A Kontinuitätssatz for $\bar{\partial}_M$ and Lewy Extendibility

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A classical circle of ideas, going back to the work of Hartogs and E. E. Levi, is centered around the problem of the simultaneous extension of all holomorphic functions on an open set Ω_1 in $\mathbf{C}^n$ ($n > 1$) to some larger open set Ω_2 , through the use of certain families of analytic discs, which sweep out $\Omega_2 - \Omega_1$ while their boundaries remain in Ω_1 . Various versions of this principle go by names such as the “Kontinuitätssatz” or the “top hat theorem.” In one way or another they all involve using either the Cauchy integral formula, or the closely related methods of holomorphic convexity and locally convergent multiple power series expansions (*e.g.*, Cartan–Thullen); see also [3]. Refined versions of these methods have been used to extend germs of holomorphic functions on M (*i.e.*, holomorphic functions defined in an open neighborhood of M) to some larger set, where M is a smooth lower-dimensional real submanifold in $\mathbf{C}^n$ (see [10], [11], [4], [7]).

In [8] and [9], H. Lewy gave the first examples of analogous extension phenomena where, instead of germs of holomorphic functions on M , the functions that are to be extended are smooth functions u on M which satisfy the tangential Cauchy–Riemann equations $\bar{\partial}_M u = 0$ to M . Such functions are called *CR* functions on M . In [8], Lewy treated the case where M is a hypersurface (real codimension one) in $\mathbf{C}^2$; the hypersurface theorem has been generalized to $\mathbf{C}^n$ ([11], [6]). In [9], Lewy gave a local example in the case where M has higher codimension: a four-dimensional M^4 in $\mathbf{C}^3$ such that all *CR* functions on M^4 extend to a fixed six-dimensional open set in $\mathbf{C}^3$ whose closure contains M .

The object of this paper is to develop a basic tool for handling such extension problems, the idea being to go up one dimension at a time. More precisely: we show that, given the “right” geometrical situation, it is always possible to extend uniquely *CR* functions from a manifold of a given dimension to *CR* functions on a manifold of one higher dimension, which contains the original manifold as part of its boundary. Thus half (the easy half) of the general problem is solved. The other (more difficult) half of the problem is to give effective

analytical criteria for determining just when the geometry is "right." We plan to follow up this work with a treatment of this second half of the problem.

The right geometrical situation is that of a top hat foliation, described in Section 2, together with a condition of holomorphic transversality, introduced in Section 6. In Theorems 1–3, we give various generalized versions of the *Kontinuitätssatz*; they are all related to a basic lemma which, essentially, asserts the vanishing of a certain cohomology group with control over supports, or what is the same: the solvability of the general inhomogeneous Cauchy problem (see [2]). The analysis here is elementary in the sense that it only involves use of the Cauchy integral formula (with some finesse) and use of the formal construction from the Cauchy–Kowalewski theorem. The problem of extending *CR* functions and the problem of extending germs of holomorphic functions are not equivalent, as is shown by simple examples in Section 3. In the final Section 9, we present, for purposes of illustration, an example of an M^4 in $\mathbf{C}^3$ whose *CR* functions all extend to a six-dimensional set. The example is such that one can see clearly what is the maximal region of extension, and how these phenomena are influenced by top hat foliations.

In order to avoid clumsy or pedantic statements of theorems, we work everywhere with C^∞ manifolds and C^∞ functions. However, all of our arguments are valid with only a few continuous derivatives; the tedious task of achieving absolute minimality in the differentiability assumptions is left to the reader.

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1. The operator $\bar{\partial}_M$ and *CR* functions. Let X be a complex analytic manifold, and let M be a locally closed C^∞ differentiable submanifold in X (all manifolds we consider are assumed to be paracompact and without boundary, unless the boundary is explicitly mentioned). The Cauchy–Riemann operator $\bar{\partial}$ on X induces an operator $\bar{\partial}_M$ on M , called the tangential Cauchy–Riemann operator to M . We give below a definition of $\bar{\partial}_M$ that is on the one hand very general (*e.g.*, the dimension of the holomorphic tangent space to M is not required to be locally constant), and on the other hand is very convenient for the purposes of this paper (see also [1], [2]).

Suppose that M has real codimension k in X . Since $\bar{\partial}_M$ is a local operator, it will be sufficient to define it in a neighborhood of an arbitrary point $p \in M$. Let U be an open neighborhood of p in X such that

$$M \cap U = \{\rho_j = 0; \quad j = 1, 2, \dots, k\},$$

$$d\rho_1 \wedge d\rho_2 \wedge \dots \wedge d\rho_k \neq 0 \quad \text{in } U,$$

where the ρ_j are C^∞ real-valued functions in U . Let $\mathfrak{J} = \mathfrak{J}(U)$ denote the "differential ideal" of C^∞ differential forms in U generated by ρ_j and $\bar{\partial}\rho_j$ ($j = 1, 2, \dots, k$); *i.e.*, $f \in \mathfrak{J}$ if and only if $f = \sum_{i=1}^k (\rho_i \alpha_i + \bar{\partial}\rho_i \wedge \beta_i)$, where the α_i , β_i are smooth forms.

Consider a function $u \in C^\infty(M)$. We define $\bar{\partial}_M u = 0$ on $M \cap U$ to mean that

$\bar{\partial}\tilde{u} \in \mathfrak{F}$ for any extension $\tilde{u} \in C^\infty(U)$ of $u|_{M \cap U}$. One needs to verify this condition only for one such extension $\tilde{u}$; it is valid for all extensions if it holds for one. It is easy to see that this condition is independent of the particular choice of the generators $\rho_i, \bar{\partial}\rho_i$ of $\mathfrak{F}$. If $\tilde{u}$ is an extension with $\bar{\partial}\tilde{u} = \sum_{i=1}^k (\rho_i \alpha_i + \bar{\partial}\rho_i \wedge \beta_i)$, then $\tilde{\tilde{u}} = \tilde{u} - \sum_{i=1}^k \rho_i \beta_i$ gives another local extension of u such that $\bar{\partial}\tilde{\tilde{u}}|_{M \cap U} = 0$ (we use the notation $g|_M = 0$, where g is a form on X , to indicate that all the coefficients of g vanish on M). Using a partition of unity to glue together the local extensions, we find: If $u \in C^\infty(M)$, then $\bar{\partial}_M u = 0$ on M if and only if there exists an extension $\tilde{u} \in C^\infty(\text{nbh } M)$ such that $\bar{\partial}\tilde{u}|_M = 0$.

The algebra of germs of holomorphic functions on K , where K is any subset of X , will be denoted by $\mathcal{O}(K)$.

By a *CR function on M* , we shall mean a function $u \in C^\infty(M)$ that satisfies $\bar{\partial}_M u = 0$ on M . We shall also speak of *CR functions on M* in the case where M has a boundary: If $M = M^\circ \cup M_0$ where M° and M_0 are disjoint locally closed manifolds without boundary (M° being the interior of M ; and M_0 , of one dimension less, being the boundary of M), then a *CR function on M* is a function $u \in C^\infty(M)$ such that $\bar{\partial}_{M^\circ} u = 0$ on M° and $\bar{\partial}_{M_0} u = 0$ on M_0 . In either case, the algebra of all *CR functions on M* will be denoted by $CR(M)$. There is a subalgebra inclusion $CR(M) \subset C^\infty(M)$ and an obvious restriction map $\mathcal{O}(M) \rightarrow CR(M)$, but the latter is not injective in general.

We will say that $\bar{\partial}_M$ has the *unique continuation property on M* iff for any sufficiently smooth function u that satisfies $\bar{\partial}_M u = 0$ on M , the vanishing of u in some connected open subset ω of M implies that u is identically zero in the component of M that contains ω .

For more about the unique continuation property, see Section 8.

Next we need to define what it means for $\bar{\partial}_M$ to act on "forms" of type $(0, 1)$ on M : the most natural way to do this is to define $\bar{\partial}_M$ so that it acts on equivalence classes of C^∞ forms on X modulo those in the ideal $\mathfrak{F}$ (see [2]). But for the purposes of this paper, we shall do the following: let $C_{(0,1)}^\infty(M)$ denote the set of all C^∞ sections of the restriction to M of the bundle of forms of type $(0, 1)$ on X ; i.e., $f \in C_{(0,1)}^\infty(M)$ means that f is a $(0, 1)$ -form on X whose coefficients are defined and of class C^∞ on M . Then we define $\bar{\partial}_M f = 0$ on M as before; it means that $\bar{\partial}\tilde{f} \in \mathfrak{F}$ for any extension $\tilde{f}$ of f ; equivalently, there exists an extension $\tilde{f} \in C_{(0,1)}^\infty(\text{nbh } M)$ such that $\bar{\partial}\tilde{f}|_M = 0$.

Finally, for $u \in C^\infty(M)$ and $f \in C_{(0,1)}^\infty(M)$, we define $\bar{\partial}_M u = f$ on M to mean that $\bar{\partial}\tilde{u} - \tilde{f} \in \mathfrak{F}$ for any extensions $\tilde{u}$ and $\tilde{f}$ of u and f , respectively. This condition is again valid for all extensions if it holds for one choice of the extensions. An equivalent formulation: given an arbitrary extension $\tilde{f} \in C_{(0,1)}^\infty(\text{nbh } M)$ of f , there exists an extension $\tilde{u} \in C^\infty(\text{nbh } M)$ of u such that $(\bar{\partial}\tilde{u} - \tilde{f})|_M = 0$.

2. Top hat foliations. For basic definitions and facts concerning foliations, see Haefliger ([5]). A top hat foliation is intended as a generalization of the set

$$(*) \quad \{|z_1| < 1, |z_2| < 1/2\} \cup \{1/2 < |z_1| < 1, |z_2| < 1\}$$

in $\mathbf{C}^2$ (a picture of the top hat can be obtained by rotating (*) about the $|z_2|$ -axis).

Let X be a complex analytic manifold of complex dimension n . Let M° , M' be locally closed differentiable submanifolds of X ; we assume that M° is open in M' , so that M° and M' have the same dimension. The closure of M° in M' will be denoted by M . For any subset $A' \subset M'$, the sets A° , A are defined by $A^\circ = M^\circ \cap A'$ and $A = M \cap A'$. This notation will be used throughout the sequel.

Definition. The pair (M, M') has a *top hat foliation* in X iff:

- i) M' is foliated into (connected) differentiable submanifolds L' , and each leaf L' is itself foliated into (connected) complex analytic submanifolds ℓ' of X , with $\dim_{\mathbf{C}} \ell' = 1$. Hence X is also foliated by the ℓ' .
- ii) Each leaf L' has a neighborhood U in M' , which is a union of the L'' 's, and such that the foliation of U by the ℓ'' 's can be defined by a single holomorphic coordinate system for X .
- iii) $\ell \subset \subset \ell'$ for each leaf ℓ .
- iv) Each $L' - L$ is connected and contains a nonvoid open set in L' which is the union of certain leaves ℓ' from the foliation of L' .
- v) $\bar{\partial}_{L'}$ has the unique continuation property on L' , for each L' .

Remarks. Assumption (ii) above means that U has a neighborhood $\tilde{U}$ in X in which there is a holomorphic coordinate system $(z_1, z_2, \dots, z_n)$, such that each ℓ' in U can be described as: {an open subset of the z_1 -plane} $\cap$ $\{(z_2, \dots, z_n) = \text{const.}\}$. Note that we have not made any explicit assumptions about the topology of the sets ℓ .

Assumption (v) above will be satisfied automatically if each L' is a complex analytic submanifold of X with $\dim_{\mathbf{C}} L' = 2$; for then $\bar{\partial}_{L'}$ is just $\bar{\partial}$ on the two-dimensional complex manifold. We shall call such a top hat foliation a *standard top hat foliation*.

3. Extension of CR functions versus extension of germs of holomorphic functions. By means of very simple examples, one can see the important distinction between the problem of extending germs of holomorphic functions and the problem of extending CR functions.

In $\mathbf{C}^2$, with the usual coordinates (z_1, z_2) , consider the sets in the $y_2 = 0$ plane determined by

$$M' : \{|z_1| < 1, 0 < x_2 < 1\},$$

$$M : \{|z_1| \leq \frac{1}{2}, \frac{1}{2} \leq x_2 < 1\},$$

$$M' - M : \{|z_1| < 1, 0 < x_2 < \frac{1}{2}\} \cup \{\frac{1}{2} < |z_1| < 1, 0 < x_2 < 1\},$$

as well as their slightly thickened versions $M'_\epsilon, M_\epsilon, M'_\epsilon - M_\epsilon$, obtained by taking their Cartesian product with $\{|y_2| < \epsilon\}$. Note that $M'_\epsilon - M_\epsilon$ is very similar to (*); the familiar classical argument using the Cauchy integral formula

(the theorem of Hartogs) shows that every holomorphic function in $M'_\epsilon - M_\epsilon$ has a unique extension to a holomorphic function in M'_ϵ . It follows that

$$\mathcal{O}(M') \xrightarrow{\sim} \mathcal{O}(M' - M),$$

where the isomorphism is given by the natural restriction.

But what about the analogous restriction map

$$CR(M') \rightarrow CR(M' - M)$$

for CR functions? It is not surjective: consider the function

$$f(z_1, z_2) = \psi(x_2) \frac{1}{z_1},$$

where $\psi(x_2)$ is any C^∞ function with $\psi \equiv 0$ for $0 < x_2 < 2/3$ and $\psi \equiv 1$ for $3/4 < x_2 < 1$. Obviously, $f \notin CR(M' - M)$, but f has no extension which is a CR function on M' .

Note that, even though the conditions (i)–(iv) above are satisfied, (M, M') fails to have a top hat foliation in $\mathbb{C}^2$ because condition (v) does not hold. On the other hand, $(M_\epsilon, M'_\epsilon)$ has a standard top hat foliation in $\mathbb{C}^2$.

Next let $\tilde{M}_0$ be the C^∞ two-manifold in $\mathbb{C}^2$ that one obtains by slightly rounding off the sharp corner from the set

$$\{|z_1| = \tfrac{1}{2}, \tfrac{1}{2} \leq x_2 < 1\} \cup \{|z_1| \leq \tfrac{1}{2}, x_2 = \tfrac{1}{2}\}$$

in the $y_2 = 0$ plane. Let $\tilde{M}$ be the slightly rounded version of the previous M . Define

$$M^5 = \tilde{M} \times \mathbb{C}, \quad M_0^4 = \tilde{M}_0 \times \mathbb{C},$$

regarding $\mathbb{C}^3 = \mathbb{C}^2 \times \mathbb{C}$. This M^5 is a C^∞ 5-manifold in 6-dimensional space (i.e., in $\mathbb{C}^3$) which has the C^∞ 4-manifold M_0^4 as its (partial) boundary. It now follows from the argument above, by looking at sections $z_3 = \text{const.}$, that the restriction

$$\mathcal{O}(M^5) \xrightarrow{\sim} \mathcal{O}(M_0^4)$$

is an isomorphism; whereas the restriction

$$CR(M^5) \rightarrow CR(M_0^4)$$

fails to be surjective. Obviously, one can invent a host of examples which have a similar character.

4. A basic lemma. In what follows, it is understood that M, M', X are as described in Section 2.

Lemma 1. Assume that (M, M') has a top hat foliation in X . Then given any $f \in C_{(0,1)}^\infty(M')$ with $\bar{\partial}_{M'} f = 0$ on M' and $\text{supp } f \subset M$, there exists a unique $u \in C^\infty(M')$ such that $\bar{\partial}_{M'} u = f$ on M' and $\text{supp } u \subset M$.

Remark. Lemma 1 implies that

$$H_M^{01}(M'; \bar{\partial}_{M'}) = 0,$$

where $H_M^{01}(M'; \bar{\partial}_{M'})$ refers to the analogue, for $\bar{\partial}_{M'}$, of the Dolbeault $(0, 1)$ -cohomology group on M' with supports in M .

Proof. Let L' be an arbitrary leaf of the foliation of M' . Then L' has a neighborhood U in M' , U has a neighborhood $\tilde{U}$ in X , and there is a holomorphic coordinate system $z = (z_1, z_2, \dots, z_n)$ in $\tilde{U}$ which defines the foliation of U into the various leaves ℓ' . Since $\bar{\partial}_{M'} f = 0$ on M' , we can find an extension $\tilde{f} \in C_{(0,1)}^\infty(\tilde{U})$ of $f|_U$ such that $\bar{\partial} \tilde{f} = 0$ on U . By assumption (iii), the restriction of $\tilde{f}$ to an arbitrary leaf ℓ' has compact support in ℓ' . Hence the extension $\tilde{f}$ can be chosen such that its restriction to each set in $\tilde{U}$ of the form $\{(z_2, \dots, z_n) = \text{const.}\}$ has compact support (as a function of z_1). In terms of the holomorphic coordinates z in $\tilde{U}$, $\tilde{f}(z) = \sum_{i=1}^n \tilde{f}_i(z) d\bar{z}_i$. Now we make use of the Cauchy integral formula in the same way as in Hörmander ([6], p. 30). Set

$$(1) \quad \tilde{u}(z) = \frac{1}{2\pi i} \iint_{\mathbb{C}} \frac{\tilde{f}_1(\xi_1, z_2, \dots, z_n)}{\xi_1 - z_1} d\xi_1 \wedge d\bar{\xi}_1.$$

The integral makes sense because the integrand has compact support; it defines a function $\tilde{u} \in C^\infty(\tilde{U})$.

By making the change of variables $t = z_1 - \xi_1$ in (1), by differentiating with respect to $\partial/\partial \bar{z}_1$ under the integral sign and then changing back to the original variables, and using the Cauchy integral formula for a function $\tilde{f}_1$ which has compact support, we obtain that

$$\frac{\partial \tilde{u}}{\partial \bar{z}_1} = \tilde{f}_1 \quad \text{on } U.$$

For $j = 2, 3, \dots, n$, we can differentiate with respect to $\partial/\partial \bar{z}_j$ under the integral sign in (1). Since $\bar{\partial} \tilde{f} = 0$ on U , it follows that $\partial \tilde{f}_1 / \partial \bar{z}_j = \partial \tilde{f}_j / \partial \bar{z}_1$ on U . Hence by evaluating everything on U and using the Cauchy integral formula for a function $\tilde{f}_j$ which has compact support, we obtain

$$\frac{\partial \tilde{u}}{\partial \bar{z}_j} = \tilde{f}_j \quad \text{on } U$$

for $j = 2, 3, \dots, n$.

Thus we have that $(\bar{\partial} \tilde{u} - \tilde{f})|_U = 0$. It follows that $\bar{\partial}_{M'} u = f$ on U , where the function $u \in C^\infty(U)$ is defined by $u = \tilde{u}|_U$.

Now consider the restriction of u to any leaf $L' \subset U$. We have that $\bar{\partial}_{L'} u = 0$ on $L' - L$ because f vanishes on $L' - L$. But it follows from the definition of $\tilde{u}$, by assumption (iv), that $u \equiv 0$ in some nonvoid open set in $L' - L$. Hence $u \equiv 0$ in $L' - L$ because $L' - L$ is connected and $\bar{\partial}_{L'}$ has the unique continuation property on L' . Thus the restriction of u to L' has its support contained in L . Application of the unique continuation property of $\bar{\partial}_{L'}$ once again to the

difference between two such solutions u shows that u is uniquely determined on each L' , hence also on U .

It follows that there exists a unique function $u \in C^\infty(M')$ with $\text{supp } u \subset M$ satisfying $\bar{\partial}_{M'} u = f$ on M' . The proof of the lemma is complete.

5. Kontinuitätssatz (Hartogs version). In what follows, we set $N' = M' - M$, so that N' is open in M' .

Theorem 1. Assume that (M, M') has a top hat foliation in X . Then the restriction map

$$CR(M') \xrightarrow{\sim} CR(N')$$

is an isomorphism.

Proof. For purposes of technical convenience in the proof, we will assume that N' is actually an open neighborhood of $M' - M^\circ$ in M' . That is no loss of generality since, by assumption (iii), each ℓ has an open neighborhood $\tilde{\ell}$ with $\ell \subset \subset \tilde{\ell} \subset \subset \ell'$; the union of the $\tilde{\ell}$ gives an $\tilde{M}$ that could be used as a replacement for M .

Any $h_0 \in CR(N')$ has an extension $w \in C^\infty(M')$ such that $w = h_0$ on a neighborhood of $M' - M^\circ$ in M' . Since $\bar{\partial}_{M'} w = 0$ there, it follows that w has an extension $\tilde{w} \in C^\infty(\text{nbh } M')$ such that $\bar{\partial} \tilde{w}|_{M'-M^\circ} = 0$ and $\tilde{w}|_{M'-M^\circ} = h_0$. Set $\tilde{f} = \bar{\partial} \tilde{w}$ and $f = \tilde{f}|_{M'}$. Then f satisfies the hypotheses of Lemma 1. Define $h = w - u$, where u is the function given by Lemma 1. It follows that $h \in C^\infty(M')$, $\bar{\partial}_{M'} h = 0$ on M' , and $h|_{M'-M} = h_0$. Also, h is unique because of the unique continuation property of $\bar{\partial}_{L'}$ on each leaf L' . This proves the theorem.

6. Kontinuitätssatz (Lewy version). Recall that M is the closure of M° in M' . In the sequel, we shall set $M_0 = M - M^\circ$.

Definition. M_0 is a holomorphically transverse hypersurface in M' iff:

- 1) M_0 is a closed C^∞ differentiable submanifold of M' , of real codimension one in M' , which has two sides in M' , one of them being M° (see [2]);
- 2) $\dim_{\mathbb{C}} HT_p(M_0) = \dim_{\mathbb{C}} HT_p(M') - 1$, $\forall p \in M_0$.

Remarks. In condition (2) above, $HT_p(M_0)$ refers to the holomorphic tangent space to M_0 at p (analogously for M'). It is the maximal complex linear subspace (when considered as a subspace of the holomorphic tangent space to X at p) of the real tangent space to M_0 at p . Let U be a small neighborhood of p in X in which there are C^∞ real-valued functions ρ, ρ_i ($1 \leq i \leq k$) such that:

$$(2) \quad \begin{aligned} M' \cap U &= \{\rho_i = 0; 1 \leq i \leq k\}, \\ d\rho_1 \wedge d\rho_2 \wedge \cdots \wedge d\rho_k &\neq 0 \quad \text{on } M' \cap U, \\ M_0 \cap U &= \{\rho = 0, \rho_i = 0; 1 \leq i \leq k\}, \\ d\rho \wedge d\rho_1 \wedge \cdots \wedge d\rho_k &\neq 0 \quad \text{on } M_0 \cap U. \end{aligned}$$

Then assumption (2) above can be expressed by the condition that, on $M_0 \cap U$,

$$\bar{\partial}\rho \text{ is linearly independent of } \{\bar{\partial}\rho_j ; 1 \leq j \leq k\}.$$

This means that M_0 is noncharacteristic in M' for the operator $\bar{\partial}_{M'}$.

Finally, as in Section 1, $\mathfrak{F}$ will denote the ideal of differential forms that are C^∞ in a neighborhood of M' , and are locally generated by ρ_j and $\bar{\partial}\rho_j$ ($1 \leq j \leq k$). This definition of $\mathfrak{F}$ is independent of the choice of the local generators ρ_j , $\bar{\partial}\rho_j$.

Theorem 2. *Assume that (M, M') has a top hat foliation in X and that $M_0 = M - M^\circ$ is a holomorphically transverse hypersurface in M' . Then the restriction map*

$$CR(M) \xrightarrow{\sim} CR(M_0)$$

is an isomorphism.

Proof. First we need a technical lemma.

Lemma 2. *Assume that M_0 is a holomorphically transverse hypersurface in M' . Then we have:*

- a) *Let a function $u \in C^\infty(M)$ be such that $u|_{M_0} = 0$ and $\bar{\partial}_{M'}u = 0$ on M . Then u vanishes to infinite order on M_0 .*
- b) *Let $g \in C_{(0,1)}^\infty(\text{nbh } M')$ be such that $g|_{M_0} = 0$ and $\bar{\partial}g = 0$ on M . Then there is a function $v \in C^\infty(\text{nbh } M')$ with $v|_{M_0} = 0$ such that, for every integer m ,*

$$g - \bar{\partial}v = O(\rho^m) + I_m,$$

where $I_m \in \mathfrak{F}$ and $O(\rho^m)$ is a term which vanishes to order m on M_0 .

Assuming for a moment the lemma, we proceed to prove Theorem 2: suppose there is some $u \in CR(M)$ with $u|_{M_0} = 0$. Then, by part (a) of the lemma,

$$w = \begin{cases} u & \text{on } M, \\ 0 & \text{on } M' - M \end{cases}$$

defines a function $w \in C^\infty(M')$ satisfying $\bar{\partial}_{M'}w = 0$ on M' . From the unique continuation property of $\bar{\partial}_{L'}$ on each leaf L' , we conclude that $w \equiv 0$ on M' . This proves injectivity.

The proof of part (a) of the lemma (see below) shows that, for the vanishing of u on M_0 up to any prescribed finite order, it is only necessary that u be sufficiently smooth, not necessarily of class C^∞ . Hence for sufficiently smooth (but not necessarily C^∞) CR functions on M , the injectivity also follows by the same argument.

To demonstrate surjectivity, consider an arbitrary $h_0 \in CR(M_0)$. This h_0 has an extension $\tilde{h}_0 \in C^\infty(\text{nbh } M')$ such that $\bar{\partial}\tilde{h}_0|_{M_0} = 0$. Set $g = \bar{\partial}\tilde{h}_0$. Then g satisfies the hypotheses of part (b) of the lemma; hence we have the v, I_m . Let $\hat{h}_0 = \tilde{h}_0 - v$; it is another C^∞ extension of h_0 which has the property that

$\bar{\partial}\hat{h}_0 = O(\rho^m) + I_m$. Define

$$f = \begin{cases} \bar{\partial}\hat{h}_0 - I_m & \text{on } M \\ 0 & \text{on } M' - M. \end{cases}$$

Note that $f \in C_{(0,1)}^{m-1}(M')$. Since $I_m \in \mathfrak{I}$ and $\bar{\partial}\mathfrak{I} \subset \mathfrak{I}$, it follows that $\bar{\partial}_{M'}f = 0$ on M' . Therefore, by Lemma 1 (more precisely, its C^{m-1} version), there is a function $u \in C^{m-1}(M')$ with $\text{supp } u \subset M$ satisfying $\bar{\partial}_{M'}u = f$ on M' . Let h be the restriction of $\hat{h}_0 - u$ to M . Then $h \in C^{m-1}(M)$, $h|_{M_0} = h_0$, and it follows that $\bar{\partial}_{M'}h = 0$ on M . Since this is true for arbitrarily large m , and since h is unique by the injectivity, we have that actually $h \in C^\infty(M)$. This completes the proof of Theorem 2.

Proof of Lemma 2. Since M_0 has two sides in M' , M_0 can be defined as $M_0 = \{\rho = 0\} \cap M'$, where ρ is some global C^∞ function on a nbh M' which satisfies (2) locally (see [2], Proposition 1.1). In what follows, let Γ be a sufficiently small neighborhood of M_0 in $\{\rho = 0\} \cap \text{nbh } M'$. Thus Γ is a piece of smooth hypersurface in X .

Let $u \in C^\infty(M)$ be as in part (a) of the lemma. Then $u = \rho\alpha_1$ for some function $\alpha_1 \in C^\infty(M)$, because u vanishes to first order on M_0 . Assume, as an inductive hypothesis, that $u = \rho^m\alpha_m$ for some $\alpha_m \in C^\infty(M)$. Extend α_m to $\tilde{\alpha}_m \in C^\infty(\text{nbh } M')$, and define $\tilde{u} \in C^\infty(\text{nbh } M')$ by the equation $\tilde{u} = \rho^m\tilde{\alpha}_m$. On the one hand, we have

$$\bar{\partial}\tilde{u} = \rho^m\bar{\partial}\tilde{\alpha}_m + m\rho^{m-1}\bar{\partial}\rho \wedge \tilde{\alpha}_m;$$

on the other hand, since $\bar{\partial}\tilde{u} \in \mathfrak{I}$ and $\bar{\partial}\tilde{u}$ must vanish to order $m-1$ on Γ , we have

$$\bar{\partial}\tilde{u} = \rho^{m-1} \sum_{i=1}^k (\rho_i a_i + \bar{\partial}\rho_i \wedge b_i).$$

It follows that

$$m\alpha_m \bar{\partial}\rho = \sum_{i=1}^k b_i \bar{\partial}\rho_i \quad \text{on } M_0.$$

Hence $\alpha_m|_{M_0} = 0$ because on M_0 , $\bar{\partial}\rho$ is linearly independent of $\{\bar{\partial}\rho_1, \bar{\partial}\rho_2, \dots, \bar{\partial}\rho_k\}$. Therefore, $\alpha_m = \rho\alpha_{m+1}$ for some $\alpha_{m+1} \in C^\infty(M)$; thus $u = \rho^{m+1}\alpha_{m+1}$. This shows that u vanishes to infinite order on M_0 .

Let $g \in C_{(0,1)}^\infty(\text{nbh } M')$ be as in part (b) of the lemma. First we calculate a formal power series in ρ for the determination of all normal derivatives of v on Γ . Since $g|_{M_0} = 0$, we have

$$g - \bar{\partial}\{\beta_0\} = \rho\gamma_1 + I_1,$$

where $\beta_0 \equiv 0$, $I_1 \in \mathfrak{I}$, and $\rho\gamma_1 = O(\rho)$. For purposes of induction, assume that

$$(3) \quad g - \bar{\partial}\left\{\sum_{k=0}^m \beta_k \rho^k\right\} = \rho^m \gamma_m + I_m,$$

where $I_m \in \mathfrak{F}$ and where each $\beta_k - \beta_k|_\Gamma$ vanishes to infinite order on Γ . Application of $\bar{\partial}$ yields, on M ,

$$0 = \rho^m \bar{\partial} \gamma_m + m \rho^{m-1} \bar{\partial} \rho \wedge \gamma_m + \bar{\partial} I_m .$$

It follows that $\bar{\partial} I_m = O(\rho^{m-1})$. Since $\bar{\partial} I_m \in \mathfrak{F}$, we have

$$m \bar{\partial} \rho \wedge \gamma_m + \sum_{i=1}^k \bar{\partial} \rho_i \wedge \gamma_i = 0 \quad \text{on } M_0 .$$

Here the γ_m, γ_i are $(0, 1)$ -forms, so we cannot argue as above where the α_m, b_i were functions. Let $p \in M_0$ be a fixed point. By relabeling, if necessary, we may assume that $\bar{\partial} \rho_1, \bar{\partial} \rho_2, \dots, \bar{\partial} \rho_k$ form a maximal set of linearly independent elements of $\{\bar{\partial} \rho_i; 1 \leq i \leq k\}$ at p . Wedging the above relation with $\bar{\partial} \rho_1 \wedge \bar{\partial} \rho_2 \wedge \dots \wedge \bar{\partial} \rho_k$, we obtain

$$\bar{\partial} \rho \wedge \bar{\partial} \rho_1 \wedge \dots \wedge \bar{\partial} \rho_k \wedge \gamma_m = 0 \quad \text{at } p .$$

Thus $\gamma_m \in \text{span} \{\bar{\partial} \rho \text{ and } \bar{\partial} \rho_i; 1 \leq i \leq k\}$ at each point of M_0 .

But on M_0 , $\bar{\partial} \rho$ is linearly independent of the $\{\bar{\partial} \rho_i; 1 \leq i \leq k\}$. Hence it follows that, on M_0 , γ_m can be expressed as $\gamma_m = a_m \bar{\partial} \rho + \xi$, where $\xi \in \text{span} \{\bar{\partial} \rho_i; 1 \leq i \leq k\}$ and the $a_m \in C^\infty(M_0)$ are uniquely determined by γ_m . Now let $\beta_{m+1} \in C^\infty(\text{nbh } M')$ be an extension of $(1/m + 1)a_m$ such that $\beta_{m+1} - \beta_{m+1}|_\Gamma$ vanishes to infinite order on Γ . It follows that

$$\gamma_m = \rho \alpha_{m+1} + (m + 1) \bar{\partial} \rho \wedge \beta_{m+1} + \eta$$

in $\text{nbh } M'$, for some $\eta \in \mathfrak{F}$. Therefore,

$$g - \bar{\partial} \left\{ \sum_{k=0}^{m+1} \beta_k \rho^k \right\} = \rho^{m+1} (\alpha_{m+1} - \bar{\partial} \beta_{m+1}) + \rho \eta + I_m = \rho^{m+1} \gamma_{m+1} + I_{m+1} ,$$

where $I_{m+1} \in \mathfrak{F}$. Thus the representation (3) holds for all m .

By the Whitney extension theorem, a function $v \in C^\infty(\text{nbh } M')$ exists which satisfies

$$\frac{1}{k!} \frac{\partial^k}{\partial \rho^k} v = \beta_k \quad \text{on } \Gamma$$

for $k = 0, 1, 2, \dots$ (see [2], Proposition 2.2). Since $\beta_0 \equiv 0$, we have $v|_\Gamma = 0$. Also,

$$v - \sum_{k=0}^m \beta_k|_\Gamma \rho^k = O(\rho^{m+1}).$$

Hence

$$g - \bar{\partial} v = g - \bar{\partial} \left\{ \sum_{k=0}^m \beta_k \rho^k \right\} + \bar{\partial} \left\{ \sum_{k=0}^m (\beta_k - \beta_k|_\Gamma) \rho^k \right\} + O(\rho^m) = O(\rho^m) + I_m ,$$

where $I_m \in \mathfrak{F}$. Here we have used (3) and the fact that each $\beta_k - \beta_k|_\Gamma$ vanishes to infinite order on Γ . This completes the proof of the lemma.

Remark. We set $L_0 = L - L^\circ$ and $\ell_0 = \ell - \ell^\circ$ (recall the notational conventions introduced in Section 2). Consider any $u \in CR(M)$ such that $u|_{M_0} = u_0 \in CR(M_0)$. Then it follows from the geometry of the top hat foliation, by applying the maximum modulus principle to each ℓ , that

$$\begin{aligned} \sup_{\ell} |u| &\leq \sup_{\ell_0} |u_0|, & \forall \ell', \\ \sup_L |u| &\leq \sup_{L_0} |u_0|, & \forall L', \\ \sup_M |u| &\leq \sup_{M_0} |u_0|. \end{aligned}$$

An analogous remark holds for the Hartogs version of the Kontinuitätssatz (replace M, M_0 by $M', N', etc.$)

7. The inhomogeneous Cauchy problem for $\bar{\partial}_M$. The method used to prove Theorem 2 actually yields a more general result. Consider the general inhomogeneous Cauchy problem for $\bar{\partial}_M$ on M with initial data prescribed on M_0 :

$$(4) \quad \begin{cases} \bar{\partial}_M u = f & \text{on } M, \\ u|_{M_0} = u_0. \end{cases}$$

Here $u_0 \in C^\infty(M_0)$ and $f \in C^\infty_{(0,1)}(M)$ are given; a solution $u \in C^\infty(M)$ is sought. The data u_0, f are called compatible iff

$$(5) \quad \begin{cases} \bar{\partial}_M f = 0 & \text{on } M, \\ \bar{\partial}_{M_0} u_0 = f_0 & \text{on } M_0, \end{cases}$$

where $f_0 = f|_{M_0}$. Clearly, these compatibility conditions are necessary.

Theorem 3. Assume that (M, M') has a top hat foliation in X and that $M_0 = M - M^\circ$ is a holomorphically transverse hypersurface in M' . Then for every prescription of compatible data, the Cauchy problem (4) has a unique solution.

Proof. The uniqueness of the solution follows from the injectivity of $CR(M) \rightarrow CR(M_0)$, as given by Theorem 2. To prove existence, let u_0, f be prescribed data satisfying (5). Then f has an extension $\tilde{f} \in C^\infty_{(0,1)}(nbh M')$ such that $\bar{\partial} \tilde{f} = 0$ on M ; similarly, u_0 has an extension $\tilde{u}_0 \in C^\infty(nbh M')$ such that $\tilde{f} - \bar{\partial} \tilde{u}_0 = 0$ on M_0 . Set $f' = \tilde{f} - \bar{\partial} \tilde{u}_0$. Then $f'|_{M_0} = 0$ and $\bar{\partial} f' = 0$ on M . Part (b) of Lemma 2 applies equally well in this case; hence there is a $v \in C^\infty(nbh M')$ with $v|_{M_0} = 0$ and $f' - \bar{\partial} v = O(\rho^m) + I_m$. Setting

$$f'' = \begin{cases} f' - \bar{\partial} v - I_m & \text{on } M, \\ 0 & \text{on } M' - M, \end{cases}$$

and arguing as in the proof of Theorem 2, there is a u'' with $u''|_{M_0} = 0$ which is a solution of $\bar{\partial}_M u'' = f''$ on M' . Let $u = \tilde{u}_0 + v + u''$ on M . Then $u|_{M_0} = u_0$, and it follows that $\bar{\partial}_M u = f$ on M . This proves Theorem 3.

8. On the unique continuation property. Conditions (i)-(iv) in the definition of a top hat foliation have a very geometric character, whereas condition (v) is a little more mysterious. Here we show that, under certain conditions, M_0 inherits the unique continuation property from M' . This makes it possible to use Theorem 2 inductively, going up one dimension at a time, provided unique continuation is known at the last stage of the process.

First, it should be observed that, in the definition of " $\bar{\partial}_M$ has the unique continuation property on M ", as given in Section 1, it is required that the equation $\bar{\partial}_M u = 0$ be satisfied on all of M . We shall now call this the *global* unique continuation property for M . By the *local* unique continuation property for M , we shall mean the following: For any open connected subset $\Omega \subset M$ and for any sufficiently smooth function u defined in Ω , which satisfies $\bar{\partial}_M u = 0$ on Ω , the vanishing of u in some open subset ω of Ω implies that u is identically zero in Ω . The local unique continuation property clearly implies the global unique continuation property, but not vice versa. In the case of a standard top hat foliation, $\bar{\partial}_{L'}$ has the local unique continuation property on L' , for each L' .

Let M, M', M_0 be as in Section 6.

Theorem 4. *Let M_0 be a holomorphically transverse hypersurface in M' , and assume that $\bar{\partial}_{M'}$ has the local unique continuation property on M' . It follows that*

- a) *if $CR(M) \simeq CR(M_0)$, then $\bar{\partial}_{M_0}$ has the global unique continuation property on M_0 ;*
- b) *if every point $p \in M_0$ has a neighborhood U in M' such that $CR(M \cap U) \simeq CR(M_0 \cap U)$, then $\bar{\partial}_{M_0}$ has the local unique continuation property on M_0 .*

Proof. Consider first assertion (a). If u_0 is a solution of $\bar{\partial}_{M_0} u_0 = 0$ on M_0 , then by hypothesis, u_0 has an extension u to M which is a solution of $\bar{\partial}_M u = 0$ on M . Suppose that u_0 vanishes in some connected open set ω_0 in M_0 . Then u vanishes to high order (or infinite order) on ω_0 , by part (a) of Lemma 2. Hence u can be extended by zero in ω , where ω is some small open set in $M' - M$ which borders on ω_0 , and u will be a solution of $\bar{\partial}_M u = 0$ on $M \cup \omega$. Then from the local unique continuation property on M' , it follows that u_0 is zero in the connected component of M_0 that contains ω_0 .

To prove part (b), let Ω_0 be an open connected set in M_0 , and let u_0 be a solution of $\bar{\partial}_{M_0} u_0 = 0$ in Ω_0 . Suppose that u_0 vanishes in some open set ω_0 in Ω_0 . Let $F = \text{supp } u_0$. By definition, F is closed in Ω_0 , and $F \neq \Omega_0$. But a local version of the argument given for part (a) shows that F consists entirely of interior points; hence F is also open. Thus F must be void. This proves the theorem.

9. An example. It is fairly easy to generate an assortment of examples of the extendibility of CR functions, in the sense of Hans Lewy, by using Theorems

2 and 4. For purposes of illustration, we give one such example below, in which it is easy to see just what is the maximal domain of global extension.

A set S in $\mathbf{C}^n$ is called a *set of holomorphy* if S has a fundamental sequence of open neighborhoods that are domains of holomorphy.

Let $z_j = x_j + iy_j$ ($j = 1, 2, 3$) denote the usual coordinates in $\mathbf{C}^3$, and consider the C^∞ functions

$$\rho_1 = (x_1 - 1)^2 + |z_2|^2 - 4,$$

$$\rho_2 = (x_1 + 1)^2 + |z_3|^2 - 4.$$

Fix $\delta > 0$, and define the sets M^4, M^6 in $\mathbf{C}^3$ by

$$M^4 : \{ \rho_1 = 0, \rho_2 = 0, |y_1| < \delta \},$$

$$M^6 : \{ \rho_1 \leq 0, \rho_2 \leq 0, |y_1| < \delta \}.$$

A calculation shows that $\bar{\partial}\rho_1 \wedge \bar{\partial}\rho_2 \neq 0$ and $d\rho_1 \wedge d\rho_2 \neq 0$ on M^4 . Hence M^4 is a C^∞ manifold of real dimension 4; it is easily visualized as $M^3 \times (-\delta, \delta)$ where M^3 is a compact manifold (topologically a 3-sphere) formed by the intersection of two cylinders. Each of the sets $\{\rho_1 < \epsilon\}$, $\{\rho_2 < \epsilon\}$, $\{|y_1| < \delta\}$ is a domain of holomorphy, so their intersection is a domain of holomorphy. Thus M^6 is a bounded 6-dimensional set of holomorphy; it can be visualized as $M^5 \times \{-\delta, \delta\}$ where M^5 is the convex hull of M^3 .

We claim that

$$(6) \quad CR(M^6) \xrightarrow{\sim} CR(M^4).$$

To see how it comes about, consider

$$M_0 = M^4,$$

$$M' : \{ \rho_2 = 0, |y_1| < \delta \},$$

$$M : \{ \rho_1 \leq 0, \rho_2 = 0, |y_1| < \delta \}.$$

We have: M_0 is a holomorphically transverse hypersurface in M' ; $\bar{\partial}_{M'}$ has the local unique continuation property on M' (this follows from part (b) of Theorem 4, because the Levi form of ρ_2 has one positive eigenvalue at each point; see [6], p. 51, or [2]). Since M' has the form $(-\delta, \delta) \times \{\text{a cylinder with generators } z_2\}$, it is easy to see that (M, M') has a top hat foliation in $\mathbf{C}^3$ (with a single leaf $L' = M$). Therefore, $CR(M) \xrightarrow{\sim} CR(M^4)$ by Theorem 2. Thus we have gone up one dimension. To go up the one remaining dimension, we repeat the process, this time with

$$M_0 = \text{previous } M,$$

$$M' : \{ \rho_1 \leq 0, |y_1| < \delta \},$$

$$M : \{ \rho_1 \leq 0, \rho_2 \leq 0, |y_1| < \delta \}.$$

Again M_0 is a holomorphically transverse hypersurface in M' , and (M, M') has a top hat foliation in $\mathbf{C}^3$ (note that $\bar{\partial}_{M'}$ is just $\bar{\partial}$ in $\mathbf{C}^3$). Hence by Theorem 2, we can go up one more dimension, and the desired result (6) follows.

Since M^6 is a set of holomorphy, this is the maximal possible extension of all CR functions from M^4 .

This M^4 also has the property of local extendibility: Let V be any sufficiently small neighborhood (in $\mathbf{C}^3$) of the point $(z_1, z_2, z_3) = (-1, 0, 2) \in M^4$. It is easy to see, by a local version of the above argument, that all CR functions on $M^4 \cap V$ extend to holomorphic functions on a fixed open set in $\mathbf{C}^3$, whose closure contains $M^4 \cap V$. This is very similar to the example of Lewy ([9]).

Next let $\delta = 0$ in the above example, and consider

$$\begin{aligned} M^3 &= M^4 \cap \{y_1 = 0\}, \\ \bar{M}^4 &: \{\rho_1 \leq 0, \rho_2 = 0, y_1 = 0\}, \\ M^5 &= M^6 \cap \{y_1 = 0\}. \end{aligned}$$

M^3 is a compact 3-manifold that is totally real (its holomorphic tangent space has dimension zero), except for a one-dimensional set of points where either $z_2 = 0$ or $z_3 = 0$. The argument we have given for (6) shows that

$$(7) \quad \mathfrak{O}(M^5) \xrightarrow{\sim} \mathfrak{O}(M^3).$$

However, in spite of (7), it is not difficult to see that neither of the maps

$$\begin{aligned} CR(\bar{M}^4) &\rightarrow CR(M^3), \\ CR(M^5) &\rightarrow CR(\bar{M}^4) \end{aligned}$$

are surjective. Thus we have another example showing the difference between CR -extendibility and the extendibility of germs of holomorphic functions.

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