The Converse to the Fixed Point Theorem of P. A. Smith: II

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An action of the cyclic multiplicative group $\mathbf{Z}_n$ on the manifold M is semi-free if either $r(t, x) = x, \forall t \in \mathbf{Z}_n$, or $r(t, x) = x$ only if $t = 1$, holds for each point x in M . Two semi-free actions $r_i : \mathbf{Z}_n \times M \rightarrow M$ ($i = 0, 1$) are equivalent if there is a semi-free action

$$r : \mathbf{Z}_n \times (M \times [0, 1], M \times \{0 \cup 1\}) \rightarrow (M \times [0, 1], M \times \{0 \cup 1\})$$

so that $r|_{M \times i}$ is equal to r_i ($i = 0, 1$).

In this paper we shall always assume that M , semi-free group actions on M , and equivalences of group actions on M are all differentiable. Under these restrictions the fixed point set, F_r , of any semi-free action $r : \mathbf{Z}_n \times M \rightarrow M$ is a differentiable submanifold of M . Furthermore the action of $\mathbf{Z}_n$ is linear in some neighborhood of F_r in M in the following sense:

(1) There is a $\mathbf{Z}_n$ -equivariant, linear bundle, τ , normal to F_r in M , such that the $\mathbf{Z}_n$ -action restricts on each fiber of τ to a linear automorphism.

(2) If $n = 2$, this restriction to each fiber of τ is just the antipodal map (multiplication by -1).

(3) If $n \neq 2$, choose (for the duration of this paper) a minimal set of integers $q_1, q_2, \dots, q_l$ prime to n , so that every primitive n^{th} root of unity is either some $e^{(2\pi i/n)q_i}$ or a complex conjugate thereof. Then τ splits into a Whitney sum $\bigoplus_{i=1}^l \tau_i$ of complex bundles, so that each factor τ_i is invariant under the action of $\mathbf{Z}_n$, and this restriction to each fiber of τ_i is just complex multiplication by $e^{(2\pi i/n)q_i}$. Furthermore the splitting $\tau \sim \bigoplus_{i=1}^l \tau_i$ is uniquely determined by the action $r : \mathbf{Z}_n \times M \rightarrow M$. Let $\{d_i : i = 1, 2, \dots, l\}$ be the complex dimensions of the fibers of $\{\tau_i : i = 1, 2, \dots, l\}$; these numbers are equivalence (of $\mathbf{Z}_n$ -actions) invariants. These results can be found on pages 104–107 of [3].

Let $C_m(d_1, d_2, \dots, d_l)$ denote the set of equivalence classes of semi-free orientation preserving (differentiable) $\mathbf{Z}_n$ -actions on S^m , where if $n \neq 2$, then $d_i, i = 1, \dots, l$, are as in (3) above; and if $n = 2$, then $l = 1$, and d_1 equals the real dimension of the normal bundle τ to F_r . $C_m(d_1, \dots, d_l)$ has an abelian

group structure: two actions are added by taking connected sums along the fixed sets; the zero element is represented by any semi-free action $r : \mathbf{Z}_n \times S^m \rightarrow S^m$ which extends to a semi-free action $r' : \mathbf{Z}_n \times D^{m+1} \rightarrow D^{m+1}$.

This paper is concerned with calculating the groups $C_m(d_1, \dots, d_l)$.

Briefly, the line of argument is as follows. First, separate the calculation of $C_m(d_1, d_2, \dots, d_l)$ into two problems (following Conner and Floyd [3]):

- (1) The calculation of the cobordism class of possible fixed point sets.
- (2) An invariant determining the free part of the action.

The invariant in (2) comes from an element in the Wall surgery obstruction group $L_{m+1}^h(\mathbf{Z}_n)$; the role of this invariant has been completely determined by W. Browder and T. Petrie [2]. The possible fixed point sets and cobordisms of fixed point sets in (1) is completely determined: by the previous discussion concerning normal bundles to fixed point sets; by the P. A. Smith fixed point theorem [11] which says that the fixed point set, F_r , of a semi-free $\mathbf{Z}_n$ -action $r : \mathbf{Z}_n \times S^m \rightarrow S^m$ satisfies

$$H_*(F_r, \mathbf{Z}_n) \cong H_*(S^j, \mathbf{Z}_n)$$

for some $j \leq m$; and by the author's previous study concerning the converse to the P. A. Smith fixed point theorem [4]. Hence the problem of computing $C_m(d_1, \dots, d_l)$ is reduced to a computation problem with a known solution, and the computation of cobordism classes of real, and almost complex $\mathbf{Z}_n$ -homology spheres. This latter problem can be solved by using a modulo n^* -torsion version of the framed surgery arguments which relate groups of homotopy spheres to stable homotopy groups of spheres [5] (n^* equals the set of primes prime to n).

Here is a result typical of those obtained by these methods: $C_m(d, 0, 0, 0, \dots, 0)$ is a finite group whenever $m = \text{even}$, $m - d \geq 5$, $m < 4d - 5$, are all satisfied.

G. Bredon has also studied the involution groups $C_m(d)$. In Chapter 8 of his forthcoming book he considers several invariants which determine non-trivial elements in $C_m(d)$; one of these coincides with the surgery invariants studied in [2].

Section 1. G denotes any one of the groups $U(d_1) \times U(d_2) \times \dots \times U(d_l)$, $SO(d)$; and $d(G) = 2(\sum_{i=1}^l d_i)$, $d(G) = d$, respectively. BG denotes the classifying space for vector bundles with structural group G .

For every integer n a graded cobordism group $\Omega_*^{G, \mathbf{Z}_n}$ is defined as follows. The set $\Omega_*^{G, \mathbf{Z}_n}$ consists of equivalence classes of j -dimensional differentiable submanifolds $M \subset S^{j+d(G)}$ whose normal bundles have a given BG -bundle structure, and which further satisfy $H_*(M, \mathbf{Z}_n) = H_*(S^j, \mathbf{Z}_n)$: two such submanifolds $M_i \subset S^{j+d(G)}$ ($i = 0, 1$) represent the same cobordism class in $\Omega^{G, \mathbf{Z}_n}$ if there is a $(j+1)$ -dimensional submanifold

$$(W, \partial_- W, \partial_+ W) \subset S^{j+d(G)} \times ([0, 1], 0, 1)$$

whose normal bundle has a given BG -bundle structure, which satisfies $H_*(W, \mathbf{Z}_n) = H_*(S^j, \mathbf{Z}_n)$, and diffeomorphisms $h_i : S^{j+d(G)} \rightarrow S^{j+d(G)} \times i$ ($i = 0, 1$) mapping M_i diffeomorphically onto $W \cap (S^{j+d(G)} \times i)$ and restricting to a BG -bundle

isomorphism on the normal bundles of these submanifolds. $\Omega_i^{G, \mathbf{Z}_n}$ has an abelian group structure coming from the connected sum operation, with the zero element represented by any $M \subset S^{i+d(G)}$ which extends to $W \subset D^{i+d(G)+1}$ satisfying $\bar{H}_*(W, \mathbf{Z}_n) = 0$, so that the normal bundle BG -structure for M extends to a normal bundle BG -structure for $W \subset D^{i+d(G)+1}$.

There is the forgetting homomorphism

$$w : \bar{C}_m(d_1, \dots, d_l) \rightarrow \Omega_{m-d(G)}^{G, \mathbf{Z}_n}$$

defined on each representative action $r : \mathbf{Z}_n \times S^m \rightarrow S^m$ to be the element in $\Omega_{m-d(G)}^{G, \mathbf{Z}_n}$ represented by the fixed point set, $F_r \subset S^m$, of r having the BG -bundle structure on its normal bundle prescribed by (2), (3) in the introductory remarks.

Let $\bar{C}_m(d_1, d_2, \dots, d_l)$ be the group of all semi-free actions $r : \mathbf{Z}_n \times S^m \rightarrow S^m$ having the standard embedding $S^{m-d(G)} \subset S^m$ for fixed point set, so that the normal bundle splitting $\tau \sim \bigoplus_{i=1}^l \tau_i$, as prescribed in (2), (3) above, extends to a specified normal bundle splitting $\tau' \sim \bigoplus_{i=1}^l \tau'_i$ of $D^{m-d(G)+1} \subset D^{m+1}$. An action $r : \mathbf{Z}_n \times S^m \rightarrow S^m$ represents the zero element in $\bar{C}_m(d_1, d_2, \dots, d_l)$ iff it extends to a semi-free action $r' : \mathbf{Z}_n \times D^{m+1} \rightarrow D^{m+1}$ having a disc $D^{m-d(G)+1} \subset D^{m+1}$ for fixed point set and acting on each summand τ'_i of τ' by multiplication by the associated primitive n^{th} root of unity. Let

$$\bar{C}_m(d_1, \dots, d_l) \xrightarrow{\sim} C_m(d_1, \dots, d_l)$$

denote the canonical map.

Lemma 1.1. *The sequence $\bar{C}_m(d_1, \dots, d_l) \xrightarrow{\sim} C_m(d_1, \dots, d_l) \xrightarrow{w} \Omega_{m-d(G)}^{G, \mathbf{Z}_n}$ is exact at $C_m(d_1, \dots, d_l)$, whenever $2(m - d(G)) + 2 < m$ is satisfied.*

Proof. It is clear that $w \circ v = 0$. That $\ker(w) \subset \text{coker}(v)$ is an easy application of the relative version of Theorem 2.1 in [4]. Q.E.D.

$\bar{C}_m(d_1, \dots, d_l)$ has been studied by W. Browder and T. Petrie [2]. They have made the following calculations. Let $L_*^h(\mathbf{Z}_n)$ be the Wall surgery obstruction group for the oriented fundamental group $\mathbf{Z}_n$. $\bar{L}_*^h(\mathbf{Z}_n)$ denotes the kernel of the mapping $L_*^h(\mathbf{Z}_n) \rightarrow L_*(0)$ induced by the augmentation map $\mathbf{Z}(\mathbf{Z}_n) \rightarrow \mathbf{Z}$.

Theorem 1.2. (Browder, Petrie)¹. *If $m \geq 5$ and $d(G) \geq 3$, then there is an isomorphism (modulo the class of finite n -torsion groups)*

$$\bar{L}_{m+1}^h(\mathbf{Z}_n) \cong C_m(d_1, \dots, d_l).$$

In the following sections we will get an equally good hold on the group $\Omega_*^{G, \mathbf{Z}_n}$.

Section 2. Here is discussed a theory for surgery on normal maps to obtain $\mathbf{Z}_n$ -homology equivalences. The real case is easier than the complex, so only the latter is discussed.

A *complex framing, mod n* , of a complex bundle is a framing for $q \cdot \xi$ where q is some positive integer prime to n . Two such framings $q_i \cdot \xi \rightarrow \mathbf{C}^{k \cdot q_i}$ ($i = 1, 2$) are equivalent, mod n , if q_3 times the first is equivalent to q_4 times the second, for some positive integers q_3, q_4 both prime to n .

A *complex surgery problem*, *mod* n , consists of a map $f : (M, \partial M) \rightarrow (N, \partial N)$ from the almost complex manifold pair $(M, \partial M)$ to the finite, $\mathbf{Z}_{(n)}(\pi(N, \partial N))$ -homological Poincaré duality pair $(N, \partial N)$, satisfying

- (a) f has unit degree, *mod* n ;
- (b) $f|_{\partial M}$ induces an isomorphism on $\mathbf{Z}_{(n)}(\pi_1(\partial N))$ -homology groups;
- (c) $\text{Tang}(M) \oplus f^*(\tau_N)$ has a specified equivalence class of complex framings, *mod* n , where τ_N is a given complex bundle over N .

Now the problem to be discussed is how to do surgery on $f : (M, \partial M) \rightarrow (N, \partial N)$ *carrying the framing of (b) along*; and what are the obstructions to making f a $\mathbf{Z}_{(n)}(\pi_1(X))$ -homology equivalence by doing these surgeries away from $f|_{\partial M}$.

Surgery below the middle dimensions. For each homotopy element $g : S^k \rightarrow M$ contained in $\ker(f_* : \pi_*(M) \rightarrow \pi_*(N))$, there is an integer q , prime to n , so that $(q \cdot g)^*(\text{Tang}(M))$ has a complex framing consistent with the *mod*- n complex framing of $(q \cdot g)^*(\text{Tang}(M) \oplus f^*(\tau_N))$. If $g : S^k \rightarrow M$ represents an element of $[G, G]$, where $G \equiv \ker(f_1 : \pi_1(M) \rightarrow \pi_1(N))$, then $q = 1$ will do.

Hence surgery may be done away from ∂M to make $\ker(f_j : \pi_j(M) \rightarrow \pi_j(N))$ a finite *abelian* group having order prime to n , for all j below the middle dimensions, and to make $\text{coker}(f_1 : \pi_1(M) \rightarrow \pi_1(N)) = 0$.

Surgery at dimension k when $\dim(N) = 2k \geq 6$. Consider the diagram

$$\begin{array}{ccc} \hat{M} & & \hat{f} \\ \downarrow p & \searrow & \\ \tilde{M} & \xrightarrow{\tilde{f}} & \tilde{N} \end{array}$$

where $\hat{}$, $\tilde{}$ denote the $\pi_1(N)$ and $\pi_1(M)$ -universal covers respectively. By the Serre–Hurewicz Theorem $\hat{f}$ induces an isomorphism of $\mathbf{Z}_{(n)}$ -homology groups through dimension $k - 1$, and its k^{th} $\mathbf{Z}_{(n)}$ -homology kernel is generated by homotopy classes. Since p is a q -fold connected covering, for q prime to n , p induces an epimorphism of $\mathbf{Z}_{(n)}$ -homology groups. So $\tilde{f}$ induces an isomorphism of $\mathbf{Z}_{(n)}$ -homology groups through dimension $k - 1$, and its k^{th} $\mathbf{Z}_{(n)}$ -homology kernel is a free $\mathbf{Z}_{(n)}(\pi_1(N))$ -module on generators $\{g_i : S^k \rightarrow M\}$ which are null in N . Use positive multiples $\{q_i \cdot g_i\}$, with q_i prime to n and $(q_i \cdot g_i)^*(\text{Tang}(M))$ the trivial complex bundle, to compute from intersection and self-intersection numbers a surgery obstruction $\sigma(f) \in L_k^A(\mathbf{Z}_{(n)}(\pi_1(N)))$ (see Chapter 5 of [12]).

We must show that $\sigma(f)$ is a cobordant invariant, and that if $\sigma(f) = 0$, then surgery can be completed.

Forget the complex structures on $\text{Tang}(M)$, τ_N , and on the *mod*- n framing for $\text{Tang}(M) \oplus \tau_N$, thereby obtaining a *real surgery problem mod* n . One and two surgeries can now be done on this surgery problem² to assure that

$$f_1 : \pi_1(M) \rightarrow \pi_1(N)$$

is an isomorphism, and $\{g_i : S^k \rightarrow M\}$ having the same intersection and self-intersection numbers still generate a basis for the $\mathbf{Z}_{(n)}(\pi_1(N))$ -kernel. Now *real* cobordism invariance for $\sigma(f)$ is proven just as in Chapter 5 of [12].

Now suppose $\sigma(f) = 0$. Let $h_i : S^k \times D^k \rightarrow M$ be framed immersions of a $\mathbf{Z}_{(n)}(\pi_1(N))$ -basis for a sub kernel K (see [12]) of $\sigma(f)$. Then $\lambda(h_i, h_i) = \sum a_g^{ii} g$, $\mu(h_i) = \sum b_g^i \cdot g$ with $\sum a_g^{ii} = 0 = \sum b_g^i$, where λ, μ are the intersection and selfintersection forms over the group ring $\mathbf{Z}(\pi_1(M))$, the summations run over all $g \in G$ with $G = \ker(f_1 : \pi_1(M) \rightarrow \pi_1(N))$,³ and the a_g^{ii}, b_g^i are integral coefficients. Let $\alpha \equiv \sum g$ denote the norm of G . Then $\lambda(\alpha \cdot h_i, \alpha h_i) = \alpha \lambda(h_i, h_i) \alpha = \alpha (\sum a_g^{ii} g) \alpha = \alpha^2 \sum a_g^{ii} = 0$ (see 5.2 (i), (ii) in [12]), and $\mu(\alpha h_i) = \alpha \mu(h_i) \alpha = \alpha (\sum b_g^i g) \alpha = \alpha^2 (\sum b_g^i) = 0$ (see 5.2 (v) in [12]). So surgery can be performed on the $\{\alpha h_i : S^k \times D^k \rightarrow M\}$. Moreover, because $|G|$ is prime to n , the $\{\alpha \cdot h_i : S^k \times D^k \rightarrow M\}$ represent a $\mathbf{Z}_{(n)}(\pi_1(N))$ basis for the given subkernel K , so surgery has now been completed.

Surgery at dimension k when $\dim(N) = 2k + 1 \geq 5$. This goes through as in Chapter 6 of [12] without any further complications.

Section 3.

Definition 3.1. Let n^* denote the set of primes, prime to the integer n . $BG_{n^* \infty}$ is the classifying space uniquely determined by the following properties:

(i) Each mapping $f : X \rightarrow BG_{n^* \infty}$ corresponds to a unique BG -bundle $\tau \rightarrow X$ over X , together with a mapping of $BG(k)$ -bundles ($G(k)$ is $U(kd_1) \times U(kd_2) \times \cdots \times U(kd_i)$ or $SO(kd)$ depending on whether G equals $U(d_1) \times \cdots \times U(d_i)$ or $SO(d)$) $k \cdot \tau \xrightarrow{h} \xi_k$, where ξ_k is the $BG(k)$ bundle over a point, k is prime to n , and $k \cdot \tau$ is the k -fold Whitney sum of τ with itself.

(ii) Two mappings $f_i : X \rightarrow BG_{n^* \infty}$, $j = 1, 2$, are homotopic iff there is a commuting diagram of $BG(k' \cdot k_1 \cdot k_2)$ -bundle mappings

$$\begin{array}{ccc}
 & k' \cdot k_1 \cdot k_2 \cdot \tau_1 & \xrightarrow{k' \cdot k_2 \cdot h_1} \\
 k' \cdot k_1 \cdot k_2 \cdot g & \downarrow & \searrow \\
 & k' \cdot k_1 \cdot k_2 \cdot \tau_2 & \xrightarrow{k' \cdot k_1 \cdot h_2} \xi_{k' \cdot k_1 \cdot k_2}
 \end{array}$$

where $k_i \cdot \tau_i \xrightarrow{h_i} \xi_{k_i}$ comes from $f_i : X \rightarrow BG_{n^* \infty}$, $\tau_1 \xrightarrow{g} \tau_2$ is a mapping of BG -bundles, and k' is prime to n .

The classifying space $BG_{n^* \infty}$ can be constructed by the usual semi-simplicial techniques. I shall indicate a geometrical construction presently.

Let $\mathbf{Z}_{n^* \infty}$ denote the quotient $\mathbf{Z}_{(n)}/\mathbf{Z}$, where $\mathbf{Z}_{(n)}$ denotes the integers localized at the ideal $n \cdot \mathbf{Z}$.

Lemma 3.2. If $G = SO(d)$ and $d > j + 1$, or if $G = U(d_1) \times \cdots \times U(d_i)$ and $2d_i > j + 1 \forall d_i \neq 0$, then there is an isomorphism

$$\pi_i(BG_{n^* \infty}) \cong (\pi_{i+1}(BG) \oplus_{\mathbf{Z}} \mathbf{Z}_{n^* \infty}) \oplus (\pi_i(BG))_{n^*},$$

where $(\pi_i(BG))_{n*}$ is the subgroup of $\pi_i(BG)$ of elements having finite order prime to n .

Proof. The forgetful map $BG_{n*\infty} \rightarrow BG$ maps $\pi_i(BG_{n*\infty})$ onto $\pi_i(BG)$, accounting for the second factor of $\pi_i(BG_{n*\infty})$.

A typical element in the first factor is accounted for as follows: Choose $x \in \pi_{i+1}(BG)$, and $1/l \in \mathbf{Z}_{n*\infty}$, and let ξ be the trivial BG bundle over S^{i+1} . $l \cdot \xi$ has one $BG(l)$ framing gotten by summing a BG framing for ξ , and a second which differs from this by $x \in \pi_{i+1}(BG)$ ($\pi_{i+1}(BG) \cong \pi_{i+1}(BG(l))$ under the hypothesis of 3.2).

The direct sum splitting for $\pi_i(BG_{n*\infty})$ occurs because $\pi_{i+1}(BG) \otimes_{\mathbf{Z}} \mathbf{Z}_{n*\infty}$ is a divisible group. Q.E.D.

Let $\tau_{G_n*\infty}$ be the universal $BG_{n*\infty}$ -bundle. I want to investigate the relationship between the homotopy groups of the Thom space $T(\tau_{G_n*\infty})$, and the cobordism groups $\Omega_*^{G_n*\infty}$. When dimensions are correct (see Lemmas 3.3, 3.4 below), the usual transversality homomorphism

$$t : \pi'_i(T(\tau_{G_n*\infty})) \rightarrow \Omega_{i-d(G)}^{G_n*\infty} / A_{i-d(G)}$$

is defined as follows. For $x \in \pi_i(T(\tau_{G_n*\infty}))$ put x in transverse position to $BG_{n*\infty} \subset T(\tau_{G_n*\infty})$ and try to complete surgery modulo n -torsion on the resulting $BG_{n*\infty}$ manifold. If the surgery obstruction vanishes, then complete surgery and let $t(x)$ be the resulting manifold. $\pi'_i(T(\tau_{G_n*\infty}))$ denotes the maximal subgroup of $\pi_i(T(\tau_{G_n*\infty}))$ on which t is defined. $A_{i-d(G)}$ denotes the minimal subgroup which, when quotiented out of $\Omega_{i-d(G)}^{G_n*\infty}$, makes t well defined.

Set $B_{i-d(G)} \equiv \text{Coker}(\pi'_i(T(\tau_{G_n*\infty})) \rightarrow \pi_i(T(\tau_{G_n*\infty})))$. Then for $n = \text{even}$

$$B_k = \begin{cases} 0 & \text{if } k \not\equiv 2 \pmod{4} \\ 0 \text{ or } \mathbf{Z}_2 & \text{if } k \equiv 2 \pmod{4}, \end{cases} \quad A_k = \begin{cases} 0 \text{ or } \mathbf{Z}_2 & \text{if } k = 1 \pmod{4}: \text{ finite} \\ \text{cyclic plus a countably infi-} \\ \text{nite group of exponent equal} \\ \text{four, if } k = 3 \pmod{4}: \\ 0 & \text{otherwise} \end{cases}$$

and for $n = \text{odd}$

$$B_k = \{0 \text{ always}, \quad A_k = \begin{cases} \text{finite cyclic, plus a countably} \\ \text{infinite group of exponent equal} \\ \text{four, if } k = 3 \pmod{4}: \\ 0 & \text{otherwise.} \end{cases}$$

These calculations follow from:

- (i) the group of homotopy spheres in any dimension ≥ 5 , which bound framed manifolds, is a finite cyclic group [5];
- (ii) for $n = \text{odd}$ $L_i(\mathbf{Z}_{(n)}(\{1\})) =$

$$\begin{cases} 0 & \text{if } j \not\equiv 0 \pmod{4} \\ \mathbf{Z} \oplus A & \text{if } j \equiv 0 \pmod{4}; \end{cases}$$

- (iii) for $n = \text{even}$, $L_{4i+2}(\mathbf{Z}_{(n)}(\{1\})) = \mathbf{Z}_2$, but for other dimensions $L_i(\mathbf{Z}_{(n)}(\{1\}))$ is as in (ii) above;
- (iv) Any surgery obstruction $\alpha \in L_{4i}(\mathbf{Z}_n\{1\})$ represented geometrically by a normal map with *closed* manifold for domain, can be represented algebraically by a *unimodular* quadratic form over the integers.⁴

Now we have these lemmas concerning the kernel and cokernel of the transversality map t . For a given j , $J_j : \pi_{j-d(G)}(G) \rightarrow \pi_j(T(\tau_{G_n^*\omega}))$ denotes the J -homomorphism defined in a fashion analogous to the stable J -homomorphism

$$J : \pi_r(SO) \rightarrow \pi_r(S^0, S^0) \quad (\equiv \lim_{n \rightarrow \infty} \pi_{n+r}(S^n)).$$

Lemma 3.3. *Let $G = \prod_{i=1}^l U(d_i)$. Suppose $j - d(G) \geq 5$, and $j - d(G) < 2d_i$ holds for all $d_i \neq 0$. Then there is an exact sequence*

$$\pi_{j-d(G)}(BG)_{n^*\omega} \xrightarrow{\epsilon} \pi'_j(T(\tau_{G_n^*\omega}))/\text{image}(J_j) \xrightarrow{t} \Omega_{j-d(G)}^{G, \mathbf{Z}_n}/A_{j-d(G)} \xrightarrow{\text{ch}} \pi_{j-d(G)}(BG)$$

satisfying, cokernel (ch) is a finite group if $j - d(G) \neq 0$ (4), and is a f.g. group with $\mathbf{Z}$ -rank equal one otherwise.

Lemma 3.4. *Let $G = SO(d)$. Suppose that $n = \text{odd}$, $j - d(G) \geq 5$, and $j - d(G) < d(G)$ all hold. Then there exists an exact sequence*

$$\pi_{j-d(G)+1}(G) \otimes \mathbf{Z}_{n^*\omega} \xrightarrow{\epsilon} \pi'_j(T(\tau_{G_n^*\omega}))/\text{image}(J_j) \xrightarrow{t} \Omega_{j-d(G)}^{G, \mathbf{Z}_n}/A_{j-d(G)} \rightarrow 0.^5$$

If $n = \text{even}$ then " $\rightarrow 0$ " may have to be changed to " $\rightarrow \mathbf{Z}_2 \rightarrow 0$ " to make the sequence exact.

Proof of 3.3. ϵ is defined as follows. Each $x \in \pi_{j-d(G)}(BG_{n^*\omega})$ corresponds to a $BG_{n^*\omega}$ -bundle, τ , over $S^{j-d(G)}$ the underlying BG -bundle which has finite order in $\pi_{j-d(G)}(BG)$. $j - d(G) < 2d_i$ for all $d_i \neq 0$ implies that $\pi_{j-d(G)}(BG)$ is torsion free. Hence τ has the trivial BG -bundle structure. So take τ to be a $BG_{n^*\omega}$ -normal bundle to the standard embedding $S^{j-d(G)} \subset S^j$, with the underlying BG -bundle extending to a BG -normal bundle for the embedding $D^{j-d(G)+1} \subset D^{j+1}$. Let $g : S^j \rightarrow T(\tau_{G_n^*\omega})$ be the map coming from the classifying map for this $BG_{n^*\omega}$ -normal bundle to $S^{j-d(G)} \subset S^j$, via the Thom construction. The homotopy class of g determines an element $\bar{x} \in \pi_j(T(\tau_{G_n^*\omega}))/\text{image}(J_j)$. Set $\epsilon(x) \equiv \bar{x}$.

$$\text{ch} : \Omega_{j-d(G)}^{G, \mathbf{Z}_n}/A_{j-d(G)} \rightarrow \pi_{j-d(G)}(BG)$$

is defined as follows: If $j - d(G) = \text{odd}$, then the hypothesis that $j - d(G) < 2d_i$ or $d_i = 0 \forall i = 1, 2, \dots, l$, implies that $\pi_{j-d(G)}(BG) = 0$, so $\text{ch} = 0$. If $j - d(G) = \text{even}$, then $\pi_{j-d(G)}(BG) = (\mathbf{Z})^{l'}$, where l' equals the number of $d_i \neq 0$. In this case $\text{ch}(x) = \prod_{i=1}^{l'} \text{ch}_i(x) \in (\mathbf{Z})^{l'}$, where $\text{ch}_i(x)$ denotes the *top* dimensional Chern class for the factor of the normal bundle to $M \subset S^j$ corresponding to the factor $U(d_i)$ of G , capped with the top dimensional homology class of M , with M any BG -submanifold of S^j representing $x \in \Omega_{j-d(G)}^{G, \mathbf{Z}_n}$.

We show the exactness at $\Omega_{j-d(G)}^{G, \mathbf{Z}_n}/A_{j-d(G)}$, and prove the statement concerning coker (ch), the rest is checked in much the same way. $\text{ch} \circ t = 0$ because any $BG_{n^*\omega}$ -bundle over a space M has finite order in $[M, \lim_{q \rightarrow \infty} BG(q)]$.

Now suppose $\text{ch}(x) = 0$. Let $M \subset S^j$ represent $x \in \Omega_{j-d(G)}^{G, \mathbb{Z}_n}$. Using $\text{ch}(x) = 0$ and $H_*(M, \mathbb{Z}_n) = H_*(S^{j-d(G)}, \mathbb{Z}_n)$, we can argue from an Atiyah–Hirzebruch spectral sequence that the BG -normal bundle to $M \subset S^j$, τ , becomes trivial when Whitney summed with itself k times, for some positive integer k prime to n . Hence τ has a $BG_{n^* \infty}$ -bundle structure compatible with its BG -structure. Now using the Thom construction we can pull x back to an element $\pi_i(T(\tau_{G_{n^* \infty}}))$. Chern classes detect all the homotopy classes $\pi_*(\lim_{q \rightarrow \infty} U(q))$, and that forgetful induced map $\pi_i(\lim_q U(q)) \rightarrow \pi_i(\lim_q SO(q))$ is an isomorphism for $i = O(4)$ and zero otherwise (modulo the class of finite groups). Furthermore, the total top Chern class of $x \in \Omega_{j-d(G)}^{G, \mathbb{Z}_n}$, $\sum_{i=1}^l \text{ch}_i(x)$, coincides with twice the top dimensional Pontrjagin class of M when $\dim(M) = O(4)$, the top Pontrjagin class of M determines the index of M which vanishes by virtue of $H_*(M, \mathbb{Z}_n) = H_*(S^{j-d(G)}, \mathbb{Z}_n)$; thus $\sum_{i=1}^l \text{ch}_i(x) = 0$ when $j - d(G) = O(4)$. These facts, and the restrictions $j - d(G) < 2d_i$ for $d_i \neq 0$, imply that the $\mathbb{Z}$ -rank of image (ch) is l' , $l' - 1$ for $j - d(G) = 2(4)$, $j - d(G) = O(4)$ respectively. Q.E.D.

Proof of 3.4. The same reasoning as in 3.3 leads to an exact sequence

$$\pi_{i-d(G)+1}(G) \otimes \mathbb{Z}_{n^* \infty} \xrightarrow{\epsilon} \pi'_i(T(\tau_{G_{n^* \infty}}))/\text{image}(J_i) \xrightarrow{t} \Omega_{i-d(G)}^{G, \mathbb{Z}_n} / A_{m-d(G)} \xrightarrow{\text{ph}} \pi_{i-d(G)}(BG),$$

where $\text{ph}(x)$ is the top Pontrjagin class of a cobordism class $x \in \Omega_{i-d(G)}^{G, \mathbb{Z}_n}$. The Hirzebruch index theorem shows that $\text{ph}(x) = 0$.

The reason we include the modulo-2-torsion hedge in 3.4 for $n = \text{even}$, is this: does $\text{Tang}(M)$ have finite order prime to n in $[M, BO]$ if $H_*(M, \mathbb{Z}_n) = H_*(S^n, \mathbb{Z}_n)$? It does except for a 2-torsion obstruction in $\pi_m(BO)$. Q.E.D.

In order to complete our investigation $\Omega_{j-d(G)}^{G, \mathbb{Z}_n}$, we must calculate $\pi_i(T(\tau_{G_{n^* \infty}}))$ and kernel (ϵ) . It is convenient at this point to introduce a geometric construction of the classifying space $BG_{n^* \infty}$. For positive integers q, k , with $d(G) < q$, let $BG_{k,q}$ be the space consisting of pairs (P, f) , where P is an F -plane through the origin of F^q having real dimension equal to $d(G)$, and $f : (F^q)^k \rightarrow (F^{d(G) \cdot a})^k$ is a linear map of F -vector spaces which restricts to a linear isomorphism on $(P)^k \subset (F^q)^k$; here F is the field of complex numbers or real numbers and $a = 1/2$ or $a = 1$, depending on whether $G = U(d)$ or $G = SO(d)$. If $q' > q$ let $BG_{k,q} \rightarrow BG_{k,q'}$ be the map induced by $F^q \subset F^{q'}$. Set $BG_k = \lim_{q \rightarrow \infty} BG_{k,q}$. If the positive integer k' factorizes as $k' = k \cdot p$, then there is a mapping $g_{k',k} : BG_k \rightarrow BG_{k'}$ corresponding to the construction which sends a linear homomorphism $f : (F^q)^k \rightarrow (F^{d(G) \cdot a})^k$ to the p -fold cartesian product map $(f)^p : (F^q)^{k'} \rightarrow (F^{d(G) \cdot a})^{k'}$. The spaces BG_k with k prime to n form a directed system with respect to the mappings $g_{k',k} : BG_k \rightarrow BG_{k'}$. Let $BG_{n^* \infty}$ be the direct limit of this system. It is not difficult to see that BG_k is the classifying space for BG -bundles whose k -fold Whitney sums with themselves have a given $BG(k)$ -bundle trivialization. Hence $BG_{n^* \infty}$ satisfies 3.1 (i), (ii). This completes the construction of $BG_{n^* \infty}$ for $G = U(d)$ or $G = SO(d)$. If $G = \prod_{i=1}^l U(d_i)$, then $BG_{n^* \infty} = \prod_{i=1}^l BU(d_i)_{n^* \infty}$.

Lemma 3.5. *For every integer m prime to n , the mapping*

$$\beta_* : H_*(BG_{n^*\infty}, \mathbb{Z}_m) \rightarrow H_*(BG, \mathbb{Z}_m)$$

is an isomorphism. Here $\beta : BG_{n^*\infty} \rightarrow BG$ is the map classifying the BG -structure of the universal $BG_{n^*\infty}$ -bundle.

Proof. Let $f_k : BG_k \rightarrow BG$ denote the composition $BG_k \rightarrow BG_{n^*\infty} \xrightarrow{\beta} BG$ of the natural inclusion with β , for each positive integer k prime to n ; i.e., f_k is the forgetful map which classifies the BG -structure of the universal BG_k -bundle. Note that each $f_k : BG_k \rightarrow BG$ is a fibration having a fiber which contains $(\lim_{q \rightarrow \infty} G(q))$ as a deformation retract. The mappings $g_{k',k} : BG_k \rightarrow BG_{k'}$ preserve the fibers of the fibrations $f_k : BG_k \rightarrow BG$, $f_{k'} : BG_{k'} \rightarrow BG$. Moreover $g_{k',k}$ restricted to a single fiber of these fibrations is homotopy equivalent to $\bar{p} : \lim_{q \rightarrow \infty} G(q) \rightarrow \lim_{q \rightarrow \infty} G(q)$, where $k \cdot p = k'$ and $\bar{p}$ denotes multiplication by p in the H -space structure of $\lim_{q \rightarrow \infty} G(q)$; i.e., $\bar{p}$ factors as the composition

$$\lim_{q \rightarrow \infty} G(q) \xrightarrow{\Delta} (\lim_{q \rightarrow \infty} G(q))^p \xrightarrow{\mu} \lim_{q \rightarrow \infty} G(q)$$

where Δ is the diagonal map and $\mu(x_1, x_2, \dots, x_p) = x_1 \cdot x_2 \cdot x_3 \cdots x_p$ for $x_i \in G(q)$. It follows that $\beta : BG_{n^*\infty} \rightarrow BG$ can be replaced by a fibration having for fiber the direct limit of the directed system of sets $(\lim_{q \rightarrow \infty} G(q))$ with respect to the maps $\bar{p} : (\lim_{q \rightarrow \infty} G(q)) \rightarrow (\lim_{q \rightarrow \infty} G(q))$ for all positive integers p prime to n . Let X denote this direct limit.

I claim that $H_*(X, \mathbb{Z}_p) = 0$ for any positive prime integer p prime to n . It suffices to show that for any $j > 0$ and $y \in H_j(\lim_{q \rightarrow \infty} G(q), \mathbb{Z}_p)$ is in the kernel of some $(\bar{p}')_* : H_j(\lim_{q \rightarrow \infty} G(q), \mathbb{Z}_p) \rightarrow H_j(\lim_{q \rightarrow \infty} G(q), \mathbb{Z}_p)$ for p' prime to n . Over all expressions of y as $y = \sum_{i \in I} x_{i,1} * x_{i,2} * \cdots * x_{i,s_i}$, where “ $*$ ” denotes the multiplication in the Pontrjagin ring structure of $H_*(\lim_{q \rightarrow \infty} G(q), \mathbb{Z}_p)$, let $d(y)$ be the minimum of $\{\max \text{ degree } (x_{i,j}) : i \in I, j = 1, \dots, s_i\}$. Clearly if $d(y) = 0$ and $\text{degree } (y) > 0$, then $y = 0$. I claim that either $d((\bar{p})_*(y)) < d(y)$ or $d(y) = 0$ holds. To see this, reason as follows: Let $y = \sum_{i \in I} x_{i,1} * x_{i,2} * \cdots * x_{i,s_i}$ be an expression for y determining $d(y)$. The mapping $\bar{p} : \lim_{q \rightarrow \infty} G(q) \rightarrow \lim_{q \rightarrow \infty} G(q)$ preserves the H -space structure of $\lim_{q \rightarrow \infty} G(q)$ so

$$(\bar{p})_* \left(\sum_{i \in I} x_{i,1} * \cdots * x_{i,s_i} \right) = \sum_{i \in I} ((\bar{p})_*(x_{i,1})) * \cdots * ((\bar{p})_*(x_{i,s_i})).$$

We must show that either $\text{degree } (x_{i,j}) = 0$ or $\text{degree } (x_{i,j}) > d((\bar{p})_*(x_{i,i}))$ holds for each $x_{i,j}$. $\bar{p}_*$ factors as

$$\begin{array}{ccc} H_*(\lim_{q \rightarrow \infty} G(q), \mathbb{Z}_p) & \xrightarrow{\bar{p}_*} & H_*(\lim_{q \rightarrow \infty} G(q), \mathbb{Z}_p) \\ & \searrow \Delta_* & \nearrow \mu_* \\ & H_*(\lim_{q \rightarrow \infty} G(q)^p, \mathbb{Z}_p) & \end{array}$$

If $\text{degree } (x_{i,i}) > 0$, then

$$\Delta_*(x_{i,i}) = \sum_{r=1}^p b_0 \times \cdots \times b_0 \times x_{i,i} \times b_0 \times \cdots \times b_0 + x'_{i,i},$$

where the $x_{i,i}$ appears in the r^{th} factor, b_0 is a generator of $H_0(\lim_{q \rightarrow \infty} G(q), \mathbf{Z}_p)$, and $x'_{i,i}$ is the sum of elements of the form $a_1 \times a_2 \times \cdots \times a_t$ with degree $(a_s) < \text{degree}(x_{i,i})$ for $s = 1, \dots, t$. But $\mu_*(\sum_{r=1}^p b_0 \times \cdots \times b_0 \times x_{i,i} \times b_0 \times \cdots \times b_0) = p \cdot x_{i,i} = 0$ in $H_*(\lim_{q \rightarrow \infty} G(q), \mathbf{Z}_p)$, so $\bar{p}_*(x_{i,i}) = \sum a_1 * a_2 * \cdots * a_t$ with degree $(a_s) < \text{degree}(x_{i,i})$ as desired.

We have shown that $\beta : BG_{n^*\infty} \rightarrow BG$ is a fibration with fiber X satisfying $\bar{H}^*(X, \mathbf{Z}_p) = 0$ for every prime integer, p , prime to n . The universal coefficient theorem, together with a spectral sequence argument, now shows that $\beta_* : H_*(BG_{n^*\infty}, \mathbf{Z}_m) \rightarrow H_*(BG, \mathbf{Z}_m)$ is an isomorphism for each m prime to n .

This completes the proof of 3.5.

Lemma 3.6. *If $G = SO(d)$ and $d > j + 1$, or if $G = \prod_{i=1}^t U(d_i)$ and $2d_i > j + 1$ for all $d_i \neq 0$, then $\bar{H}_j(BG_{n^*\infty}, \mathbf{Z})$ is a torsion group every element of which has order prime to n .*

Proof. The Serre–Hurewicz Theorem, with 3.2, shows that $\bar{H}_j(\widehat{BG}_{n^*\infty}, \mathbf{Z})$ are torsion groups every element of which has order prime to n . 3.2 and a calculation show that $\bar{H}_*(K(\pi_1(BG_{n^*\infty})), \mathbf{Z})$ is a torsion group every element of which has order prime to n . The reader can convince himself that $\pi_1(BG_{n^*\infty})$ acts trivially on $H_*(\widehat{BG}_{n^*\infty}, \mathbf{Z})$. Now a standard spectral sequence argument completes the proof. Q.E.D.

Lemma 3.7. *Let ϵ be as in 3.3, 3.4. Then $\ker(\epsilon)$ is finite whenever $k \equiv j - d(G)$ is in the following ranges:*

- (1) *If $G = \prod_{i=1}^t U(d_i)$, then $2d_i > k + 2$ for all $d_i \neq 0$.*
- (2) *If $G = SO(d)$, then $d > k + 2$.*

Proof. I will prove 3.7 for $G = U(d)$, the general case being only slightly more complicated.

Since $\pi_i(U(d)) = 0$ for $i = 0 \vee 2 \nmid (4)$ we need only consider the cases when $k \neq 0, 2 \nmid (4)$.

Consider the diagram

$$\begin{array}{ccc}
 \pi_k(BG_{n^*\infty}) & \xrightarrow{h} & H_k(BG_{n^*\infty}, \mathbf{Z}) \\
 \uparrow \alpha & & \downarrow \theta \\
 & & H_{k+1}(BG_{n^*}, \mathbf{Z}_{n^*}) \\
 & & \downarrow \beta_{k+1} \\
 \pi_{k+1}(BG) \otimes \mathbf{Z}_{n^*\infty} & \xrightarrow{h'} & H_{k+1}(BG, \mathbf{Z}_{n^*\infty}),
 \end{array}$$

where α is the isomorphism of 3.2, h is the Hurewicz map, h' is the Hurewicz map tensored with $1_{\mathbf{Z}_{n^*\infty}}$, β_{k+1} is induced by the forgetful map $BG_{n^*\infty} \rightarrow BG$, and θ is the isomorphism coming from the universal coefficient theorem (see 3.6). This diagram commutes. Also, since the Hurewicz map $\pi_*(BG) \rightarrow H_*(BG, \mathbf{Z})$ has zero kernel in dimensions $< d(G)$, h' has finite kernel. Hence h has finite kernel.

Now 3.7 follows from commutativity in the diagram

$$\begin{array}{ccc} \pi'_{k+d(G)}(T(\tau_{G_n^*\omega})) & \xrightarrow{\bar{h}} & H_{k+d(G)}(T(\tau_{G_n^*\omega}), \mathbf{Z}) \\ \uparrow \epsilon & & \uparrow \psi \\ \pi_k(BG_{n^*\omega}) & \xrightarrow{h} & H_k(BG_{n^*\omega}, \mathbf{Z}), \end{array}$$

where ψ is the Thom isomorphism.

Q.E.D.

Let $\pi_k(S^0, S^0)$ denote the k^{th} stable stem (recall that these groups are finite). For any abelian group K and any positive integer n , K_n , K_{n^*} denote the subgroups of K whose elements have finite order equal a divisor of some power of n or have order prime to n respectively.

Lemma 3.8. *Let j be a positive integer satisfying*

(1) $j < 2d - 4$ when $G = SO(d)$.

(2) $j < 2d_i + d(G) - 4$ for all $d_i \neq 0$ when $G = \prod_{i=1}^l U(d_i)$.

Then there is an isomorphism

$$\pi_j(T(\tau_{G_n^*\omega})) \cong \pi_{j+1}(T(\tau_G)) \otimes_{\mathbf{Z}} \mathbf{Z}_{n^*\omega} \oplus (\pi_j(T(\tau_G)))_{n^*} \oplus (\pi_{j-d(G)}(S^0, S^0))_n. \quad ^6$$

Proof. Since $d(G)$ denotes the real dimension of the disc fibers of each of the bundles $\tau_{G_n^*\omega}$ and τ_G , $T(\tau_{G_n^*\omega})$ and $T(\tau_G)$ are both $d(G) - 1$ connected. Via the mapping cylinder construction we can think of the forgetful map $\bar{\beta} : T(\tau_{G_n^*\omega}) \rightarrow T(\tau_G)$ as an inclusion $T(\tau_{G_n^*\omega}) \subset T(\tau_G)$. Set $X = T(\tau_G)/T(\tau_{G_n^*\omega})$. Then the suspension $\sum(X)$ is $d(G)$ -connected. The stability condition $j < 2d(G) - 4$ assures that there is an exact sequence

$$\pi_{k+1}(X \wedge M_q) \rightarrow \pi_k(\sum(X)) \xrightarrow{a_*} \pi_k(\sum(X)) \rightarrow \pi_k(X \wedge M_q)$$

for any positive integers k, q with $k \leq j + 2$, where M_q is the Moore space obtained by gluing D^2 to S^1 along a degree q map $\partial D^2 \rightarrow S^1$, and q_* coincides with multiplication by the integer q . $H_*(X \vee M_q, \mathbf{Z}) = 0$ for q prime to n (3.5). We may assume that $d(G) > 4$, for otherwise Lemma 3.8 is trivial. In particular X , and thus $X \wedge M_q$ also, is now 1-connected. So $\pi_*(X \wedge M_q) = 0$ for q prime to n . Now using the above exact sequence we deduce that $q_* : \pi_k(\sum(X)) \rightarrow \pi_k(\sum(X))$ is an isomorphism for all q prime to n and $k \leq j + 2$: this desuspends to the same result for $q_* : \pi_k(X) \rightarrow \pi_k(X)$, $k \leq j + 1$.

There are exact sequences

$$\begin{aligned} 1) \quad & \pi_{j+1}(T(\tau_{G_n^*\omega})) \longrightarrow \pi_{j+1}(T(\tau_G)) \xrightarrow{r_{j+1}} \pi_{j+1}(X) \\ & \xrightarrow{\partial} \pi_j(T(\tau_{G_n^*\omega})) \xrightarrow{\bar{\beta}_j} \pi_j(T(\tau_G)) \xrightarrow{r_j} \pi_j(X), \\ 2) \quad & \pi_{j+1}(T(\tau_{G_n^*\omega})/S^{d(G)}) \longrightarrow \pi_j(S^{d(G)}) \longrightarrow \pi_j(T(G_{n^*\omega})) \longrightarrow \pi_j(T(\tau_{G_n^*\omega})/S^{d(G)}), \end{aligned}$$

where in 2) $S^{d(G)} \subset T(\tau_{G_n^*\omega})$ comes from the natural inclusion of the Thom space of the BG_{n^*} -bundle over a point. From 2) and 3.6, $\pi_j(T(\tau_{G_n^*\omega}))$ splits as

$$\pi_j(T(\tau_{G_n^*\omega}))_{n^*} \oplus \pi_{j-d(G)}(S^0, S^0)_n,$$

thus accounting for the last factor in 3.8. From the previous paragraph $\pi_{i+1}(X)$ is divisible by any q prime to n , so image $(\partial) \wedge \pi_i(T(\tau_{G_n^{\infty}}))_{n^*}$ is a divisible subgroup. Let B denote a complementary summand for it in $\pi_i(T(\tau_{G_n^{\infty}}))_{n^*}$. From the previous paragraph, $\pi_i(X)_{n^*} = 0$. So $\pi_i(T(\tau_G))_{n^{\infty}} \subset \ker(r_i)$; but then $B \cong \pi_i(T(\tau_G))_{n^*}$ accounting for the second factor in 3.8. Finally, the previous paragraph also showed that $\pi_{i+1}(X)_{n^*} = 0$. Hence, if $K \subset \pi_{i+1}(X)$ is the subgroup generated by elements $\{r_{i+1}(y)/q: q \text{ prime to } n, y \in R_{i+1}\}$ where R_{i+1} is a maximal torsion free direct sum of $\pi_{i+1}(T(\tau_G))$ —then $K \cong (\mathbf{Z}_{(n)})^r$ with $r = \text{rank}(R_{i+1})$ and $K \rightarrow \pi_{i+1}(X) \rightarrow \pi_{i+1}(X)/\pi_{i+1}(X)_n$ is an isomorphism. So image $(\partial) \cap \pi_i(T(\tau_{G_n^{\infty}}))_{n^*} \cong (\mathbf{Z}_{n^{\infty}})^r$, accounting for the first factor in 3.8. Q.E.D.

Section 4. In this section we list some amusing results of our previous calculations.

Let $\Omega_i^{U, \mathbf{Z}_n}$ denote the group of equivalence classes of almost complex manifolds M satisfying $H_*(M, \mathbf{Z}_n) \cong H_*(S^j, \mathbf{Z}_n)$: two such manifolds M_1, M_2 are equivalent iff there is an almost complex cobordism, W , from M_1, M_2 satisfying $H_*(W, \mathbf{Z}_n) \cong H_*(S^j, \mathbf{Z}_n)$. Similarly there is the cobordism group $\Omega_*^{U, R}$ for any $\mathbf{Z}$ -module R .

Let E_i denote the group of j -dimensional exotic spheres, and $P_i \subset E_i$ the subgroup of exotic spheres which bound framed manifolds. $A_i \subset \Omega_i^{U, \mathbf{Z}_n}$ is the subgroup generated by $\mathbf{Z}_n$ -homology spheres bounding $BU_{n^{\infty}}$ manifolds.

Theorem 4.0. *If $j \geq 5$, there are these isomorphisms:*

$$\begin{aligned} \Omega_i^{U, \mathbf{Z}_n}/A_i &\cong (\Omega_{i+1}^U \otimes_{\mathbf{Z}} \mathbf{Z}_{n^{\infty}})/(\mathbf{Z}_{n^{\infty}}) \oplus (E_i/P_i)_n & \text{for } j = \text{odd}; \\ \Omega_i^{U, \mathbf{Z}_n} &\cong (E_i)_n & \text{for } j = 0(4); \\ \Omega_i^{U, \mathbf{Z}_n} &\cong (E_i)_n \oplus \mathbf{Z} & \text{for } j = 2(4). \end{aligned}$$

Proof. Apply 2.2, 3.3, 3.7, 3.8.

Q.E.D.

Corollary 4.1. *The geometric inclusion $E_i/P_i \rightarrow \Omega_i^{U, \mathbf{Z}_n}/A_i$ has $(E_i/P_i)_{n^*}$ for kernel.*

Corollary 4.2. *For $j \geq 5$ there are these isomorphisms:*

$$\begin{aligned} \Omega_i^{U, \mathbf{Q}}/A_i &\cong (\Omega_{i+1}^U \otimes_{\mathbf{Z}} \mathbf{Q}/\mathbf{Z})/(\mathbf{Q}/\mathbf{Z}) & \text{for } j = \text{odd}; \\ \Omega_i^{U, \mathbf{Q}} &\cong 0 & \text{for } j = 0(4); \\ \Omega_i^{U, \mathbf{Q}} &\cong \mathbf{Z} & \text{for } j = 2(4). \end{aligned}$$

Similar results hold for $G = SO, STOP, SPL$. In particular, for $n = \text{odd}$ and $j \geq 5$, there are these pleasing formulae:

$$\begin{aligned} \Omega_i^{SPL, \mathbf{Z}_n}/A_i &\cong (\Omega_i^{SPL})_{n^*} \oplus (\Omega_{i+1}^{SPL} \otimes_{\mathbf{Z}} \mathbf{Z}_{n^{\infty}})/C_i \otimes_{\mathbf{Z}} \mathbf{Z}_{n^{\infty}}, \\ \Omega_i^{SPL, \mathbf{Q}}/C_i &\cong (\Omega_{i+1}^{SPL} \otimes_{\mathbf{Z}} (\mathbf{Q}/\mathbf{Z}))/C_i \otimes_{\mathbf{Z}} (\mathbf{Q}/\mathbf{Z}) \end{aligned}$$

where

$$C_i = \begin{cases} \mathbf{Z} & \text{if } j = 3(4) \\ 0 & \text{otherwise} \end{cases}.$$

Lemma 4.3. *Let K be any 1-connected CW complex satisfying $\bar{H}_*(K, \mathbf{Z}_n) = 0$. If $j \geq 5$, then any element in the oriented bordism group $x \in \Omega_j(K, k_0) (k_0 \in K)$ can be represented by a map $g : M \rightarrow K$ from a closed differentiable manifold M satisfying $H_*(M, \mathbf{Z}_n) \cong H_*(S^j, \mathbf{Z}_n)$.*

Proof. Recall that $\Omega_j(K, k_0) \equiv \lim_{q \rightarrow \infty} \pi_{i+q}(T(\tau_{SO(q)}) \wedge K)$. According to 3.5, and the Thom isomorphism theorem, for each i and l with l prime to n ,

$$\bar{\beta}_{i+q} : H_{i+q}(T(\tau_{SO(q)n^*\infty}), \mathbf{Z}_l) \rightarrow H_{i+q}(T(\tau_{SO(q)}), \mathbf{Z}_l)$$

becomes an isomorphism when q becomes large enough.

Thus $(\bar{\beta} \wedge 1_K)_{i+q} : H_{i+q}(T(\tau_{SO(q)n^*\infty}) \wedge K) \rightarrow H_{i+q}(T(\tau_{SO(q)}) \wedge K)$ becomes an isomorphism for q large enough. Choose q large enough so that $(\bar{\beta} \wedge 1_K)_{i+q}$ is an isomorphism for all $0 \leq i \leq j$, and so that $x \in \Omega_j(K, k_0)$ is represented by $h : S^{i+q} \rightarrow T(\tau_{SO(q)}) \wedge K$. The Whitehead theorem shows that

$$\bar{\beta} \wedge 1_K : T(\tau_{SO(q)n^*\infty}) \wedge K \rightarrow T(\tau_{SO(q)}) \wedge K$$

is a (stable) j -equivalence, so h pulls back to $r : S^{i+q} \rightarrow T(\tau_{SO(q)n^*\infty}) \wedge K$. The usual transversality applies to r to give a mapping $g : M \rightarrow K$ from a closed $BSO(q)n^*\infty$ -manifold M to K , which represents the bordism class $x \in \Omega_j(K, k_0)$. The hypotheses of 4.3 imply that $j = \dim(M) \geq 5$, and that $\pi_*(K)$ is all n^* -torsion. Without loss of generality we can take $q > \dim(M)$. Using the $BSO(q)n^*\infty$ normal bundle structure of M to generate the normal bundle framings for surgery spheres, do surgery on the map $g : M \rightarrow K$ to kill $\bar{H}_*(M, \mathbf{Z}_n)$ below the top dimension. Possibly there is a surgery obstruction. If so, replace $g : M \rightarrow K$ by $g \# c : M \# (-M) \rightarrow K$, where $c : -M \rightarrow K$ collapses $-M$ to $k_0 \in K$. Q.E.D.

Section 5. In Section 1 I described an exact sequence

$$\bar{C}_m(d_1, \dots, d_l) \xrightarrow{\sim} D_m(d_1, \dots, d_m) \xrightarrow{w} \Omega_{m-d(G)}^{G, \mathbf{Z}_n^*} \quad (1.1);$$

the left end is well known from [2]; the greater part of Section 1 was devoted to the investigation of the right end $\Omega_{m-d(G)}^{G, \mathbf{Z}_n^*}$. In this Section I shall investigate kernel (v) , cokernel (w) , and complete the calculations of the groups $C_m(d_1, \dots, d_l)$.

I begin with the statements of two theorems whose proofs shall be put aside until a more convenient time.

Theorem A. *Let $r : \mathbf{Z}_n \times S^m \rightarrow S^m$ be a (differentiable) semi-free $\mathbf{Z}_n$ -action ($n \neq 2$) having fixed point set, F_r . Let τ be the complex normal bundle to F_r in S^m , where the complex structure of τ comes from the given $\mathbf{Z}_n$ -action (see the introductory remarks). Then τ has finite order in $[F_r, BU]$.*

The following is a useful converse to Theorem A. Let $G = \prod_{i=1}^l U(d_i)$, where l is half the number of primitive n^{th} roots of unity.

Theorem B. Suppose $d(G) \geq 6$. Let $x \in \pi_i(BG)$ correspond to a BG -bundle over S^i the underlying BU -bundle of which is stably trivial. Then there exists a semi-free $\mathbf{Z}_n$ -action $r : \mathbf{Z}_n \times S^m \rightarrow S^m$ ($m = j + d(G)$) having the j -sphere for fixed point set, and determining a BG -bundle structure on the normal bundle to S^i in S^m (as in the introduction) which corresponds to $k \cdot x \in \pi_i(BG)$ for some integer $k > 0$.

There is the following lemma concerning kernel (v) .

Lemma 5.1. Suppose that $m - d(G) \geq 5$ and one of these conditions holds:

(1) $G = SO(d)$, and $m < 2d(G) - 5$,

or

(2) $G = \prod_{i=1}^l U(d_i)$, and $m < d(G) + 2d_i - 5$ for all $d_i \neq 0$.

Then kernel $(v : \bar{C}_m(d_1, \dots, d_l) \rightarrow C_m(d_1, \dots, d_l))$ is a finite group if $m - d(G) \neq 1$ (4) or if $G = SO(d)$, and is the direct sum of $\mathbf{Z}$ with a finite group otherwise.

Proof. Choose a representative action $r : \mathbf{Z}_n \times S^m \rightarrow S^m$ for $x \in \text{kernel } (v)$. $x \in \text{kernel } (v)$ means that $r : \mathbf{Z}_n \times S^m \rightarrow S^m$ extends to a semi-free action $r' : \mathbf{Z}_n \times D^{m+1} \rightarrow D^{m+1}$. Since the fixed point set for r' , $F_{r'}$, satisfies $\bar{H}_*(F_{r'}, \mathbf{Z}_n) = 0$, after replacing x by some positive multiple $k \cdot x$, if need be, there will be a BG -cobordism $(W, V) \subset (D^{m+1}, \partial D^{m+1}) \times [0, 1]$ from $(F_{r'}, F_r) \subset (D^{m+1}, \partial D^{m+1}) \times 0$, so that V is a product cobordism $\bar{H}_*(W, \mathbf{Z}_n) = 0$, and $\partial_+ W$ is a disc (see §3). Apply a relative version of Theorem 2.1 in [4] to extend r' to $r'' : \mathbf{Z}_n \times (D^{m+1}, \partial D^{m+1}) \times [0, 1] \rightarrow (D^{m+1}, \partial D^{m+1}) \times [0, 1]$ having W for fixed point set and acting as multiplication by the various primitive n^{th} roots of unity on the various complex bundle factors of the BG normal bundle. Let $\oplus_i \tau_i$ denote the complex splitting associated to the normal bundle to the fixed point set of $r : \mathbf{Z}_n \times S^m \rightarrow S^m$, and let $\oplus_i \tau'_i$ be the specified extension of this complex splitting to the normal bundle of $D^{m+1-d(G)} \subset D^{m+1}$. The extension of r to $r''|_{\mathbf{Z}_n \times (\partial D^{m+1} \times [0, 1] \cup D^{m+1} \times 1)}$ provides another extension, $\oplus_i \tau''_i$, of the complex splitting $\oplus_i \tau_i$ to the normal bundle of $D^{m+1-d(G)} \subset D^{m+1}$. If $\oplus_i \tau''_i$ and $\oplus_i \tau'_i$ are the same extension, then $r''|_{\mathbf{Z}_n \times (\partial D^{m+1} \times [0, 1] \cup D^{m+1} \times 1)}$ would provide a null cobordism for $k \cdot x$ in $\bar{C}_m(d_1, d_2, \dots, d_l)$. In general, apply theorems A, B above to see that kernel (v) calculates as claimed in 5.1, modulo a torsion group. Finally, the groups $\bar{C}_m(d_1, d_2, \dots, d_l)$ are finitely generated (1.2), so kernel (v) calculates as claimed: here I use an unpublished result of C. T. C. Wall which says that the surgery groups $L_*^h(\pi)$ are finitely generated in every dimension provided π is a finite group. Q.E.D.

There is the following analogous result for cokernel (w) .

Lemma 5.2. Suppose that $m - d(G) \geq 5$, $d(G) \geq 6$, and one of these conditions is satisfied:

(1) $G = SO(d)$, and $m < 2d - 4$.

(2) $G = \prod_{i=1}^l U(d_i)$, and $m < 2d_i + d(G) - 4$ for all $d_i \neq 0$.

Then cokernel $(w : C_m \rightarrow \Omega_{m-d(G)}^{G, \mathbf{Z}_n})$ is a finite group when $G = SO(d)$ or when $2 + m - d(G) \neq 0$ (4), and is a f.g. group having $\mathbf{Z}$ -rank equal one otherwise.

Lemma 5.2 is deduced directly from Lemma 5.1 and the following proposition:

Proposition 5.2. *Under the dimension restrictions of Lemma 5.2, there is a long exact sequence*

$$C_m \xrightarrow{\omega} \Omega_{m-d(G)}^{G, Z_n} \xrightarrow{\partial} \bar{C}_{m-1} \xrightarrow{v} C_{m-1} \xrightarrow{\omega} \dots$$

where ω, v are defined in 1.1 above.

Proof of Proposition 5.2. Here is how ∂ is defined: let $M \subset S^m$ represent $x \in \Omega_{m-d(G)}^{G, Z_n}$, cut an open ball $B \subset S^m$ —which intersects M transversely in an open ball—from (S^m, M) to obtain (D^m, W) ; define $r : Z_n \times D^m \rightarrow D^m$, having $W \subset D^m$ for fixed point set, by letting r act as the various primitive n^{th} roots of unity on the various complex vector bundle factors of the normal bundle to $W \subset D^m$, then use 2.1 in [4] to extend this to an action on all of D^m . Now $r : Z_n \times \partial D^m \rightarrow \partial D^m$, together with the extension to the normal bundle for $M \cap \bar{B} \subset \bar{B}$ of the complex splitting of the normal bundle for the fixed point set of $r|_{Z_n \times \partial D^m}$, represents the element $\partial(x) \in \bar{C}_{m-1}$. It is a routine exercise to check the exactness of the sequence. Q.E.D.

The following theorems give a good converse to the P. A. Smith Theorem, for differentiable actions on spheres.

Theorem 5.3. *Let M be a compact, almost complex manifold satisfying $H_*(M, Z_n) \cong H_*(S^j, Z_n)$, for some $j \geq 5$. Let m be a positive integer satisfying $2j + 4 < m$, $m - j = \text{even}$. If*

$$(1) \ j = 0 \ (4),$$

or

(2) $j + 2 = 0 \ (4)$ and the complex normal bundle for M has finite order in $[M, BU]$, then there exists an exotic sphere Σ^j , and a semi-free action $r : Z_n \times S^m \rightarrow S^m$ having $M \# \Sigma^j$ for fixed point set.⁷

Proof. According to 4.0 $M \# \Sigma^j$ is zero in Ω_j^{U, Z_n} for some Σ^j with trivial BU -bundle. Let W be a null cobordism, embedded $(W, \partial W) \subset (D^{m+1}, \partial D^{m+1})$ and use 2.1 in [4] to construct $Z_n \times D^{m+1} \rightarrow D^{m+1}$ having W for fixed point set and acting by multiplication by $e^{2\pi i/n}$ on the normal bundle to W in D^{m+1} . The restriction $Z_n \times \partial D^{m+1} \rightarrow \partial D^{m+1}$ is the desired action. Q.E.D.

Theorem 5.4. *Let M be an almost complex manifold satisfying $H_*(M, Z_n) = H_*(S^j, Z_n)$, for $j \geq 5$, $j = 1 \ (4)$. Let m satisfy $m > 2j + 4$, $m - j = \text{even}$. There exists an exotic sphere Σ^j so that $M \# \Sigma^j$ is the fixed point set of a semi-free Z_n -action $r : Z_n \times S^m \rightarrow S^m$.*

Proof. $M \# \Sigma^j$ is infinitely divisible in Ω_j^{U, Z_n} for some exotic sphere Σ^j with trivial BU bundle (4.0). Hence $(M \# \Sigma^j)$ vanishes in $\bar{C}_{m-1}((m-j)/2)$. Q.E.D.

Let $\mathfrak{L}$ denote the class of finite n -torsion groups. Let $C_m(d_1, d_2, \dots, d_l)$ be a group of equivalence classes of semi-free Z_n -actions on S^m defined in the introduction, with $G = \prod_{i=1}^l U(d_i)$ or $SO(d)$ for associated structural group. Recall $d(G) \equiv \sum_{i=1}^l 2d_i$ or $d(G) = d$ as the case may be.

Theorem 5.5. *Suppose both $m - d(G) \geq 5$, $m < 2d_i + d(G) - 5$ for all $d_i \neq 0$, are satisfied. Then there is, mod $\mathfrak{L}$, an exact sequence*

$0 \rightarrow F_{m+1} \rightarrow \tilde{C}(d_1, d_2, \dots, d_l) \xrightarrow{v} C_m(d_1, d_2, \dots, d_l) \xrightarrow{w} \Omega_{m-d(G)}^{G, \mathbb{Z}_n} \rightarrow F_m \rightarrow 0$,
 where $\tilde{C}_m$ is calculated in 1.2, $\Omega_{m-d(G)}^{G, \mathbb{Z}_n}$ is calculated in 4.0 for $G = U(d)$, and

$$F_m = \begin{cases} \mathbb{Z} & \text{if } m - d(G) = 2(4) \\ 0 & \text{otherwise} \end{cases}$$

Mod $\mathfrak{L}$ the equations of 4.0 hold with U replaced by G whenever $j \leq m - d(G)$.

Remark 5.9. Actually the mod $\mathfrak{L}$ exact sequences in 5.5 are mod $\mathfrak{L}$ *split exact*. The complete argument is not given here, but here is a brief description:

$$C_m \xrightarrow{w} \bar{\Omega}_{m-d(G)}^{G, \mathbb{Z}_n} \rightarrow 0$$

is mod $\mathfrak{L}$ exact. Let $M \subset S^m$ represent $x \in \bar{\Omega}_{m-d(G)}^{G, \mathbb{Z}_n}$. Cut an open ball from S^m , which intersects M transverse in an open ball, and denote by (D^m, K) the remaining portion of (S^m, M) . Let $r : \mathbb{Z}_n \times D^m \rightarrow D^m$ be the semi-free $\mathbb{Z}_n$ -action having K as fixed point set and restricting to complex multiplication by various primitive n^{th} roots of unity on the various complex bundle factors of the BG -normal bundle to K in D^m . $r : \mathbb{Z}_n \times D^m \rightarrow D^m$ extends to a unique *piecewise linear* semi-free action $r' : \mathbb{Z}_n \times S^m \rightarrow S^m$ having $M \subset S^m$ for fixed point set, by capping $r|_{\mathbb{Z}_n \times \partial D^m}$ off via the *cone construction*. The obstruction to smoothing the $\mathbb{Z}_n$ -action $r' : \mathbb{Z}_n \times S^m \rightarrow S^m$, modulo $r'|_{\mathbb{Z}_n \times D^m}$, can be shown to lie in a finite group depending only on n, m and G . If $r_1, r_2 : \mathbb{Z}_n \times S^m \rightarrow S^m$ are two smoothings for r' , the obstruction to making r_1, r_2 (smoothly) concordant also lie in a finite group depending only on n, m , and G . Hence there is a mod $\mathfrak{L}$ right inverse $w' : \bar{\Omega}_{m-d(G)}^{G, \mathbb{Z}_n} \rightarrow C_m$ to $w : C_m \rightarrow \bar{\Omega}_{m-d(G)}^{G, \mathbb{Z}_n}$ as claimed, defined by sending each $x \in \bar{\Omega}_{m-d(G)}^{G, \mathbb{Z}_n}$ into the $\mathbb{Z}_n$ -action obtained by smoothing the coned action constructed above.

Remark 5.10. Let $x \in C_m$, and $y \in \tilde{C}_m$ corresponding to $y \in \bar{L}_{m+1}^h(\mathbb{Z}_n)$ (see 1.2 above). There is a simple geometric construction for getting from an action $r : \mathbb{Z}_n \times S^m \rightarrow S^m$ representing x to one representing $x + y$. Let $O(S^m)$ denote the orbit space of $r : \mathbb{Z}_n \times S^m \rightarrow S^m$. Then $\pi_1(O(S^m - \tau_{F_r})) = \mathbb{Z}_n$, where F_r is the fixed point set of the given action and τ_{F_r} is an equivariant tubular neighborhood of F_r in S^m . Vary $1 : O(S^m - \tau_{F_r}) \rightarrow O(S^m - \tau_{F_r})$ thru a surgery cobordism which realizes the surgery obstruction $y \in \bar{L}_{m+1}^h(\mathbb{Z}_n)$ (see 5.8, 6.5 in [12]), and denote by X the resulting manifold. Let $\hat{X}$ be the universal covering space for X . τ_{F_r} glues to $\hat{X}$ along $\partial(\tau_{F_r})$ to give S^m ; actually $\hat{X} \cup \tau_{F_r}$ may be an exotic sphere, but mod $\mathfrak{L}$ this difficulty may be ignored. The action of $\pi_1(X)$ ($= \mathbb{Z}_n$) on $\hat{X}$ glues to the action $r|_{\mathbb{Z}_n \times \tau_{F_r}}$ to give a semi-free action $r' : \mathbb{Z}_n \times S^m \rightarrow S^m$, with $F_r \cdot r'$ represents $x + y$ in C_m .

Coupling this geometric construction with 5.9, we now have a way of generating (mod $\mathfrak{L}$) all the elements in C_m from $\Omega_{m-d(G)}^{G, \mathbb{Z}_n}$ and $L_{m+1}^h(\mathbb{Z}_n)$.

The groups $L_{m+1}^h(\mathbb{Z}_n)$ have been computed in many instances. For instance $L_{m+1}^h(\mathbb{Z}_n)$ is a finite two-torsion group for $m + 1$ and n both odd [7]. For n equal an odd prime and $m + 1$ even, $\bar{L}_{m+1}(\mathbb{Z}_n)$ is torsion free of rank $(n - 1)/2$ [8, 12].

For $n = 2 \bar{L}_{m+1}^h(\mathbf{Z}_n)$ equals $\mathbf{Z}$ when $m + 1$ is even and zero otherwise [13]. Using these calculations we can derive explicit expressions for the groups $C_m(d_1, \dots, d_l)$. In particular we have

Corollary 5.11. *For $C_m(d_1, \dots, d_l)$ as in 5.5, $C_m(d_1, \dots, d_l)$ is a finite group if $l' = 1$ and $m = \text{even}$. Here l' equals the number of d_i with $d_i \neq 0$.*

FOOTNOTES

1. I have before me work by M. Sebastiania [10] in which $\bar{C}_m$ is studied for the isolated fixed point case; and also an unpublished manuscript by M. Rothenberg and J. Sondow in which a Whitehead torsion invariant is involved with $\bar{C}_m$.

2. The difference in the calculations $\pi_1(0) = \mathbf{Z}_2$, $\pi_1(U) = \mathbf{Z}$ accounts for this.

3. In general $\lambda(h_i, h_j) = \sum \bar{a}_x \cdot x$, where the summation runs over $x \in \pi_1(N)$, and $\bar{a}_x$ has the form of the special case for $\lambda(h_i, h_j)$ considered above. The above argument now shows $\lambda(\alpha h_i, \alpha h_j) = 0$ for the general case.

4. The $\mathbf{Z}$ -summand in $L_{4j}(\mathbf{Z}_n(\{1\}))$ is determined by the index; A is determined by discriminant and Hasse-Symbol type invariants associated to quadratic forms representing surgery elements (see §63, §66, §72 of *Introduction to quadratic forms* by O. T. O'Meare.) Note that these exponent four invariants vanish on quadratic forms which are unimodular over the integers. I am indebted to C. T. C. Wall for reminding me of the Hasse and discriminant invariants involved in the classification of quadratic forms over the rationals.

5. This sequence is exact for $G = STOP, SPL$ as well.

6. This formula holds for $G = STOP$ or $G = SPL$ as well.

7. Σ^j is equipped with the trivial BU -structure.

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