) = \lim_{\pi} \Sigma_{\pi} S = \lim_{\pi \geq \rho} \Sigma_{\pi} S = \sum_{i=1}^{n(\rho)} S_i E(\rho_i),$$

which completes the proof.

The mapping of Lemma 3.1 also allows us to “transfer” the spectral measure E and the Cauchy valuation c from Y to X .

3.3. Proposition. For any $\bar{\sigma} \in \bar{\Sigma}$ let $\bar{E}(\bar{\sigma}) = E(\sigma)$ and $\bar{c}(\bar{\sigma}) = c(\sigma)$ where σ is the unique preimage in Σ of $\bar{\sigma}$. Then $\bar{c}$ is a Cauchy valuation in $\bar{\Sigma}$, $\bar{E}$ is a spectral measure on $\bar{\Sigma}$ and the Cauchy integral defined by the 5-tuple $(X, X, \bar{\Sigma}, \bar{c}, \bar{E})$ is equivalent to the Cauchy integral defined by the 5-tuple (X, Y, Σ, c, E) in the sense that the two integrals are defined on the same set of functions and coincide when defined.

Proof. The fact that $\bar{E}$ is a spectral measure on $\bar{\Sigma}$ and $\bar{c}$ a Cauchy valuation is an immediate consequence of the isomorphic character of the mapping from σ to $\bar{\sigma}$. Now for any T

$$(3.3) \quad \Sigma_{\bar{p}} T = \sum_{i=1}^{n(\bar{\rho})} T(\bar{c}(\bar{\rho}_i)) \bar{E}(\bar{\rho}_i) = \sum_{i=1}^{n(\rho)} T(c(\rho_i)) E(\rho_i) = \Sigma_{\rho} T,$$

and since the mapping from Σ to $\bar{\Sigma}$ preserves ordering (by refinement), the limit of the $\Sigma_{\bar{p}} T$ over the $\bar{\Sigma}$ -partitions must exist simultaneously and coincide with the limit of the $\Sigma_{\rho} T$ over the Σ -partitions; hence

$$(3.4) \quad (c) \int_Y T(x) dE(y) = (\bar{c}) \int_X T(x) d\bar{E}(x)$$

for all T . As such, if we say that two integrals are equivalent if they are defined on the same class of functions and have values which coincide when defined, the proof is complete.

3.4. Corollary. Every Cauchy integral is equivalent to a Cauchy integral wherein $X = Y$ and $c(\sigma) \in \sigma$.

Note that for the right Cauchy integral $\bar{\Sigma}$ is composed of left-open right-closed intervals on $\mathbf{R}^+$. Hence indicator functions of the form $\chi_{(s, t]}$ are assured to be c -integrable by Proposition 3.3, but the function $\chi_{[0, 1]}$ of Example 2.2 is not assured to be c -integrable.

Since the $\bar{\Sigma}$ -simple functions are c -integrable it is natural to define a measurable function as the limit of a sequence of such simple functions.

3.5. Definition. For each $x \in X$ and bounded linear operator A , let

$$\|A\|_x^c = \lim_{\sigma \in \Sigma(x)} \|AE(\sigma)\|$$

and for each operator valued function T defined on X let

$$\|T\|_{\infty}^c = \sup_{x \in X} \|T(x)\|_x^c.$$

$\|\cdot\|_x^c$ is well defined since $\|AE(\sigma)\|$ is a decreasing function of σ , and is a seminorm since $\|\cdot\|$ is a norm. Similarly, $\|\cdot\|_{\infty}^c$ is a seminorm.

3.6. Definition. An operator valued function T on X is c -totally measurable, if there exists a sequence of operator valued $\bar{\Sigma}$ -simple functions, S^i , such that

$$\lim_{i \rightarrow \infty} \|S^i - T\|_{\infty}^c = 0.$$

c -total measurability is essentially the same concept used by Dinculeanu [4] except that by taking our limits in $\|\cdot\|_\infty^c$, rather than $\|\cdot\|_\infty$, the approximation is oblivious to "parts" of $T(x)$ which are orthogonal to the range of the measure E . As such, c -total measurability incorporates a generalized measure-zero concept analogous to that used by Robertson and Rosenberg [11] for matrix valued measures.

With these definitions we may now state our main theorem on the c -total measurability of c -integrable functions.

3.7. Theorem. *Let T be a c -integrable function on X . Then:*

- (i) T is c -totally measurable.
- (ii) $\|(c)\int_Y T(x) dE(y)\| \geq \|T\|_\infty^c$.
- (iii) $\|\hat{A} - T\|_\infty^c = 0$ where $\hat{A}$ is the constant function equal to $(c)\int_Y T(x) dE(y)$ for all $x \in X$.

Note that unlike the Lebesgue integral [4], c -total measurability does not imply c -integrability, since the c -integral is an unbounded operator (with respect to the topology induced by $\|\cdot\|_\infty$ and hence also with respect to the topology induced by $\|\cdot\|_\infty^c$ which is coarser than that induced by $\|\cdot\|_\infty$). The proof of the theorem requires the following lemma.

3.8. Lemma. *For any $\bar{\Sigma}$ -simple function $S = \sum_{i=1}^{n(\rho)} S_i \chi_{\bar{p}_i}$*

$$\left\| (c) \int_Y S(x) dE(y) \right\| \geq \|S\|_\infty^c.$$

Proof. For any $x \in X$ let $\bar{\sigma}$ be an element of $\bar{\Sigma}(x)$ contained in a single element of $\bar{p}$, say $\bar{p}_k$. Axiom (ii) in the definition of $\bar{c}$ assures that such a $\bar{\sigma}$ will exist, whereas the fact that $x = \bar{c}(\bar{\sigma}) \in \bar{\sigma} \subset \bar{p}_k$ implies that $S(x) = S_k$. Now

$$\begin{aligned} (3.4') \quad \left\| \sum_{i=1}^{n(\bar{p})} S_i E(\rho_i) \right\| &\geq \left\| \sum_{i=1}^{n(\bar{p})} S_i \bar{E}(\bar{p}_i) \right\| \|\bar{E}(\bar{\sigma})\| \geq \left\| \sum_{i=1}^{n(\bar{p})} S_i \bar{E}(\bar{p}_i) \bar{E}(\bar{\sigma}) \right\| \\ &= \|S_k \bar{E}(\bar{\sigma})\| = \|S(x) \bar{E}(\bar{\sigma})\| \geq \|S(x)\|_\infty^c. \end{aligned}$$

Here the first inequality follows from the fact that $\|\bar{E}(\bar{\sigma})\| \leq 1$ and $E(\rho_i) = \bar{E}(\bar{p}_i)$, the second from the multiplicative triangle inequality, the third from the fact that $\bar{\sigma} \subset \bar{p}_k$ and that the other $\bar{p}_i$ are disjoint from $\bar{p}_k$, and the last from the fact that $\|S(x) \bar{E}(\bar{\sigma})\|$ is a decreasing function for $\bar{\sigma} \in \bar{\Sigma}(x)$. Finally, taking the sup over all x on both sides of (3.4) we obtain the desired result.

Proof of the theorem. If the Cauchy integral of T exists, the partial sums

$$(3.5) \quad \Sigma_\rho T = \sum_{i=1}^{n(\rho)} T(c(\rho_i)) E(\rho_i) = \sum_{i=1}^{n(\rho)} T(\bar{c}(\bar{p}_i)) \bar{E}(\bar{p}_i)$$

converge, and we may define the $\bar{\Sigma}$ -simple functions

$$(3.6) \quad S_{\bar{p}} = \sum_{i=1}^{n(\bar{p})} T(\bar{c}(\bar{p}_i)) \chi_{\bar{p}_i}.$$

We now propose to extract the appropriate sequence of simple functions from the $S_{\bar{\rho}}$. Since the $\Sigma_{\rho}T$ converge, there exists a partition π^1 such that for ρ , $\theta > \pi^1$ $\|\Sigma_{\rho}T - \Sigma_{\theta}T\| < 1$. Let $S^1 = S_{\pi^1}$. Given $S^i = S_{\pi^i}$, $i = 1, 2, \dots, n$, let ϕ be a partition such that for ρ , $\theta > \phi$: $\|\Sigma_{\rho}T - \Sigma_{\theta}T\| < 1/n + 1$, and let $\pi^{n+1} > \{\phi; \pi^i, i = 1, 2, \dots, n\}$. Such a π^{n+1} exists since the partitions are directed by refinement and we may let $S^{n+1} = S_{\pi^{n+1}}$. Now, by construction, the Cauchy integrals of the S^i converge to the Cauchy integral of T (since they form a sub-net of the $\Sigma_{\rho}T$), and hence the lemma implies that the S^i form a Cauchy sequence in the $\|\cdot\|_{\infty}^c$ topology. It remains to show that they converge to T .

By construction for any N and $i \geq N$, $\|S^i - S^N\|_{\infty}^c < 1/N$, whereas for any $\rho > \pi^N$, $\|\Sigma_{\rho}T - \Sigma_{\pi^N}T\| < 1/N$. Hence for any $x \in X$, let $\sigma \in \Sigma(x)$ be contained in an element of π^N , say π_j^N , and decompose $\pi_j^N \setminus \sigma$ as

$$(3.7) \quad \pi_j^N \setminus \sigma = \bigcup_{r=1}^k \mu_r.$$

We may thus define a new partition ρ of Y by

$$\rho = \{\pi_i^N, i = 1, 2, \dots, j-1, j+1, \dots, n(\pi^N); \mu_r, r = 1, 2, \dots, k; \sigma\}$$

which refines π^N . As such

$$(3.8) \quad 1/N > \|\Sigma_{\rho}T - \Sigma_{\pi^N}T\| \geq \|S_{\bar{\rho}} - S_{\pi^N}\|_{\infty}^c \geq \|S_{\bar{\rho}}(x) - S_{\pi^N}(x)\|_x^c \\ = \|T(x) - S_{\pi^N}(x)\|_x^c.$$

Here the last equality results from the fact the $\bar{c}(\bar{\sigma}) = c(\sigma) = x \in \bar{\sigma}$ implying that $S_{\bar{\rho}}(x) = T(\bar{c}(\bar{\sigma})) = T(x)$. We thus have for any $i \geq N$,

$$(3.9) \quad \|T(x) - S^i(x)\|_x^c = \|T(x) - S_{\pi^N}(x) + S^N(x) - S^i(x)\|_x^c \\ \leq \|T(x) - S_{\pi^N}(x)\|_x^c + \|S^N(x) - S^i(x)\|_x^c \leq 2/N,$$

showing that the S^i converge to T , as required for c -total measurability of T . This proves (i). Of course, since the Cauchy integrals of the S^i converge in the norm topology to A and the S^i converge in the $\|\cdot\|_{\infty}^c$ topology to T , (ii) follows immediately from the lemma. Finally, (ii) implies that the integral operator is a one-to-one transformation on the space of c -integrable functions modulo equivalences in the seminorm topology induced by $\|\cdot\|_{\infty}^c$; hence (iii) is implied by the fact that the integrals of T and $\hat{A}$ coincide, and the proof of the theorem is complete.

Since the Cauchy integral defines an unbounded linear operator, an equality opposite that of (ii) is not in general possible. If, however, the range of T is appropriately restricted such an inequality can be obtained. For this purpose we define the following seminorms.

3.9. Definition. For each $x \in X$ and bounded linear operator A , let

$$\|A\|_x^* = \lim_{\sigma \in \Sigma(x)} \|A\| \|E(\sigma)\|,$$

and for each operator valued function T defined on X , let

$$||T||_{\infty}^{\circ} = \sup_{x \in X} ||T(x)||_x^{\circ}.$$

Since $||E(\sigma)||$ is either zero or one, $||A||_x^{\circ}$ is equal to $||A||$ or it is zero, depending on whether or not $\Sigma(x)$ is eventually [9] in a set of E -measure zero. As such, both $||\cdot||_x^{\circ}$ and $||\cdot||_{\infty}^{\circ}$ are seminorms, the latter serving as an "essential sup norm" modulo the appropriate sets of E -measure zero for the Cauchy integral.

3.10. Corollary. *Let T be a c -integrable function defined on X which satisfies*

$$E(\sigma)T(x) = T(x)E(\sigma); \quad x \in X \text{ and } \sigma \in \Sigma.$$

Then

$$||T||_{\infty}^{\circ} \geq \left\| (c) \int_Y T(x) dE(y) \right\| \geq ||T||_{\infty}^{\circ}.$$

The proof of the corollary involves a straightforward manipulation of the partial sums combined with the orthogonality of the ranges of the $E(\rho_i)$'s, and will not be given. Although the above corollary yields an upper bound on the norm of the Cauchy integral, it does not assure that the integral will be a bounded linear operator since $||T||_{\infty}^{\circ}$ may be strictly greater than $||T||_{\infty}^{\circ}$. This can, however, be achieved by a further restriction on the values of T .

3.11. Corollary. *Let T be an operator valued function defined on X which satisfies*

$$E(\sigma)T(x) = T(x)E(\sigma); \quad x \in X \text{ and } \sigma \in \Sigma$$

and

$$||T(x)E(\sigma)|| = ||T(x)|| ||E(\sigma)||; \quad x \in X \text{ and } \sigma \in \Sigma.$$

Then

- (i) T is c -integrable if and only if T is c -totally measurable.
- (ii) $||T||_{\infty}^{\circ} = ||(c) \int_Y T(x) dE(y)||$.
- (iii) $||T||_{\infty}^{\circ} = ||T||_{\infty}^{\circ}$.

Proof. If

$$(3.10) \quad ||T(x)E(\sigma)|| = ||T(x)|| ||E(\sigma)||; \quad x \in X \text{ and } \sigma \in \Sigma,$$

then

$$(3.11) \quad ||T(x)||_x^{\circ} = \lim_{\sigma \in \Sigma(x)} ||T(x)E(\sigma)|| = \lim_{\sigma \in \Sigma(x)} ||T(x)|| ||E(\sigma)|| = ||T(x)||_{\infty}^{\circ}.$$

Hence

$$(3.12) \quad ||T||_{\infty}^{\circ} = ||T||_{\infty}^{\circ},$$

and (ii) follows from Corollary 3.10. As such, the c -integral is continuous in the topology induced by $||\cdot||_{\infty}^{\circ}$ for this class of function, hence c -total measurability implies c -integrability, and the proof is complete.

Although the class of functions to which Corollary 3.11 applies would seem to be rather restrictive, they appear quite commonly in a number of applications [14]. Of course, scalar valued functions are included (*i.e.*, functions whose values are operators of the form kI where k is a scalar). More generally, if $H = L_2(Y, K, \mu)$, K a Hilbert space, and $E(\sigma)$ is taken to be multiplication by the indicator function of σ , then the operation of "multiplying" a function in $L_2(Y, K, \mu)$ by an operator on K satisfies the hypotheses of the corollary. Finally, if Y is a second countable, locally compact abelian topological group and U is a unitary group on Y such that the imprimitivity equality

$$(3.13) \quad U^y E(\sigma) = E(\sigma + y) U^y$$

holds for all $y \in Y$, then those operators which commute with both E and U satisfy the hypotheses of the corollary [14].

IV. Appendix—proof of Lemma 1.

Proof. By construction the mapping is onto. On the other hand if $\bar{\sigma} = \bar{\mu}$ we consider the set

$$(4.1) \quad \sigma \setminus \mu = \bigcup_{k=1}^n \delta_i$$

which is non-void if $\sigma \not\subset \mu$. Now in this case $\delta_1 \subset \sigma$; hence $c(\delta_1) \in \bar{\sigma} = \bar{\mu}$. Thus there must be a set $\Delta \subset \mu$ such that $c(\Delta) = c(\delta_1)$. As such, both δ_1 and Δ are in $\Sigma(c(\delta_1))$ and there exists a non-void set $\omega \in \Sigma(c(\delta_1))$ such that

$$(4.2) \quad \emptyset \neq \omega \subset \delta_1 \cap \Delta \subset \sigma \setminus \mu \cap \mu = \emptyset,$$

which, being contradictory, assures that $\sigma \subset \mu$. Upon involving a symmetric argument we have $\mu \subset \sigma$, hence $\bar{\sigma} = \bar{\mu}$ implies that $\sigma = \mu$, showing that the mapping is one-to-one.

If $\sigma \subset \mu$, all subsets of σ are also subsets of μ ; hence $\bar{\sigma} \subset \bar{\mu}$. Since $\sigma \cap \mu \subset \mu$ and $\sigma \cap \mu \subset \sigma$, this implies that $\sigma \cap \mu \subset \bar{\sigma} \cap \bar{\mu}$. Conversely, if $x \in \bar{\sigma} \cap \bar{\mu}$ there exists in $\Sigma(x)$ sets $\delta \subset \sigma$ and $\Delta \subset \mu$ and $\omega \subset \delta \cap \Delta \subset \sigma \cap \mu$; hence $x = c(\omega) \in \sigma \cap \mu$. Thus $\bar{\sigma} \cap \bar{\mu} \subset \sigma \cap \mu$, which combined with our previous containment yields

$$(4.3) \quad \bar{\sigma} \cap \bar{\mu} = \overline{\sigma \cap \mu}.$$

Hence intersections are preserved by the mapping, and $\bar{\Sigma}$ is indeed closed under intersections.

If $\rho = \{\rho_i\}_{i=1}^{n(\rho)}$ is a Σ -partition of Y , then by the above intersection condition the $\bar{\rho}_i$ are disjoint. Moreover, for any $x \in X$, $\Sigma(x)$ is eventually contained in a single element of ρ , say ρ_i . Letting ω be such a set contained in ρ_i we have $x = c(\omega) \in \bar{\rho}_i$ showing that the $\bar{\rho}_i$ cover X and hence $\bar{\rho} = \{\bar{\rho}_i\}_{i=1}^{n(\rho)}$ is a partition of X . Finally, if

$$(4.4) \quad \sigma \setminus \mu = \bigcup_{i=1}^k \delta_i,$$

we may write

$$(4.5) \quad Y \setminus \sigma = \bigcup_{i=1}^m \Delta_i,$$

whence $\{\delta_i, i = 1, 2, \dots, k; \bar{\Delta}_i, i = 1, 2, \dots, m; \mu\}$ is a partition of $\bar{Y}$, and by the above argument $\{\bar{\delta}_i, i = 1, 2, \dots, k; \bar{\Delta}_i, i = 1, 2, \dots, m; \bar{\mu}\}$ is a partition of X . Now by the above intersection and containment arguments the $\bar{\delta}_i$ must partition $\bar{\sigma} \setminus \bar{\mu}$, showing that

$$(4.6) \quad \bar{\sigma} \setminus \bar{\mu} = \bigcup_{i=1}^k \delta_i$$

as required for $\bar{\Sigma}$ to be closed under complements and complements to be preserved by our mapping from Σ to $\bar{\Sigma}$. The proof is thus complete.

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