Asymptotic Estimates for Solutions of Linear Systems of Ordinary Differential Equations Having Multiple Characteristic Roots

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§1 Introduction. In a well known and by now classical paper [8], N. Levinson in 1948 gave a general theorem for asymptotically estimating a complete set of linearly independent solutions for a system of differential equations of the form

$$(1.1) \quad y'(t) = [A + V(t) + R(t)]y(t), \quad t \in (0, \infty),$$

where A is a constant square matrix with simple characteristic roots, $V(t) \rightarrow 0$ as $t \rightarrow \infty$ and is absolutely continuous on some interval $[t_0, \infty)$, and the elements of $R(t)$ are Lebesgue integrable. In 1954, M. A. Naimark, in his well known book on linear differential operators [9], showed how such types of asymptotic theorems could be used to obtain results on the difficult and important deficiency index problem for ordinary self-adjoint differential operators.

In 1966, one of us [3] (but see [4] for a corrected treatment), by generalizing and refining Naimark's techniques, was able to use Levinson's asymptotic theorem to compute the deficiency index of a wide class of self-adjoint differential operators. In the same year M. V. Fedorjuk [7] proved an asymptotic theorem for the solutions of a class of even order differential operators and used this to compute their deficiency indices. Fedorjuk's asymptotic theorem is not a consequence of Levinson's theorem, nor does it contain the latter. In [5] an asymptotic theorem is given which contains Levinson's theorem and will also essentially yield Fedorjuk's theorem.

All of the asymptotic theorems mentioned above assume that the matrix A in (1.1) has only simple eigenvalues. However, recent investigations on the deficiency index of a certain class of fourth order differential operators has indicated the necessity of being able to obtain asymptotic theorems when A has multiple eigenvalues. Such types of theorems were obtained in [6], [10], and [11] for special fourth order systems in which A has two simple eigenvalues and one double eigenvalue.

In this paper we shall obtain a theorem about asymptotic estimates of solutions of systems of the form (1.1) in which the characteristic polynomial of A is allowed to have multiple roots. The exact conditions will be given in the next section. Although the theorem we obtain in this paper does not contain the special asymptotic theorems mentioned in the previous paragraph, nevertheless it is an asymptotic theorem of a fairly general nature which allows multiple characteristic roots for A , and it does contain Levinson's theorem as well as a theorem obtained in [2].

§2. Statement of theorem. Suppose

$$(2.1) \quad y'(t) = [A + V(t) + R(t)]y(t)$$

is a system of differential equations defined on $(0, \infty)$. We suppose that A is a constant $n \times n$ matrix whose minimal polynomial is of degree n , and is of the form

$$\chi(\lambda) = \prod_{k=1}^m (\lambda - \lambda^k)^{n_k}, \quad \lambda^j \neq \lambda^k \quad \text{if } j \neq k, \quad \sum_{k=1}^m n_k = n.$$

Let $r + 1 = \max n_k$. Suppose that $V(t)$ is an $n \times n$ matrix satisfying:

(2.2) $V(t)$ is absolutely continuous on each compact subinterval in $(0, \infty)$,

(2.3) $t^r |V'(t)| \in L^1[t_0, \infty)$, and $V(t) \rightarrow 0$ as $t \rightarrow \infty$,

where t_0 is some sufficiently large number, and $|V'(t)|$ denotes the sum of the absolute values of the entries of $V'(t)$. Suppose further, that $R(t)$ is a Lebesgue measurable $n \times n$ matrix satisfying the condition

$$(2.4) \quad t^r |R(t)| \in L^1[t_0, \infty).$$

Let the roots of $\det(A + V(t) - \lambda I) = 0$ be denoted by $\{\lambda_i^j(t)\}$, where $1 \leq i \leq m$ and $1 \leq j \leq n_i$. We assume that it is possible to choose the $\lambda_i^j(t)$ so that they are absolutely continuous and for $1 \leq j \leq n_i$, $\lambda_i^j(t) \rightarrow \lambda^i$ as $t \rightarrow \infty$. Assume also that the eigenvalues $\lambda_i^j(t)$ satisfy the following:

$$(2.5) \quad \int_{t_0}^{\infty} t^r |\lambda_i^{j'}(t)| dt < \infty, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n_i, \quad \text{or equivalently,}$$

$$(2.5') \quad \int_{t_0}^{\infty} t^r \{ |\lambda_i^{j'}(t) - \lambda_k^{l'}(t)| + |\lambda_k^{l'}(t)| |\lambda_\mu^i(t) - \lambda_\nu^l(t)| \} dt < \infty,$$

$$1 \leq i, l \leq m, \quad 1 \leq j, k \leq n_i, \quad 1 \leq \mu, \nu \leq n_l.$$

Finally, let

$$\mu_{ka} = \lambda_1^k(t) + \frac{q-1}{t}.$$

Theorem 2.1. Suppose that (2.2)–(2.5) hold for equation (2.1). Suppose, further, that for each fixed (k, q) , $1 \leq k \leq m$, $1 \leq q \leq n_k$, all of the pairs (j, p) , $1 \leq j \leq m$, $1 \leq p \leq n_j$, fall into one of two classes I_{ka} and J_{ka} , where

$$(2.6) \quad (j, p) \in I_{ka} \text{ if } \int_{t_1}^t \operatorname{Re} [\mu_{ka}(\sigma) - \mu_{jp}(\sigma)] d\sigma \rightarrow \infty \text{ as } t \rightarrow \infty$$

and

$$\int_{\tau}^t \operatorname{Re} [\mu_{ka}(\sigma) - \mu_{jp}(\sigma)] d\sigma > -K \text{ for } t \geq \tau > 0,$$

$$(2.7) \quad (j, p) \in J_{ka} \text{ if } \int_{\tau}^t \operatorname{Re} [\mu_{ka}(\sigma) - \mu_{jp}(\sigma)] d\sigma < K \text{ for } t \geq \tau > 0,$$

where K is a constant. Then there exists $t_0 \geq 0$, eigenvectors q_i corresponding to λ^i , $1 \leq i \leq m$, and n linearly independent functions $y_i^i(t)$, $1 \leq i \leq m$, $1 \leq j \leq n_i$, which are solutions of (2.1) satisfying

$$y_i^i(t)t^{-i+1} \exp \left[- \int_{t_0}^t \lambda_1^i(\tau) d\tau \right] \rightarrow q_i \text{ as } t \rightarrow \infty.$$

We will require the following result.

Lemma 2.1. *Let condition (2.5) be satisfied for $t_0 \geq 1$, $r \geq 1$. Then*

$$(2.8) \quad \int_{t_0}^{\infty} [\lambda^i - \lambda_i^i(t)] dt$$

converges to a finite value for $1 \leq i \leq m$, $1 \leq j \leq n_i$.

Proof. We know that

$$|\lambda^i - \lambda_i^i(t)| = \left| \int_t^{\infty} \lambda_i^{i'}(\tau) d\tau \right| \leq \int_t^{\infty} |\lambda_i^{i'}(\tau)| d\tau.$$

Integrating both sides of this inequality, and changing the order of integration, we obtain

$$\int_{t_0}^{\infty} |\lambda^i - \lambda_i^i(t)| dt \leq \int_{t_0}^{\infty} dt \int_t^{\infty} |\lambda_i^{i'}(\tau)| d\tau \leq \int_{t_0}^{\infty} \tau |\lambda_i^{i'}(\tau)| d\tau.$$

Since, for $t_0 \geq 1$, $r \geq 1$, we have

$$\int_{t_0}^{\infty} \tau |\lambda_i^{i'}(\tau)| d\tau \leq \int_{t_0}^{\infty} \tau^r |\lambda_i^{i'}(\tau)| d\tau,$$

our result follows from (2.5).

It is immediate from Lemma 2.1 that the following holds:

Corollary 2.1. *Let condition (2.5) be satisfied for $r \geq 1$ and $t_0 \geq 1$. Then*

$$(2.9) \quad \int_{t_0}^{\infty} [\lambda_i^i(t) - \lambda_k^i(t)] dt$$

converges to a finite value for $1 \leq i \leq m$, $1 \leq j, k \leq n_i$.

Before proceeding to the proof of Theorem 2.1, let us discuss some of the hypotheses.

First, let us remark that if the matrix A has simple eigenvalues, then we always have $n_i = n_k = p = q = 1$, and so conditions (2.6) and (2.7) reduce to the corresponding conditions in [1].

Now, if we can show that (2.5) is equivalent to (2.5'), it will follow that our theorem is in fact a generalization of previous theorems, since (2.5') is trivially satisfied under the hypotheses of [1], or [2]. Clearly, since each $\lambda_i^i(t)$ is bounded, (2.5) implies (2.5'). To demonstrate the converse, let $\lambda_{a_i}^i(t)$ be fixed, $1 \leq i \leq m$. Let $\chi(\lambda, t)$ denote the characteristic polynomial of $A + V(t)$. Then

$$(-1)^n \chi(\lambda, t) = \prod_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n_i}} (\lambda - \lambda_i^j(t)) = \sum_{l=0}^m f_l(t) \lambda^{n-l}.$$

Set

$$g(\lambda, t) = \prod_{1 \leq i \leq m} (\lambda - \lambda_{a_i}^i(t))^{n_i} = \sum_{l=0}^n g_l(t) \lambda^{n-l}.$$

Then

$$f_l(t) = (-1)^l \sum \lambda_{i_1}^{i_1}(t) \lambda_{i_2}^{i_2}(t) \cdots \lambda_{i_l}^{i_l}(t),$$

where the summation is taken over all l -fold products with $i_1 \leq i_2 \leq \cdots \leq i_l$, $1 \leq j_k \leq n_{i_k}$, $1 \leq i_k \leq m$, $1 \leq k \leq l$, so that if $i_p = i_q$, then $j_p < j_q$ if $p < q$. The function $g_l(t)$ is of the same form as $f_l(t)$, except that each $\lambda_{i_k}^{i_k}(t)$ is replaced by the eigenvalue $\lambda_{a_e}^e(t)$ where $e = i_k$.

We will restrict our analysis to the terms $f_2(t)$ and $g_2(t)$, the analysis of the other terms being similar. Let us compute $f_2'(t)$ and $g_2'(t)$. To do this, we look at a representative term in the sum defining $f_2(t)$. We have

$$[\lambda_{i_p}^{i_p}(t) \lambda_{i_q}^{i_q}(t)]' = \lambda_{i_p}^{i_p'}(t) \lambda_{i_q}^{i_q}(t) + \lambda_{i_p}^{i_p}(t) \lambda_{i_q}^{i_q'}(t).$$

Corresponding to this term, in the sum defining $g_2(t)$, is the term

$$[\lambda_{a_s}^s(t) \lambda_{a_d}^d(t)]' = \lambda_{a_s}^{s'}(t) \lambda_{a_d}^d(t) + \lambda_{a_s}^s(t) \lambda_{a_d}^{d'}(t) \quad \text{where } s = i_p, d = i_q.$$

Hence, using the fact that each $\lambda_j^i(t)$ is bounded on $[t_0, \infty)$, t_0 sufficiently large, we have

$$f_2'(t) - g_2'(t) = O\left(\sum_{\substack{1 \leq i \leq m \\ 1 \leq j \leq n_i}} |\lambda_j^{i'}(t) - \lambda_{a_i}^{i'}(t)|\right) + O\left(\sum_{\substack{1 \leq i, j \leq m \\ 1 \leq k \leq n_i}} |\lambda_{a_i}^{i'}(t)| |\lambda_k^i(t) - \lambda_{a_i}^i(t)|\right),$$

where O is the Hardy-Littlewood symbol. Hence, if (2.5') holds it follows that $t^r(f_2'(t) - g_2'(t)) \in L^1$. But by condition (2.3) we know that $t^r f_2'(t) \in L^1$, since $f_2(t)$ is a coefficient in the characteristic polynomial of $A + V(t)$. It follows that $t^r g_2'(t) \in L^1$. As we noted before, a similar analysis shows that $t^r g_l(t) \in L^1$ for $1 \leq l \leq m$. Thus

$$t^r \frac{\partial g(\lambda, t)}{\partial t} \bigg|_{\lambda=c} \in L^1 \quad \text{where } c = \lambda_{a_k}^k(t).$$

Let us set

$$G_k(\lambda, t) = \frac{\partial^{n_k-1} g(\lambda, t)}{\partial \lambda^{n_k-1}}, \quad t \geq t_0.$$

Clearly,

$$t^r \frac{\partial G_k(\lambda, t)}{\partial t} \Big|_{\lambda=c} \in L^1.$$

Further, since $G_k(\lambda_{a_k}^k(t), t) = 0$ we have

$$\frac{d}{dt} G_k(\lambda_{a_k}^k(t), t) = \frac{\partial G_k(\lambda, t)}{\partial \lambda} \Big|_c \cdot \lambda_{a_k}^{k'}(t) + \frac{\partial G_k(\lambda, t)}{\partial t} \Big|_c = 0.$$

Thus

$$\lambda_{a_k}^{k'}(t) = - \frac{\partial G_k(\lambda, t)}{\partial \lambda} \Big|_c / n! \prod_{\substack{j=1 \\ j \neq k}}^m (\lambda_{a_k}^k(t) - \lambda_{a_j}^j(t)).$$

Since the denominator is bounded away from zero as $t \rightarrow \infty$ we see that $t^r \lambda_{a_k}^{k'}(t) \in L^1$.

Finally, we give an example to show that hypothesis (2.3) does not necessarily imply (2.5).

Example 2.1. Let $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Thus A has a double eigenvalue at $\lambda = 1$. Let

$$V(t) = \begin{bmatrix} 0 & 0 \\ t^{-3/2} & 0 \end{bmatrix}.$$

Clearly, $V(t) \rightarrow 0$ as $t \rightarrow \infty$, $t |V(t)| \notin L^1$, and $t |V'(t)| \in L^1$. However, the roots of $A + V(t)$, which we can denote by $\lambda_1(t)$ and $\lambda_2(t)$, are given by

$$\lambda_1(t) = 1 + t^{-3/4} \quad \lambda_2(t) = 1 - t^{-3/4}.$$

Note that $t |\lambda_i'(t)| = \frac{3}{4} t^{-3/4} \notin L^1$, $i = 1, 2$.

§3. Proof of special case. We will restrict ourselves, at first, to a differential system of the form

$$(3.1) \quad y'(t) = [\Lambda(t) + R_1(t)]y(t)$$

where R_1 satisfies (2.4), and $\Lambda(t) = \text{diag} [\Lambda_1(t), \Lambda_2(t), \dots, \Lambda_m(t)]$. For $t \geq t_0$ each $\Lambda_i(t)$ is an $n_i \times n_i$ matrix of the form

$$(3.2) \quad \Lambda_i(t) = \begin{bmatrix} \lambda_1^i(t) & 1 & 0 & \cdots & 0 \\ & \lambda_2^i(t) & 1 & \cdots & 0 \\ & & \ddots & \ddots & \vdots \\ 0 & & & \ddots & 1 \\ & & & & \lambda_{n_i}^i(t) \end{bmatrix}.$$

Let $Q(t) = \text{diag} [Q_1(t), Q_2(t), \dots, Q_m(t)]$, where $Q_i = \text{diag} [1, t^{-1}, t^{-2}, \dots, t^{-n_i+1}]$. Perform the change of variables

$$(3.3) \quad y(t) = Q(t)w(t).$$

We then obtain

$$(3.4) \quad w'(t) = [M(t) + R_2(t)]w(t)$$

where $R_2 = Q^{-1}R_1Q$ and $M = Q^{-1}AQ - Q^{-1}Q'$. The matrix M is of the form $M = \text{diag} [M_1(t), M_2(t), \dots, M_m(t)]$, where each block M_i is an $n_i \times n_i$ matrix of the form

$$M_i(t) = \begin{bmatrix} \lambda_1^i(t) & \frac{1}{t} & 0 & 0 & \cdots & 0 \\ & \lambda_2^i(t) + \frac{1}{t} & \frac{1}{t} & 0 & \cdots & 0 \\ & & \lambda_3^i(t) + \frac{2}{t} & \frac{1}{t} & \cdots & 0 \\ & & & \ddots & \ddots & \vdots \\ 0 & & & & \ddots & \frac{1}{t} \\ & & & & & \lambda_{n_i}^i(t) + \frac{n_i - 1}{t} \end{bmatrix}.$$

Since we have assumed that $t^r |R_1(t)| \leq L^1$, it follows that $|R_2| = |Q^{-1}R_1Q| \leq L^1$.

In order to examine the asymptotic behavior of the nonhomogeneous equation (3.4) let us first construct a fundamental matrix for the homogeneous system

$$(3.5) \quad w'(t) = M(t)w(t).$$

Clearly, there exists a fundamental matrix $\Phi(t)$ for (3.5) of the form $\Phi(t) = \text{diag} [\Phi_1(t), \Phi_2(t), \dots, \Phi_m(t)]$, where $\Phi_i(t)$ is a fundamental matrix for the system

$$(3.6) \quad u'(t) = M_i(t)u(t).$$

Let us, therefore, construct a fundamental matrix $\Phi_i(t)$ for (3.6). For simplicity of notation, let i be fixed, and let $p = n_i$. If u_p is the column vector $(u_{1p}, u_{2p}, \dots, u_{pp})$, then by quadratures we obtain

$$\begin{aligned} u_{pp}(t) &= \exp \left[\int_{t_0}^t \left[\lambda_p^i(\tau) + \frac{p-1}{\tau} \right] d\tau \right], \\ u_{p-1,p}(t) &= \exp \left[\int_{t_0}^t \left[\lambda_{p-1}^i(\tau) + \frac{p-2}{\tau} \right] d\tau \right] \\ &\quad \cdot \int_{t_0}^t \frac{1}{\tau} \exp \left[\int_{t_0}^{\tau} \left[\lambda_p^i(\sigma) - \lambda_{p-1}^i(\sigma) + \frac{1}{\sigma} \right] d\sigma \right] d\tau. \end{aligned}$$

But,

$$\int_{i_0}^t \frac{1}{\tau} \exp \left[\int_{i_0}^{\tau} \left[\lambda_p^i(\sigma) - \lambda_{p-1}^i(\sigma) + \frac{1}{\sigma} \right] d\sigma \right] d\tau$$

is asymptotic to $c_{p-1,p}^i t$ for some non-zero constant $c_{p-1,p}^i$ by (2.9). Thus we have

$$u_{p-1,p}(t) = c_{p-1,p}^i \exp \left[\int_{i_0}^t \left[\lambda_{p-1}^i(\tau) + \frac{p-1}{\tau} \right] d\tau \right] [1 + o(1)].$$

Proceeding in this way, in general we have

$$u_{i,p}(t) = c_{i,p}^i \exp \left[\int_{i_0}^t \left[\lambda_i^i(\tau) + \frac{p-1}{\tau} \right] d\tau \right] [1 + o(1)], \quad c_{i,p}^i \neq 0, \quad 1 \leq j \leq p,$$

where it is understood that the $o(1)$ term depends upon i, j and p .

Next, let u_{p-1} denote the column vector $(u_{1,p-1}, u_{2,p-1}, \dots, u_{p-1,p-1}, 0)$ and consider the system $u'_{p-1} = M_i u_{p-1}$. This is just a $(p-1) \times (p-1)$ system of the same form as (3.6), and proceeding as above we obtain

$$u_{i,p-1}(t) = c_{i,p-1}^i \exp \left[\int_{i_0}^t \left[\lambda_i^i(\tau) + \frac{p-2}{\tau} \right] d\tau \right] [1 + o(1)],$$

$$c_{i,p-1}^i \neq 0, \quad 1 \leq j \leq p-1.$$

Thus we obtain a fundamental matrix $\Phi_i(t)$ whose (j, k) th element is given by

$$c_{i,j,k}^i \exp \left[\int_{i_0}^t \left[\lambda_i^i(\tau) + \frac{k-1}{\tau} \right] d\tau \right] [1 + o(1)],$$

where $c_{i,j,k}^i \neq 0$ if $j \leq k$ and $c_{i,j,k}^i = 0$ if $j > k$. Moreover, by (2.9), we may replace every $\lambda_i^i(t)$ by $\lambda_1^i(t)$. In this way, the (j, k) th element of $\Phi_i(t)$ is given by

$$(3.7) \quad c_{i,j,k}^i \exp \left[\int_{i_0}^t \left[\lambda_1^i(\tau) + \frac{k-1}{\tau} \right] d\tau \right] [1 + o(1)],$$

where the $o(1)$ term depends upon i, j and k , and $c_{i,j,k}^i \neq 0$ if and only if $j \leq k$, but where the constants in (3.7) are, in general, different than those given previously. Finally, $\Phi(t) = \text{diag} [\Phi_1, \Phi_2, \dots, \Phi_m]$ is a fundamental matrix for (3.5).

In order to be able to solve (3.4) by variation of constants, it will be necessary to compute the matrix $\Phi^{-1}(t)$. To do this, observe that

$$(\Phi^{-1}(t))' = -\Phi^{-1}(t)\Phi'(t)\Phi^{-1}(t) = -\Phi^{-1}(t)M(t).$$

For any matrix $B(t)$, let $B_*(t)$ denote its transpose. We then have

$$(3.8) \quad (\Phi_*^{-1}(t))' = -M_*(t)\Phi_*^{-1}(t).$$

We can obtain a fundamental matrix $\psi(t)$ for (3.8) in a manner similar to the way in which we constructed Φ . Let $\psi = \text{diag} [\psi_1, \psi_2, \dots, \psi_m]$ where ψ_i is a

fundamental matrix for the system $v' = -M_{i,*}v$. Let v_i denote the column vector $(0, 0, \dots, 0, v_{i,i}, v_{i+1,i}, \dots, v_{p,i})$. Then, as previously, we have

$$v_{k,i}(t) = \tilde{d}_{k,i}^i \exp \left[- \int_{t_0}^t \left[\lambda_i^i(\tau) + \frac{j-1}{\tau} \right] d\tau \right] [1 + o(1)]$$

where $\tilde{d}_{k,i}^i \neq 0$ if and only if $k \geq j$, $1 \leq j$, $k \leq n_i$. Now, making use of (2.9), we may replace every $\lambda_j^i(t)$ by $\lambda_1^i(t)$. In this way, the $(k, j)^{\text{th}}$ element of $\psi_i(t)$ is given by

$$\tilde{d}_{k,i}^i \exp \left[- \int_{t_0}^t \left[\lambda_1^i(\tau) + \frac{j-1}{\tau} \right] d\tau \right] [1 + o(1)]$$

where the constants $\tilde{d}_{k,i}^i$ may be different than those given previously. The matrix $\psi(t) = [\psi_1, \psi_2, \dots, \psi_m]$ is thus a fundamental matrix for (3.8). To compute $\Phi^{-1}(t)$, we need only observe that $\Phi^{-1}(t) = \psi_*^{-1}(t)$, since they both satisfy (3.8), and $\Phi^{-1}(t_0) = \psi_*^{-1}(t_0) = I$. If we let $d_{i,k}^i = \tilde{d}_{k,i}^i$, then the $(j, k)^{\text{th}}$ entry of the i^{th} submatrix of $\Phi^{-1}(t)$ is

$$(3.9) \quad d_{i,k}^i \exp \left[- \int_{t_0}^t \left[\lambda_1^i(\tau) + \frac{j-1}{\tau} \right] d\tau \right] [1 + o(1)]$$

where $d_{i,k}^i \neq 0$ if and only if $j \leq k$.

Now $\Phi(t)\Phi^{-1}(\tau) = \text{diag} [\Phi_1(t)\Phi_1^{-1}(\tau), \Phi_2(t)\Phi_2^{-1}(\tau), \dots, \Phi_m(t)\Phi_m^{-1}(\tau)]$, where the (μ, ν) element of $\Phi_i(t)\Phi_i^{-1}(\tau)$ is given by

$$(3.10) \quad \sum_{l=\mu}^{\nu} c_{\mu,l}^i d_{l,\nu}^i \exp \left[\int_{t_0}^t \left[\lambda_1^i(\sigma) + \frac{l-1}{\sigma} \right] d\sigma \right] \\ \cdot \exp \left[- \int_{t_0}^{\tau} \left(\lambda_1^i(\sigma) + \frac{l-1}{\sigma} \right) d\sigma \right] [1 + o_{\mu\nu}(1)]$$

provided $1 \leq \mu \leq \nu \leq n_i$. The (μ, ν) element of $\Phi(t)\Phi^{-1}(\tau)$ is zero if $\mu > \nu$.

Before we may apply Levinson's technique, we require the following well known result.

Suppose $\Phi(t)$ is a solution to an $n \times n$ matrix equation $\Phi'(t) = A(t)\Phi(t)$, with column vectors $\varphi_1, \varphi_2, \dots, \varphi_n$. If B is a constant $n \times n$ matrix, the k^{th} column vector of ΦB is $\sum_{i=1}^n b_{ik}\varphi_i$, and hence if we write $\Phi B = \Phi_1 + \Phi_2$ where the k^{th} column of Φ_1 is $\sum_{i=1}^{n_k} b_{i1,k}\varphi_{i1}$, then $\Phi_1' = A\Phi_1$ and $\Phi_2' = A\Phi_2$.

Now let (k, q) be fixed. Write $\Phi(t)\Phi^{-1}(\tau) = \Phi_1(t, \tau) + \Phi_2(t, \tau)$ in the following manner. The non-zero components of $\Phi_1(t, \tau)$ consist of blocks down the main diagonal so that the (μ, ν) entry of the i^{th} block consists of the sum of those elements indexed by l in the sum (3.10) for which $(\mu, l) \in I_{kq}$, and the (μ, ν) entry of the i^{th} block of Φ_2 consists of the sum of the remaining elements. Notice that for fixed ν every (μ, ν) entry contains the sum over the same indices l for any μ , and hence the above result is applicable. Suppose, therefore, that we can construct a solution of the integral equation

$$(3.11) \quad w(t) = \Phi(t)e_{k_q} + \int_{t_0}^t \Phi_1(t, \tau)R_2(\tau)w(\tau) d\tau - \int_t^\infty \Phi_2(t, \tau)R_2(\tau)w(\tau) d\tau$$

where e_{k_q} is the unit vector with a one in the position $\sum_{i=1}^{k-1} n_i + q$, and zeros elsewhere. It follows that any solution of (3.11) will also be a solution of (3.4).

We will establish the existence of a solution of (3.11) by applying Levinson's technique using successive approximations. Define $w_0(t) = 0$, and

$$(3.12) \quad w_{i+1}(t) = \Phi(t)e_{k_q} + \int_{t_0}^t \Phi_1(t, \tau)R_2(\tau)w_i(\tau) d\tau - \int_t^\infty \Phi_2(t, \tau)R_2(\tau)w_i(\tau) d\tau.$$

Then $w_1(t) = \Phi(t)e_{k_q}$, and so

$$(3.13) \quad |w_1(t) - w_0(t)| \leq C \exp \left[\int_{t_0}^t \operatorname{Re} \left[\lambda_1^k(\tau) + \frac{q-1}{\tau} \right] d\tau \right] [1 + o(1)]$$

where C is a constant which depends upon the $c_{l,m}^i$, $d_{l,m}^i$ and the value of the constant in (2.9). Since each element of $\Phi(t)\Phi^{-1}(\tau)$ is of the form (3.10), and since if $(j, p) \in I_{k_q}$, then (2.6) holds, it follows that

$$(3.14) \quad |\Phi_1(t, \tau)R_2(\tau)| \leq Ce^K |R_2(\tau)| \exp \left[\int_\tau^t \operatorname{Re} \left[\lambda_1^k(\sigma) + \frac{q-1}{\sigma} \right] d\sigma \right] \cdot [1 + o(1)]$$

for $t_0 \leq \tau \leq t$. Similarly, we have

$$(3.15) \quad |\Phi_2(t, \tau)R_2(\tau)| \leq Ce^K |R_2(\tau)| \exp \left[- \int_t^\tau \operatorname{Re} \left(\lambda_1^k(\sigma) + \frac{q-1}{\sigma} \right) d\sigma \right] \cdot [1 + o(1)]$$

for $\tau \geq t$.

Let t_0 be chosen so large that for $t, \tau \geq t_0$, the terms of the form $1 + o(1)$ in (3.13), (3.14) and (3.15) are uniformly bounded by 2, and

$$(3.16) \quad 4Ce^K \int_{t_0}^\infty |R_2(\tau)| d\tau < 1/2.$$

Applying (3.13), (3.14), (3.15) and (3.16) to (3.12), we have

$$\begin{aligned} & |w_2(t) - w_1(t)| t^{-q+1} \exp \left[- \int_{t_0}^t \operatorname{Re} \lambda_1^k(\tau) d\tau \right] \\ & \leq 2Ce^K \left(\int_{t_0}^t + \int_t^\infty \right) |R_2(\tau)| |w_1(\tau) - w_0(\tau)| \\ & \quad \cdot \exp \left[- \int_{t_0}^\tau \operatorname{Re} \left[\lambda_1^k(\sigma) + \frac{q-1}{\sigma} \right] d\sigma \right] d\tau \\ & \leq 4C^2 e^K \int_{t_0}^\infty |R_2(\tau)| d\tau < C(1/2). \end{aligned}$$

In general

$$\begin{aligned}
 & |w_{i+1}(t) - w_i(t)| t^{-q+1} \exp \left[- \int_{t_0}^t \operatorname{Re} \lambda_1^k(\tau) d\tau \right] \\
 & \leq 2Ce^K \left(\int_{t_0}^t + \int_t^\infty \right) |R_2(\tau)| |w_i(\tau) - w_{i-1}(\tau)| \\
 & \quad \cdot \exp \left[- \int_{t_0}^\tau \operatorname{Re} \left[\lambda_1^k(\sigma) + \frac{q-1}{\sigma} \right] d\sigma \right] d\tau \\
 & \leq 4C^2 e^K (1/2)^{i-1} \int_{t_0}^\infty |R_2(\tau)| d\tau \leq C(1/2)^i.
 \end{aligned}$$

Proceeding in the same manner as the proof in Coddington and Levinson [1, p. 92], we are able to conclude the existence of a function $w(t)$ which is a solution of (3.12). Moreover, by the techniques of the proof in [1], it will follow that

$$\lim_{t \rightarrow \infty} \left\{ w(t) t^{-q+1} \exp \left[- \int_{t_0}^t \lambda_1^k(\tau) d\tau \right] - \gamma_{kq} \right\} = 0$$

where γ_{kq} is the n -vector given by $\gamma_{kq} = (0, \dots, 0, c_{1,q}^k, c_{2,q}^k, \dots, c_{a,q}^k, 0, \dots, 0)$ and the coefficients $c_{i,q}^k$ are those used in (3.7), provided that we make use of the fact that for any fixed $t_1 \geq t_0$, each term of the matrix

$$t^{-q+1} \exp \left[- \int_{t_0}^t \operatorname{Re} \lambda_1^k(\sigma) d\sigma \right] \Phi_1(t, \tau)$$

converges to zero as $t \rightarrow \infty$ uniformly for $\tau \in [t_0, t_1]$.

Recall that we had made the change of variables $y(t) = Q(t)w(t)$ in (3.3). Applying $Q(t)$ to the vector $w(t)$ just obtained, we find a solution $\tilde{y}(t)$ of (3.1) which is asymptotic to the column vector

$$\exp \left[\int_{t_0}^t \lambda_1^k(\tau) d\tau \right] (0, \dots, 0, c_{1,q}^k t^{q-1}, c_{2,q}^k t^{q-2}, \dots, c_{a,q}^k, 0, \dots, 0).$$

If we now set $y(t) = \tilde{y}(t)/c_{1,q}^k$, we have

$$y(t) t^{-q+1} \exp \left[- \int_{t_0}^t \lambda_1^k(\tau) d\tau \right] \sim e_{k,1}.$$

§4. Reduction of general case. The proof of Theorem 2.1 will be completed upon establishing the following lemma.

Lemma 4.1. *Suppose the matrix $A + V(t)$ satisfies the conditions of Theorem 2.1. Then for all t sufficiently large there exists an absolutely continuous and invertible matrix $B(t)$ such that $t^r |B^{-1}(t)B'(t)| \in L^1$, $B^{-1}(t)[A + V(t)]B(t) = \Lambda(t)$, where $\Lambda(t) = \operatorname{diag} [\Lambda_1(t), \Lambda_2(t), \dots, \Lambda_m(t)]$ and each $\Lambda_i(t)$ is a "Jordan block" of the form (3.2). Moreover, $B(t) \rightarrow B$ and $B^{-1}(t) \rightarrow B^{-1}$ as $t \rightarrow \infty$, where $B^{-1}AB$ is a Jordan canonical form for A .*

Proof. Since the minimal and characteristic polynomials of A are of the same degree it is possible to choose m cyclic vectors $\{b_{n_i}^i\}$, $i = 1, 2, \dots, m$, having the following property:

For each fixed i , if we set

$$b_j^i = (A - \lambda^i)^{n_i - j} b_{n_i}^i, \quad j = 1, 2, \dots, n_i,$$

then the vectors $\{b_j^i\}$ are linearly independent. Let B denote the matrix whose columns are $(b_1^1, b_2^1, \dots, b_{n_1}^1, b_1^2, b_2^2, \dots, b_{n_2}^2, \dots, b_1^m, \dots, b_{n_m}^m)$. Then $B^{-1}AB$ is a Jordan canonical form for A .

Let V_i denote the subspace spanned by $\{b_j^i : j = 1, 2, \dots, n_i\}$. Let $\pi_i(A) = \prod_{j \neq i} (A - \lambda^j)^{n_j}$. Then $\pi_i(A)$ is invertible when restricted to V_i . Set $h_i = \pi_i^{-1}(A)b_{n_i}^i$, where by $\pi_i^{-1}(A)$ we mean the inverse of the restriction of $\pi_i(A)$ to V_i .

Let us write the characteristic polynomial of $A + V(t)$, (up to a factor of $(-1)^n$), as

$$\chi(\lambda, t) = \pi_i(\lambda, t) \prod_{1 \leq k \leq n_i} (\lambda - \lambda_k^i(t))$$

where

$$\pi_i(\lambda, t) = \prod_{\substack{j \neq i \\ 1 \leq k \leq n_j}} (\lambda - \lambda_k^j(t)).$$

Set $b_{n_i}^i(t) = \pi_i(A + V(t), t)h_i$; now define $b_j^i(t)$ inductively by

$$(4.1) \quad b_j^i(t) = (A + V(t) - \lambda_{j+1}^i(t))b_{j+1}^i(t), \quad j = 1, 2, \dots, n_i - 1.$$

Then since $\pi_i(A + V(t), t) \rightarrow \pi_i(A)$ and $\lambda_j^i(t) \rightarrow \lambda_j^i$ as $t \rightarrow \infty$, it follows that

$$b_j^i(t) \rightarrow b_j^i \text{ as } t \rightarrow \infty, \quad 1 \leq i \leq m, \quad 1 \leq j \leq n_i.$$

Thus, for sufficiently large t , it follows that the vectors $b_j^i(t)$ are linearly independent. Moreover, by (4.1)

$$[A + V(t)]b_j^i(t) = \lambda_j^i(t)b_j^i(t) + b_{j-1}^i(t), \quad j > 1,$$

while by the Cayley-Hamilton theorem,

$$\begin{aligned} [A + V(t) - \lambda_1^i(t)]b_1^i(t) &= \prod_{1 \leq k \leq n_i} (A + V(t) - \lambda_k^i(t))b_{n_i}^i(t) \\ &= \prod_{1 \leq k \leq n_i} (A + V(t) - \lambda_k^i(t))\pi_i(A + V(t), t)h_i = \chi(A + V(t), t)h_i = 0. \end{aligned}$$

It follows, therefore, that if we let $B(t)$ be the matrix whose columns are

$$(b_1^1(t), b_2^1(t), \dots, b_{n_1}^1(t), b_1^2(t), b_2^2(t), \dots, b_{n_2}^2(t), \dots, b_1^m(t), \dots, b_{n_m}^m(t)),$$

then $B^{-1}(t)[A + V(t)]B(t) = \Lambda(t)$ as desired.

Since $B^{-1}(t) \rightarrow B^{-1}$, it is clear that $B^{-1}(t)$ is bounded in a neighborhood of infinity. Thus, if we can show that $t^r |B'(t)| \in L^1$, we will be done. But the elements

of $B'(t)$ are linear functions of $\{\lambda_i^{i'}(t)\}$ and the entries of $V'(t)$, with coefficients which are bounded in a neighborhood of infinity. But $t^r |V'(t)| \in L^1$ by (2.3), and $t^r |\lambda_i^{i'}(t)| \in L^1$ by (2.5). Hence the lemma is proved.

The proof of Theorem 2.1 now follows simply by making the change of variables $y(t) = B(t)x(t)$. Then equation (2.1) becomes

$$x'(t) = [B^{-1}(t)(A + V(t))B(t) + B^{-1}(t)R(t)B(t) - B^{-1}(t)B'(t)]x(t).$$

The matrix $B^{-1}(t)[A + V(t)]B(t)$ is in the form of §3, while $t^r |B^{-1}RB - B^{-1}B'| \in L^1$.

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