On Nonlinear Parabolic Variational Inequalities

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Let K be a closed convex subset of a real reflexive separable Banach space V with $0 \in K$ and let A be a nonlinear monotone coercive operator mapping bounded sets of $L^p(0, T; V)$ into bounded sets of $L^q(0, T; V^*)$. Brezis [2], Browder [4], Lions [8] and others have shown the existence of an element u in $L^p(0, T; V)$ with $u(t)$ in K for almost all t , such that

$$\int_0^T (v' + Au - f, v - u) dt \geq 0$$

for all v in $L^p(0, T; V)$, v' in $L^q(0, T; V^*)$, $v(t)$ in K and $v(0) = 0$.

The main purpose of this paper is to study the regularity of the solution u when A is a nonlinear elliptic operator written in divergence form and V is a subspace of $W^{m,p}(G)$. Following Lions, we shall make a systematic use of the penalty function.

The notations and the assumptions are given in Section 1. The global regularity of u with respect to the time-variable is shown in Section 2. The results obtained, contain those of Brezis [2] and extend the local regularity results of Visik [11]. In Section 3, the case when K is a closed convex subset of both V and H with $V \subset H$ is considered. With some additional hypotheses, the regularity of u with respect to the space-variables is obtained by using an imbedding theorem of Dubinskii [6].

The regularity of solutions of linear elliptic and parabolic inequalities has been studied by Lions [8], [9], Brezis–Stampacchia [3], Lewy–Stampacchia [7].

Section 1. Let G be a bounded open subset of E^n with a boundary ∂G of class C^∞ . Set $D_i = i^{-1} \partial / \partial x_i$; $j = 1, \dots, n$ and for each n -tuple $\alpha = (\alpha_1, \dots, \alpha_n)$ of non-negative integers, we write

$$D^\alpha = \prod_{i=1}^n D_i^{\alpha_i} \quad \text{with} \quad |\alpha| = \sum_{i=1}^n \alpha_i.$$

$W^{m,p}(G)$ is the real reflexive separable Banach space:

$$W^{m,p}(G) = \{u: u \in L^p(G), \quad D^\alpha u \in L^p(G), \quad |\alpha| \leq m\}$$

with the norm:

$$\|u\|_{m,p} = \left\{ \sum_{|\alpha| \leq m} \|D^\alpha u\|_{L^p(G)}^p \right\}^{1/p}; \quad 2 \leq p < \infty.$$

$W_0^{m,p}(G)$ is the completion of $C_0^\infty(G)$, the family of all infinitely differentiable functions with compact support in G , in the $\|\cdot\|_{m,p}$ norm. Let V be a closed subspace of $W^{m,p}(G)$ with $W_0^{m,p}(G) \subset V \subset W^{m,p}(G)$ and let $H = L^2(G)$. The pairing between V and its dual V^* is denoted by $(\cdot, \cdot)$. Let $[0, T]$ be a compact interval of the real line R . The derivative of u with respect to t will be denoted by $D_t u$ or simply by u' when there is no confusion possible.

$L^p(0, T; V)$ is the space of equivalence classes of functions $u(t)$ from $[0, T]$ to V which are L^p -integrable. It is a real reflexive separable Banach space with the norm:

$$\|u\|_{L^p(0,T;V)} = \left\{ \int_0^T \|u(t)\|_V^p dt \right\}^{1/p}.$$

$L^2(0, T; H)$ is defined in a similar fashion.

We consider nonlinear partial differential operators on G with coefficients depending on t in R , of the form:

$$A(u) = \sum_{|\alpha| \leq m} D^\alpha A_\alpha(x, t, u, \cdot, D^m u).$$

Assumption (I). (i) Each A_α is continuous in x on G and continuously differentiable in all the other arguments.

(ii) There exist a positive constant C and a function $g(x, t) \in L^2(R; H)$ such that

$$|A_\alpha(x, t, \zeta)| + |D_t A_\alpha(x, t, \zeta)| + (1 + |\zeta|) \sum_{|\beta| \leq m} |A_{\alpha\beta}(x, t, \zeta)| \leq |g(x, t)| + C |\zeta|$$

where $\zeta = \{\zeta_\alpha : |\alpha| \leq m\}$ and $A_{\alpha\beta} = \partial A_\alpha / \partial \zeta_\beta$.

$$(iii) \sum_{|\alpha|, |\beta| \leq m} A_{\alpha\beta}(x, t, \zeta) \eta_\alpha \eta_\beta \geq c \sum_{|\alpha| \leq m} \eta_\alpha^2.$$

c is a positive constant.

We shall also consider nonlinear partial differential operators A satisfying the following assumption.

Assumption (II). (i) Each A_α depends only on (x, ζ) , is continuous in x on G and continuously differentiable in ζ .

(ii) There exists a positive constant C such that

$$|A_\alpha(x, \zeta)| + \sum_{|\beta| \leq m} |\zeta| |A_{\alpha\beta}(x, \zeta)| \leq C |\zeta|^{p-1}, \quad 2 \leq p < \infty.$$

$$(iii) \sum_{|\alpha|, |\beta| \leq m} A_{\alpha\beta}(x, \zeta) \eta_\alpha \eta_\beta \geq \sum_{|\alpha| \leq m} \phi(\zeta) \eta_\alpha^2$$

where $\phi(\zeta) = c_0 + c_1 |\zeta|^{p-2}$. $c_0 \geq 0$ and c_1 is a positive constant.

Section 2. The main results of this section are the following two theorems.

Theorem 1. Let K be a closed convex subset of V with $0 \in K$ and let $A(u)$ be an operator satisfying Assumption (I). Suppose that $f \in L^2(0, T; V^*)$, $f' \in L^2(0, T; V^*)$ and $f(0) = 0$. Then there exists a unique u in $L^2(0, T; V)$, u' in $L^2(0, T; V)$ with $u(t)$ in K for almost all t and $u(0) = 0$, such that

$$\int_0^T (u' + Au - f, v - u) dt \geq 0$$

for all v in $L^2(0, T; V)$, v' in $L^2(0, T; V^*)$, $v(t)$ in K and $v(0) = 0$.

(V is a subspace of $W^{m,2}(G)$ with $W_0^{m,2}(G) \subset V \subset W^{m,2}(G)$.)

We have also:

Theorem 2. Let A be an elliptic operator satisfying Assumption (II) and let K be a closed convex subset of V with $0 \in K$. Suppose that

- (a) $(Av - Aw, v - w) \geq c_0 \|v - w\|_{1,2}^2$ for all v, w in V .
 c_0 is a positive constant.
- (b) $f \in L^q(0, T; V^*)$, $f' \in L^2(0, T; (W^{m,2}(G))^*)$ and $f(0) = 0$.

Then there exists a unique u in $L^p(0, T; V)$, u' in $L^2(0, T; W^{m,2}(G))$ with $u(t)$ in K for almost all t and $u(0) = 0$, such that

$$\int_0^T (u' + Au - f, v - u) dt \geq 0$$

for all v in $L^p(0, T; V)$, v' in $L^q(0, T; V^*)$, $v(t)$ in K and $v(0) = 0$. Moreover, $D_i(|D^\alpha u|^{(p-2)/2} D^\alpha u) \in L^2(0, T; H)$ for $|\alpha| \leq m$. If the constant c_0 of (a) is null and if $f' \in L^2(0, T; H)$, then the unique solution u of the above inequality is such that $u' \in L^\infty(0, T; H)$ with $D_i(|D^\alpha u|^{(p-2)/2} D^\alpha u) \in L^2(0, T; H)$ for $|\alpha| \leq m$.

Theorem 1 will be derived from

Theorem 3. Let B be the penalty function associated with the convex K and the gauge function $\varphi(r) = r$. Suppose all the hypotheses of Theorem 1 are satisfied. Then for each ϵ , $0 < \epsilon < 1$, there exists a unique solution u_ϵ in $L^2(0, T; V)$ of the equation:

$$\epsilon(u'_\epsilon + Au_\epsilon) + Bu_\epsilon = \epsilon f, \quad u_\epsilon(0) = 0.$$

Moreover,

$$\|u_\epsilon\|_{L^2(0,T;V)} + \|u'_\epsilon\|_{L^2(0,T;V)} + \epsilon^{-1} \int_0^T (Bu_\epsilon, u_\epsilon) dt \leq C.$$

C is a constant independent of ϵ .

Proof of Theorem 1 using Theorem 3.

1) It is known that the penalty function B associated with the convex K and with the gauge function $\varphi(r) = r$ exists. B is a semi-continuous, monotone

operator mapping bounded sets of V into bounded sets of V^* . Moreover, $K = \{v: v \text{ in } V, Bv = 0\}$. Since $0 \notin K$, we have $(Bv, v) \geq 0$ for all v in V .

2) From the weak compactness of the unit ball in a Hilbert space, we get, by taking subsequences if necessary: $u_\epsilon \rightarrow u$ weakly in $L^2(0, T; V)$, $u'_\epsilon \rightarrow u'$ weakly in $L^2(0, T; V)$ and $Bu_\epsilon \rightarrow h$ weakly in $L^2(0, T; V^*)$ as $\epsilon \rightarrow 0$.

It is clear that $Bu_\epsilon = \epsilon(f - u'_\epsilon - Au_\epsilon) \rightarrow 0$ in $\mathfrak{D}'(0, T; V^*)$. Since B is monotone and $\int_0^T (Bu_\epsilon, u_\epsilon) dt \leq C\epsilon$, it follows from a standard argument of the theory of monotone operators that $h = Bu = 0$. Hence $u(t) \in K$ for almost all t .

3) We have

$$\int_0^T (u'_\epsilon + Au_\epsilon + \epsilon^{-1}Bu_\epsilon - f, u_\epsilon - v) dt = 0$$

for all v in $L^2(0, T; V)$, v' in $L^2(0, T; V^*)$ and $v(0) = 0$.

Thus if $v \in K$, we get

$$\int_0^T (u'_\epsilon + Au_\epsilon + \epsilon^{-1}Bu_\epsilon - f, u_\epsilon - v) dt - \epsilon^{-1} \int_0^T (Bv, u_\epsilon - v) dt = 0.$$

Hence,

$$\int_0^T (u'_\epsilon + Au_\epsilon - f, u_\epsilon - v) dt \leq 0.$$

Applying Aubin's theorem [1], we obtain

$$\int_0^T (u', u - v) dt + \limsup \int_0^T (Au_\epsilon, u_\epsilon - v) dt \leq \int_0^T (f, u - v) dt.$$

Take $v = u$ and we have

$$\limsup \int_0^T (Au_\epsilon, u_\epsilon - u) dt \leq 0.$$

Since A is monotone, a standard argument gives: $Au_\epsilon \rightarrow Au$ weakly in $L^2(0, T; V^*)$ and

$$\limsup \int_0^T (Au_\epsilon, u_\epsilon) dt \leq \int_0^T (Au, u) dt.$$

Therefore

$$\int_0^T (u' + Au - f, v - u) dt \geq 0$$

for all v in $L^2(0, T; V)$, v' in $L^2(0, T; V^*)$, $v(t) \in K$ and $v(0) = 0$.

4) It is easy to check that the solution u is unique.

Proof of Theorem 3.

1) To prove the theorem, we use a technical device introduced by Browder in [5].

Let $F = \{u: u \in L^2(R; V) \cap C(R; V) \cap C_0^1(R; H)\}$ and denote by L_0 the linear mapping from $D(L_0) = F$ into $L^2(R; V^*)$ such that

$$\int_R (L_0 u, v) dt = \int_R (u', v) dt \quad \text{for all } v \text{ in } L^2(R; V).$$

L is the closure of L_0 as a mapping with domain in $L^2(R; V)$ and range in $L^2(R; V^*)$.

If $\hat{f} \in L^2(R; V^*)$, then it was shown in [5] that there exists a unique solution v_ϵ in $D(L) \cap L^2(R; V)$ of the equation:

$$\epsilon(Lv_\epsilon + Av_\epsilon) + Bv_\epsilon = \epsilon\hat{f}.$$

Moreover, if $\hat{f}(t) = 0$ for $t \leq 0$, then $v_\epsilon(t) = 0$ for $t \leq 0$. Let $u_\epsilon(t)$ be the restriction of $v_\epsilon(t)$ to $[0, T]$. Then u_ϵ is the unique solution of the equation:

$$\epsilon(u'_\epsilon + Au_\epsilon) + Bu_\epsilon = \epsilon\hat{f} \quad \text{on } [0, T]; \quad u_\epsilon(0) = 0.$$

Let $\tilde{f}(t) = f(t)$ on $[0, T]$, $\tilde{f}(t) = 0$ for $t \leq 0$ and $\tilde{f}(t) = f(T)$ for $t \geq T$. Denote by $\zeta(t) \in C_0^\infty(R)$, $\zeta(t) = 1$ on $[0, T]$, $\zeta(t) = 0$ for $t \leq -2$ and $t \geq 2T$. We take $\tilde{f} = \zeta(t)\tilde{f}(t)$. Then $\tilde{f}$ is in $L^2(R; V^*)$ and $D_t\tilde{f}$ is in $L^2(R; V^*)$.

2) We show that $\|v'_\epsilon\|_{L^2(R; V)} \leq C$.

Let $h > 0$ and set $d_h\varphi = (\varphi(t+h) - \varphi(t))/h$. Then

$$\int_R (D_t(d_hv_\epsilon) + d_h(Av_\epsilon), d_hv_\epsilon) dt + \epsilon^{-1} \int_R (d_hBv_\epsilon, d_hv_\epsilon) dt = \int_R (d_h\tilde{f}, d_hv_\epsilon) dt.$$

But $(d_h(Bv_\epsilon), d_hv_\epsilon) = h^{-2}(B(v_\epsilon(t+h)) - B(v_\epsilon(t)), v_\epsilon(t+h) - v_\epsilon(t)) \geq 0$. Thus

$$\begin{aligned} \int_R (d_h(Av_\epsilon), d_hv_\epsilon) dt &\leq \|d_h\tilde{f}\|_{L^2(R; V^*)} \cdot \|d_hv_\epsilon\|_{L^2(R; V)} \\ &\leq C \|d_hv_\epsilon\|_{L^2(R; V)} \{ \|f\|_{L^2(0, T; V^*)} + \|f'\|_{L^2(0, T; V^*)} \}. \end{aligned}$$

C is a constant independent of both ϵ and h .

We have

$$d_h(A_\alpha) = \sum_{|\beta| \leq m} \hat{A}_{\alpha\beta}(x, t, v_\epsilon, \cdot, D^m v_\epsilon) D^\beta(d_hv_\epsilon) + \hat{A}_\alpha(x, t, v_\epsilon, \cdot, D^m v_\epsilon)$$

where

$$\hat{A}_{\alpha\beta}(x, t, v, \cdot, D^m v) = \int_0^1 \partial A_\alpha(x, t, v_\eta, \cdot, D^m v_\eta) / \partial (D^\beta v_\eta) d\eta$$

and

$$\hat{A}_\alpha(x, t, v, \cdot, D^m v) = \int_0^1 \partial A_\alpha(x, t, v_\eta, \cdot, D^m v_\eta) / \partial t d\eta$$

with $v_\eta = (1 - \eta)v + \eta v(\cdot, t + h)$. Therefore

$$\begin{aligned} \int_R (d_h(Av_\epsilon), d_h v_\epsilon) dt &= \sum_{|\alpha|, |\beta| \leq m} \int_Q \hat{A}_{\alpha\beta} D^\alpha(d_h v_\epsilon) D^\beta(d_h v_\epsilon) dx dt \\ &+ \sum_{|\alpha| \leq m} \int_Q \hat{A}_\alpha D^\alpha(d_h v_\epsilon) dx dt = E_1 + E_2. \end{aligned}$$

Q denotes the cylinder $R \times G$.

It follows from Assumption (I) that $E_1 \geq c \|d_h v_\epsilon\|_{L^2(R; V)}^2$. On the other hand, an easy computation gives $|E_2| \leq M \|d_h v_\epsilon\|_{L^2(R; V)}$. Hence $\|d_h v_\epsilon\|_{L^2(R; V)} \leq C$. C is a constant independent of both h and ϵ . A standard argument gives $\|v'_\epsilon\|_{L^2(R; V)} \leq C$ and it follows that $\|u'_\epsilon\|_{L^2(0, T; V)} \leq C$.

Proof of Theorem 2. The proof is similar to that of Theorem 1. Let B be the penalty function associated with the convex K and with the gauge function $\varphi(r) = r^{p-1}$. As in the proof of Theorem 3, there exists a unique solution of the global boundary-value problem:

$$\epsilon(Lv_\epsilon + Av_\epsilon) + Bv_\epsilon = \epsilon \hat{f}.$$

The restriction u_ϵ of v_ϵ to $[0, T]$ is the unique solution of

$$\epsilon(u'_\epsilon + Au_\epsilon) + Bu_\epsilon = \epsilon f, \quad u_\epsilon(0) = 0.$$

1) Consider the case when $\phi(\xi) = c_0 + c_1 |\xi|^{p-2}$ with $c_0 > 0$.

A proof identical to that of Theorem 3 gives

$$\begin{aligned} c_0 \|d_h v_\epsilon\|_{L^2(R; W^{m,2}(G))}^2 + c_1 \sum_{|\alpha| \leq m} \int_Q |D^\alpha v_\epsilon|^{p-2} |D^\alpha(d_h v_\epsilon)|^2 dx dt \\ \leq \sum_{|\beta|, |\alpha| \leq m} \int_Q \hat{A}_{\alpha\beta}(x, v_\epsilon, \cdot, D^m v_\epsilon) D^\alpha(d_h v_\epsilon) D^\beta(d_h v_\epsilon) dx dt. \end{aligned}$$

The expression on the right hand side is majorized by $C \|d_h v_\epsilon\|_{L^2(R; W^{m,2}(G))}$. Thus

$$\|d_h v_\epsilon\|_{L^2(R; W^{m,2}(G))} + \sum_{|\alpha| \leq m} \int_Q |D^\alpha v_\epsilon|^{p-1} |D^\alpha(d_h v_\epsilon)|^2 dx dt \leq C.$$

The different constants C are all independent of both h and ϵ . From the weak compactness of the unit ball in a Hilbert space, we obtain, by taking subsequences if necessary: $d_h v_\epsilon \rightarrow v'_\epsilon$ weakly in $L^2(R; W^{m,2}(G))$, $|D^\alpha v_\epsilon|^{(p-2)/2} D^\alpha(d_h v_\epsilon) \rightarrow g'_\epsilon$ weakly in $L^2(R; H)$ for $|\alpha| \leq m$, as $h \rightarrow 0$.

Moreover

$$\|v'_\epsilon\|_{L^2(R; W^{m,2}(G))} + \|g'_\epsilon\|_{L^2(R; H)} \leq M.$$

We show that $g'_\epsilon = |D^\alpha v_\epsilon|^{(p-2)/2} D^\alpha v'_\epsilon$.

Consider the case $|\alpha| = m$. We have: $D^m(d_h v_\epsilon) \rightarrow D^m v'_\epsilon$ weakly in $L^2(R; H)$. On the other hand, since $v_\epsilon \in L^p(R; W^{m,p}(G))$, $|D^m v_\epsilon|^{(p-2)/2} \in L^{2p/(p-2)}(Q)$. Hence for any $\varphi \in C_0^\infty(Q)$, we get

$$\int_Q |D^m v_\epsilon|^{(p-2)/2} D^m(d_h v_\epsilon) \cdot \varphi dx dt \rightarrow \int_Q |D^m v_\epsilon|^{(p-2)/2} D^m v'_\epsilon \cdot \varphi dx dt.$$

So $g^m = |D^m v_\epsilon|^{(p-2)/2} D^m v'_\epsilon$ in $\mathfrak{D}'(Q)$.

2) If $c_0 = 0$, then the proof is slightly different and the results are weaker.

Set $d_{-h}\varphi = (\varphi(t) - \varphi(t-h))/h$ with $h > 0$ and consider

$$\int_{-1}^t (d_{-h}v'_\epsilon + d_{-h}(Av_\epsilon) + \epsilon^{-1} d_{-h}(Bv_\epsilon), d_{-h}v_\epsilon) dt = \int_{-1}^t (d_{-h}f, d_{-h}v_\epsilon) dt.$$

After some computations as in the first part, we get

$$\frac{1}{2} \|d_{-h}v_\epsilon(t)\|_H^2 \leq C + \int_0^t \|d_{-h}v_\epsilon\|_H^2 dt.$$

C is independent of both ϵ and h .

Hence $\|v'_\epsilon\|_{L^\infty(0,T;H)} \leq M$.

3) Now let $\epsilon \rightarrow 0$. A proof as in that of Theorem 1 gives the wanted result.

Remark. In Theorem 2, if some additional regularity hypotheses on A_ϵ are made, we may assume that A_ϵ depends also on t .

Section 3. We shall now study the regularity of solutions of nonlinear parabolic inequalities, with respect to the space variables.

Theorem 4. Suppose all the hypotheses of Theorem 1 are satisfied. Let K be a closed convex subset of both V and H with $V = W_0^{m,2}(G)$ and let B be the penalty function associated with K , considered as a closed convex subset of H . Suppose further that

(i) $\|Bu_\epsilon\|_{L^2(0,T;H)} \leq C\epsilon$ where u_ϵ is the unique solution of

$$\epsilon(u'_\epsilon + Au_\epsilon) + Bu_\epsilon = \epsilon f, \quad u_\epsilon(0) = 0.$$

(ii) $f \in L^2(0, T; H)$, $f' \in L^2(0, T; V^*)$ with $f(0) = 0$.

Then there exists a unique u in $L^2(0, T; V) \cap L^2(0, T; W^{m+1,2}(G))$ with u' in $L^2(0, T; V)$, $u(t)$ in K for almost all t and $u(0) = 0$, such that

$$\int_0^T (u' + Au - f, v - u) dt \geq 0$$

for all v in $L^2(0, T; V)$, v' in $L^2(0, T; V^*)$, $v(t)$ in K and $v(0) = 0$.

Proof. 1) It follows from the hypotheses of the theorem that $\epsilon^{-1}Bu_\epsilon \rightarrow h$ weakly in $L^2(0, T; H)$ as $\epsilon \rightarrow 0$. Thus from Theorem 1 (or more precisely from the proof of Theorem 1), we get $u' + Au = f - h = g$ with g in $L^2(0, T; H)$.

2) First, we show that u is in $L^2(0, T; W^{m+1,2}(G_1))$ for any G_1 with $cl(G_1) \subset G$.

Let $\eta(x)$ be a $C_0^\infty(G)$ function with $\eta(x) = 1$ on G_1 , $\eta(x) = 0$ for x outside of G_2 with $cl(G_1) \subset G_2 \subset cl(G_2) \subset G$.

We have

$$\int_0^T (d_{i\kappa}u', \eta^{2m} d_{i\kappa}u) dt + \int_0^T (d_{i\kappa}(Au), \eta^{2m} d_{i\kappa}u) dt = \int_0^T (d_{i\kappa}g, \eta^{2m} d_{i\kappa}u) dt.$$

Thus

$$\int_0^T (d_{ih}(Au), \eta^{2m} d_{ih}u) dt \leq C \|g\|_{L^2(0,T;H)} \|\eta^{2m} d_{ih}u\|_{L^2(0,T;W^{1,2}(G))}.$$

C is independent of h .

Consider

$$d_{ih}(A_\alpha) = \sum_{|\beta| \leq m} \hat{A}_{\alpha\beta}(x, t, u, \cdot, D^m u) D^\beta(d_{ih}u) + \hat{A}_{\alpha i}(x, t, u, \cdot, D^m u).$$

A standard argument gives

$$\begin{aligned} E_1 &= \sum_{|\alpha|, |\beta| \leq m} \int_Q \hat{A}_{\alpha\beta} D^\alpha(d_{ih}u) D^\beta(\eta^{2m} d_{ih}u) dx dt \\ &\geq c_1 \sum_{|\alpha| \leq m} \|\eta^m D^\alpha(d_{ih}u)\|_{L^2(0,T;H)}^2 - c_2 \|u\|_{L^2(0,T;V)}^2. \end{aligned}$$

c_1 and c_2 are two positive constants independent of h . Consider

$$E_2 = \sum_{|\alpha| \leq m} \int_Q \hat{A}_{\alpha i} D^\alpha(d_{ih}u \cdot \eta^{2m}) dx dt.$$

We have

$$|E_2| \leq c_3 \|u\|_{L^2(0,T;V)} \sum_{|\alpha| \leq m} \|\eta^m D^\alpha(d_{ih}u)\|_{L^2(0,T;H)}.$$

Thus

$$\sum_{|\alpha| \leq m} \|\eta^m D^\alpha(d_{ih}u)\|_{L^2(0,T;H)} \leq C.$$

Hence $u \in L^2(0, T; W^{m+1,2}(G_1))$.

3) We shall now show the regularity of u near the boundary.

Since $u' + Au = g$ and u' is in $L^2(0, T; H)$, we may write $Au = H$ with $H(x, t)$ in $L^2(0, T; H)$.

The boundary ∂G is of class C^∞ . We map a neighborhood of ∂G into $B^+(R) = \{y: |y| < R, y_1 > 0\}$ with the boundary on $\{y: |y| < R, y_1 = 0\}$.

Let $\eta(y) \in C_0^\infty(\text{cl}(B^+(R)))$ with $\eta(y) = 1$ on $B^+(R_1)$, $\eta(y) = 0$ outside of $B^+(R_2)$ with $R_1 < R_2 < R$.

Then, a proof as in part 2) gives

$$\sum_{|\alpha| \leq m} \sum_{j=2}^n \int_0^T \int_{B^+(R)} \eta(y) |D^\alpha D_j u|^2 dy dt \leq C.$$

To compute the normal derivative, we consider the equation. We may write

$$D_1^m A_{\alpha'}(y, t, u, \cdot, D^m u) = H(y, t) - \sum_{\substack{|\beta| \leq m-1 \\ \alpha \neq \alpha'}} D^\beta \left(\sum_{j=2}^n D_j A_\alpha \right),$$

with $\alpha' = (m, 0, \dots, 0)$. But

$$D_i A_\alpha = \sum_{|\gamma| \leq m} A_{\alpha\gamma} D^\gamma D_i u + A_{\alpha i} \quad \text{with } j = 2, \dots, n.$$

From the previous part, we know that $D_j A_\alpha \in L^2(0, T; L^2(B^+(R_2)))$ for $j = 2, \dots, n$. Since $H(y, t)$ is in $L^2(0, T; L^2(B^+(R)))$, we have by a well-known argument of the theory of elliptic operators:

$$\begin{aligned} \|D_1 A_{\alpha'}\|_{L^2(B^+(R_2))} &\leq C \left\{ 1 + \|A_\alpha\|_{L^2(B^+(R_2))} + \sum_{j=2}^n \|D_j A_\alpha\|_{L^2(B^+(R_2))} \right\} \\ &\leq C \left\{ 1 + \|u\|_{W^{m,2}(B^+(R))} + \sum_{j=2}^n \|D_j u\|_{W^{m,2}(B^+(R_2))} \right\}. \end{aligned}$$

α' is such that $\alpha' = (m, 0, \dots, 0)$. On the other hand,

$$A_{\alpha', \alpha'} D_1^{m+1} u = D_1 A_{\alpha'} - A_\alpha^1 - \sum_{\substack{|\gamma| \leq m \\ \gamma \neq \alpha'}} A_\alpha D^\gamma (D_1 u).$$

Condition (iii) of Assumption (I) gives

$$\int_0^T \int_{B^+(R)} \eta(y) |D_1^{m+1} u|^2 dx dt \leq C.$$

The theorem is proved.

Theorem 5. Suppose all the hypotheses of Theorem 2 are satisfied and A_α are independent of x . Let K be a closed convex subset of both V and H with $V = W_0^{m,p}(G)$ and $0 \in K$. Let B be the penalty function associated with the convex K considered as a subset of H . Suppose further that

- (i) $\|Bu_\epsilon\|_{L^2(0,T;H)} \leq C\epsilon$ where u_ϵ is the unique solution of the equation: $\epsilon(u'_\epsilon + Au_\epsilon) + Bu_\epsilon = \epsilon f$, $u_\epsilon(0) = 0$.
- (ii) $f \in L^2(0, T; H)$, $f' \in L^2(0, T; (W^{m,2}(G))^*)$ and $f(0) = 0$.
- (iii) $(Av - Aw, v - w) \geq c_0 \|v - w\|_{1,2}^2$ for all v, w in V with $c_0 > 0$.

Then there exists a unique u in $L^p(0, T; V) \cap L^q(0, T; W^{m+1,q}(G))$, u' in $L^2(0, T; W^{m,2}(G))$ with $u(t)$ in K for almost all t and $u(0) = 0$ such that

$$\int_0^T (u' + Au - f, v - u) dt \geq 0$$

for all v in $L^p(0, T; V)$, v' in $L^q(0, T; V^*)$ and $v(0) = 0$ with $v(t)$ in K . Moreover, $|D^\alpha u|^{p-2} D_i D^\alpha u \in L^q(0, T; L^q(G))$ for $|\alpha| \leq m$ and $j = 1, \dots, n$.

Proof. 1) The proof of the interior regularity of u with respect to the space-variables is analogous to that of Theorem 4. One shows

$$\sum_{|\alpha| \leq m} \sum_{j=1}^n \int_0^T \int_{G_1} |D^\alpha u|^{p-2} |D_j D^\alpha u|^2 dx dt < \infty.$$

G_1 is any subset of G with $\partial(G_1) \subset G$.

2) To show that u is regular near the boundary of G , we use an imbedding theorem of Dubinskii [6] (Lemma 6).

Since the boundary ∂G is of class C^∞ , we map each neighborhood of ∂G into $B^+(R) = \{y: |y| < R, y_1 > 0\}$ with the boundary on $\{y: |y| < R, y_1 = 0\}$.

The tangential derivatives, i.e., with respect to $(y_2, \dots, y_n)$, are obtained in the same fashion as in the first part. It remains only to compute the normal derivative. We may write

$$D_1^m A_\alpha(y, t, u, \cdot, D^m u) = f - u' - \sum_{|\beta| \leq m-1} D^\beta \left(\sum_{j=2}^n D_j A_\alpha \right)$$

where $\alpha = (m, 0, \dots, 0)$.

It follows from Lemma 6 of Dubinskii [6] and from the estimates on the tangential derivatives that $D_j A_\alpha \in L^q(0, T; L^q(B^+(R_2)))$ for $j = 2, \dots, n$. A well-known result of the theory of elliptic operators gives $D_1 A_\alpha \in L^q(0, T; L^q(B^+(R_2)))$ with $R_2 < R$ and $\alpha = (m, 0, \dots, 0)$. On the other hand,

$$A_{\alpha\alpha} D_1^{m+1} u = D_1 A_\alpha - \sum_{\substack{|\beta| \leq m \\ \beta \neq \alpha}} A_{\alpha\beta} D^\beta D_1 u.$$

Condition (iii) Of Assumption (II) gives

$$|D_1^{m+1} u| + |D^m u|^{p-2} |D_1^{m+1} u| \leq |A_{\alpha\alpha} D_1^{m+1} u| \leq |D_1 A_\alpha| + \sum_{\substack{|\beta| \leq m \\ \alpha \neq \beta}} |A_{\alpha\beta} D^\beta D_1 u|.$$

Using again Lemma 6 of [6], we obtain $D_1^{m+1} u \in L^q(0, T; L^q(B^+(R_1)))$, $|D^\gamma u|^{p-2} D^\gamma D_1 u \in L^q(0, T; L^q(B^+(R_1)))$ where $|\gamma| \leq m$ and $R_1 < R_2 < R$.

The theorem is proved.

We shall now give some applications.

Theorem 6. Let A be as in Theorem 1 with $m = 1$ and $V = W_0^{1,2}(G)$. Let K be the set $\{v: v \text{ in } V, v \geq 0 \text{ a.e. on } G\}$. Then if $f \in L^2(0, T; H)$, f' in $L^2(0, T; V^*)$ and $f(0) = 0$, there exists a unique u in $L^2(0, T; V) \cap L^2(0, T; W^{2,2}(G))$, $u(t)$ in K for almost all t , $u(0) = 0$ and u' in $L^2(0, T; V)$, such that

$$\int_0^T (u' + Au - f, v - u) dt \geq 0$$

for all v in $L^2(0, T; V)$, v' in $L^2(0, T; V^*)$, $v(t)$ in K and $v(0) = 0$.

Proof. To prove the theorem we apply Theorem 4. It is clear that K is a closed convex subset of both V and $H = L^2(G)$. It suffices to verify hypothesis (i) of Theorem 4.

Let v be in H . Define the penalty function B by $Bv = -v^-$. Then if $v \in V$, Bv is also in V .

Consider the equation: $\epsilon(u'_\epsilon + Au_\epsilon) + Bu_\epsilon = \epsilon f$, $u_\epsilon(0) = 0$. It is easy to see that $-(Au_\epsilon, u_\epsilon^-) \geq 0$ and thus

$$\int_0^T (Bu_\epsilon, -u_\epsilon^-) dt = \|u_\epsilon^-\|_{L^2(0, T; H)}^2 \leq \epsilon \|f\|_{L^2(0, T; H)} + \epsilon \|u'_\epsilon\|_{L^2(0, T; H)}.$$

Hence $\|Bu_\epsilon\|_{L^2(0, T; H)} \leq C\epsilon$.

Applying Theorem 4, we get the wanted results.

Theorem 7. Let A be as in Theorem 2 with $m = 1$ and $V = W_0^{1,p}(G)$. Let K be the set $\{v: v \text{ in } V, v \geq 0 \text{ a.e. on } G\}$. Then for any f in $L^2(0, T; H)$, f' in $L^2(0, T; (W^{1,2}(G))^*)$ and $f(0) = 0$, there exists a unique solution u in $L^p(0, T; V) \cap L^q(0, T; W^{m+1,q}(G))$, u' in $L^2(0, T; H)$ and $u(t)$ in K for almost all t , $u(0) = 0$ such that

$$\int_0^T (u' + Au - f, v - u) dt \geq 0$$

for all v in $L^p(0, T; V)$, v' in $L^q(0, T; V^*)$, $v(t)$ in K and $v(0) = 0$.

Proof. K is a closed convex subset of both V and $H = L^2(G)$. Let v be in H and define the penalty function B by: $Bv = -v^-$. Then B maps V into V .

The theorem follows from Theorem 5. Indeed, it is easy to show that $\|Bu_\epsilon\|_{L^2(0, T; H)} \leq C\epsilon$ where u_ϵ is the unique solution of $\epsilon(u'_\epsilon + Au_\epsilon) + Bu_\epsilon = \epsilon f$, $u_\epsilon(0) = 0$.

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